A Note on the Specification of Error Structures in Latent Interaction Models

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**Xiulin Mao¹, Jeffrey R. Harring¹, and
Gregory R. Hancock¹**

Abstract

Latent interaction models have motivated a great deal of methodological research, mainly in the area of estimating such models. Product-indicator methods have been shown to be competitive with other methods of estimation in terms of parameter bias and standard error accuracy, and their continued popularity in empirical studies is due, in part, to their straightforward implementation and relative ease of estimation in mainstream structural equation modeling software. In recent years, the impact of different specifications of the mean structure of the structural model has been the focus of a fair amount of investigation in this area. Yet the effects of misspecification of the error structure of the observed variables implied by the model have not been investigated. In this study, the authors demonstrate algebraically the ramifications of misspecifying these error structures for the unconstrained product-indicator approach. Recommendations to practitioners based on these results are discussed.

Keywords

error structures, latent interaction models, unconstrained method, model misspecification

A latent interaction model is a type of nonlinear structural equation model in which at least one exogenous latent variable enters the structural model in a nonlinear fashion (Moosbrugger, Schermelleh-Engel, Kelava, & Klein, 2009). Although researchers have been interested in estimating the interactions between latent variables for some time, the real wave of such research interest was triggered by the pioneering work of

¹University of Maryland, College Park, MD, USA

Corresponding Author:

Xiulin Mao, University of Maryland, 1230 Benjamin Building, College Park, MD 20742-1115, USA
Email: xmao@umd.edu

Kenny and Judd (1984), who suggested that functions of observed variables mimicking functions of latent variables be integrated into the measurement model.

The aforementioned latent interaction model includes both latent endogenous and latent exogenous variables, as denoted in Equation 1, using the LISREL notation (Jöreskog & Yang, 1996),

$$\eta = \gamma_0 + \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_1\xi_2 + \zeta, \quad (1)$$

where η is a latent criterion variable; γ_0 is an intercept term; ξ_1 and ξ_2 are latent predictor variables; $\xi_1\xi_2$ is the interaction term; and γ_1 , γ_2 , and γ_3 are the regression coefficients corresponding to the linear predictors and the interaction term, respectively. The disturbance term ζ represents the regression residual.

Methods for estimating the interaction model in Equation 1 are numerous (Cudeck, Haring, & du Toit, 2009), but in this article, we focus on so-called product-indicator methods, which were originally proposed by Kenny and Judd (1984), and specifically the unconstrained approach (Marsh, Wen, & Hau, 2004). Within this framework, using a matched-pairs strategy suggested by Marsh et al. (2004), a latent cross-product term would be operationalized by creating product terms of the observed indicators of the exogenous latent variables. Suppose for illustrative purposes and without loss of generality that there are two latent exogenous latent variables in the model (ξ_1, ξ_2), and each latent exogenous variable has three indicators, x_1, x_2, x_3 and x_4, x_5, x_6 , respectively. The interaction term $\xi_1\xi_2$ will also have three product indicators, formed using a matched-pair strategy, namely, x_1x_4, x_2x_5 , and x_3x_6 . This latent interaction model under the unconstrained method is depicted in Figure 1.

The unconstrained approach is a simplification of constrained product-indicator approaches (Algina & Moulder, 2001; Jöreskog & Yang, 1996) in which factor loadings, the variance of the latent interaction term, and error variances of the product indicators that could be constructed using nonlinear constraints of other model parameters are freely estimated. In addition, given the assumption that the exogenous factors have means of zero, the mean of the interaction term is simply the covariance between the exogenous factors $\kappa_3 = E(\xi_1\xi_2) = E[(\xi_1 - \mu_{\xi_1})(\xi_2 - \mu_{\xi_2})] = \text{cov}(\xi_1, \xi_2) = \phi_{21}$ (see Appendix A for a detailed outline of these constraints). As in Wall and Amemiya (2001), the unconstrained approach allows for nonnormality among the exogenous latent predictors and the corresponding latent interaction term by freeing the covariances between these components.¹

The basic assumptions underlying the unconstrained approach with three indicators for each latent variable are as follows:

1. ξ_1 and ξ_2 are bivariate normal with means κ_1 and κ_2 , respectively;
2. $\delta_k \sim N(0, \theta_k)$, for $k = 1, \dots, 6$ and are independent of ξ_j for $j = 1, 2$;
3. $\varepsilon_l \sim N(0, \theta)$, for $l = 1, \dots, 3$ and are independent of δ_k for $k = 1, \dots, 6$;
4. $\zeta \sim N(0, \psi)$, and is independent of both δ_k and ξ_j for $k = 1, \dots, 6$ and $j = 1, 2$; and
5. $\text{cov}(\delta_k, \delta_{k'}) = 0$ for $k \neq k'$.

Figure 1. Schematic representation of a latent variable interaction model with three observed variable indicators for both exogenous, (ξ_1 , ξ_2), and endogenous, η , latent variables.

In a series of articles, Marsh and colleagues (Lin, Wen, Marsh, & Lin, 2010; Marsh et al., 2004, 2007; Marsh, Wen, & Hau, 2006) discussed nuances regarding the mean structure and centering for product-indicator methods. Absent from their deliberations regarding latent interaction models, however, was any discussion of the role that proper specification of the error structure of the observed variable indicators and covariance structure of the exogenous latent variables under multivariate normality may play in facilitating the estimation of unbiased coefficients of the structural regression model. For example, Marsh et al. (2004, 2007) stated that if the matched-pair strategy were used, the 9×9 covariance matrix, Θ_δ , whose elements are the variances and covariances between the observed variable error terms, would be diagonal: covariances fixed at zero. This statement is true if the multivariate normality assumption is satisfied (see, e.g., Algina & Moulder, 2001; Jöreskog & Yang, 1996). However, when the assumption of multivariate normality is released, it is not confined to the first-order exogenous latent variables only. Instead, the multivariate normality assumption applies to both the first-order exogenous latent variables and their indicators' error terms. Furthermore, other methods based on the unconstrained model, such as the doubly mean-centered approach (Lin et al., 2010) and the residual centering approach (Little, Bovaird, & Widaman, 2006), did not include these error

covariances either. Therefore, a “patch” is technically necessary to make the unconstrained model and methods based on the unconstrained model properly specified.

In this article, we demonstrate algebraically the ramifications of misspecifying these error structures for the unconstrained product-indicator approach and implications this may have on drawing inferences regarding the interaction effect. To support the findings from the analytic derivations, we further investigate the consequences of misspecification of the error structure and misspecification due to assuming that the joint distribution of the observed and latent variables is multivariate normal by conducting a small population analysis in which we quantify the degree to which these misspecifications adversely affect the interact effect.

Mathematical Derivation of Elements in the Error Covariance Matrix

As a starting point, for the model depicted in Figure 1, consider the covariance between δ_2 and δ_8 . The product of indicators x_2 and x_5 implies that $x_2x_5 = \tau_{x_{25}} + \lambda_{x_{25}}\xi_1\xi_2 + \delta_8$, from which $\delta_8 = x_2x_5 - \lambda_{x_{25}}\xi_1\xi_2 - \tau_{x_{25}}$ is derived. The covariance can be specified as

$$\theta_{82} = \text{Cov}[\delta_2, (x_2x_5 - \lambda_{x_{25}}\xi_1\xi_2 - \tau_{x_{25}})].$$

Because $x_2 = \tau_{x_2} + \lambda_{x_2}\xi_1 + \delta_2$, $x_5 = \tau_{x_5} + \lambda_{x_5}\xi_2 + \delta_5$, $\delta_8 = x_2x_5 - \lambda_{x_{25}}\xi_1\xi_2 - \tau_{x_{25}}$, and the $\text{Cov}(\delta_2, \tau_{x_{25}}) = 0$,

$$\begin{aligned} \theta_{82} &= \text{Cov}[\delta_2, ((\tau_{x_2} + \lambda_{x_2}\xi_1 + \delta_2)(\tau_{x_5} + \lambda_{x_5}\xi_2 + \delta_5) - \lambda_{x_7}\xi_1\xi_2)] \\ &= \text{Cov}[\delta_2, ((\lambda_{x_2}\lambda_{x_5} - \lambda_{x_{25}})\xi_1\xi_2 + \lambda_{x_2}\xi_1\delta_5 + \lambda_{x_5}\xi_2\delta_2 + \delta_2\delta_5 + \tau_{x_2}\tau_{x_5} + \tau_{x_2}\lambda_{x_5}\xi_2 \\ &\quad + \tau_{x_2}\delta_5 + \tau_{x_5}\lambda_{x_2}\xi_1 + \tau_{x_5}\delta_2)] \\ &= \text{Cov}[\delta_2, (\lambda_{x_2}\lambda_{x_5} - \lambda_{x_{25}})\xi_1\xi_2] + \text{Cov}[\delta_2, \lambda_{x_2}\xi_1\delta_5] + \text{Cov}[\delta_2, \lambda_{x_5}\xi_2\delta_2] + \text{Cov}[\delta_2, \delta_2\delta_5] \\ &\quad + \text{Cov}[\delta_2, \tau_{x_2}\tau_{x_5}] + \text{Cov}[\delta_2, \tau_{x_2}\lambda_{x_5}\xi_2] + \text{Cov}[\delta_2, \tau_{x_2}\delta_5] + \text{Cov}[\delta_2, \tau_{x_5}\delta_2]. \end{aligned}$$

On the basis of Bohrnstedt and Goldberger (1969), the covariance between a random variable and the product of two random variables can be solved as

$$\text{Cov}(xy, v) = E(x)\text{Cov}(y, v) + E(y)\text{Cov}(x, v) + E(\Delta x \Delta y \Delta v). \quad (2)$$

Defining $\Delta\lambda = \lambda_{x_2}\lambda_{x_5} - \lambda_{x_{25}}$, on the basis of Equation 2, the first piece of θ_{82} , $\text{Cov}[\delta_2, (\lambda_{x_2}\lambda_{x_5} - \lambda_{x_{25}})\xi_1\xi_2]$, can be reexpressed as

$$\begin{aligned} \text{Cov}[\delta_2, (\lambda_{x_2}\lambda_{x_5} - \lambda_{x_{25}})\xi_1\xi_2] &= \Delta\lambda E(\xi_1)\text{Cov}(\xi_2, \delta_2) \\ &\quad + \Delta\lambda E(\xi_2)\text{Cov}(\xi_1, \delta_2) + \Delta\lambda E(\Delta\delta_2 \Delta\xi_1 \Delta\xi_2). \end{aligned}$$

Moreover, given distributional assumptions about δ and ξ

$$\Delta\delta_2 = \delta_2 - E(\delta_2) = \delta_2$$

$$\Delta\xi_1 = \xi_1 - E(\xi_1) = \xi_1$$

$$\Delta\xi_2 = \xi_2 - E(\xi_2) = \xi_2$$

yields

$$\text{Cov}[\delta_2, (\lambda_{x_2}\lambda_{x_5} - \lambda_{x_{25}})\xi_1\xi_2] = \Delta\lambda E(\delta_2\xi_1\xi_2).$$

Similarly,

$$\text{Cov}[\delta_2, \lambda_{x_2}\xi_1\delta_5] = \lambda_{x_2}E(\delta_2\xi_1\delta_5)$$

$$\text{Cov}[\delta_2, \lambda_{x_5}\xi_2\delta_2] = \lambda_{x_5}E(\delta_2\xi_2\delta_2)$$

$$\text{Cov}[\delta_2, \delta_2\delta_5] = E(\delta_2\delta_2\delta_5),$$

and

$$\text{Cov}[\delta_2, \tau_{x_2}\tau_{x_5}] = 0$$

$$\text{Cov}[\delta_2, \tau_{x_2}\lambda_{x_5}\xi_2] = 0$$

$$\text{Cov}[\delta_2, \tau_{x_2}\delta_5] = 0$$

$$\text{Cov}[\delta_2, \tau_{x_5}\delta_2] = \tau_{x_5}\theta_{\delta_2}.$$

Therefore, by substitution,

$$\theta_{82} = \text{Cov}(\delta_2, \delta_8) = \Delta\lambda E(\delta_2\xi_1\xi_2) + \lambda_{x_2}E(\delta_2\xi_1\delta_5) + \lambda_{x_5}E(\delta_2\xi_2\delta_2) + E(\delta_2\delta_2\delta_5) + \tau_{x_5}\theta_{\delta_2}.$$

Now, on the basis of Bohrnstedt and Goldberger, if δ_2 , ξ_1 , ξ_2 are multivariate normal, all third moments equal zero. Therefore,

$$\theta_{82} = \tau_{x_5}\theta_{\delta_2}.$$

If indicator variables are centered, $\tau_{x_5} = 0$; thus, $\theta_{82} = 0$. However, if ξ_1 and ξ_2 are not assumed to be bivariate normally distributed, as claimed to be allowed by the unconstrained approach, the third moment will not vanish:

$$\Delta\lambda E(\delta_2\xi_1\xi_2) \neq 0.$$

Furthermore, when the broader assumption of multivariate normality is released, the nonnormality applies not only to the first-order exogenous latent variables but also to the error terms $(\delta_1, \dots, \delta_6)$. Therefore,

$$\text{Cov}[\delta_2, \lambda_{x_2}\xi_1\delta_5] \neq 0$$

$$\text{Cov}[\delta_2, \lambda_{x_5}\xi_2\delta_2] \neq 0$$

$$\text{Cov}[\delta_2, \delta_2\delta_5] \neq 0.$$

Thus, as the above mathematical derivations show, when the multivariate normality assumption is not satisfied, the error covariance elements are not zero, even when the matched-pair strategy is used to create the product indicators in the mean-centered unconstrained model proposed by Marsh et al. (2004, 2006). Whether the exclusion of these elements can affect parameter estimation is examined through an empirical population analysis in the next section.

Empirical Population Analysis

Notably different from a simulation study, a population analysis is not focused on the behavior of statistics from repeated samplings but rather on the behavior of the parameter(s) in the population (see, e.g., Bandalos & Leite, 2013). The analysis for the population study considers each condition independently, as one set of data will be generated for this condition, and the generated data are then regarded as the population data.

The design of this population analysis was based on crossing four manipulated independent variables, yielding 36 conditions overall: 3 (observed indicator variable reliability levels) $\times$ 3 (latent variable distributions) $\times$ 2 (error term distributions for the observed variables) $\times$ 2 (centering or noncentering). The population size for each cell was set to be $N = 500,000$.

Indicator Reliability

For the fully latent interaction model in which the structural model and measurement model are both accommodated in the larger model and estimated simultaneously, unreliability of the indicators may pose fewer threats (Harring, Weiss, & Hsu, 2012). However, it would be important to find out the possible influences the unreliability of the indicators might have on estimation of the models when they are specified properly and when they are misspecified. The reliabilities of the indicators in measurement model were set to be 0.95 (extremely high), 0.75 (medium), and 0.45 (low), with the indicator reliabilities set to be equal across all indicators in each condition.

Latent Variable and Error Distributions

To determine and quantify the impact of the misspecification on parameter estimation, different nonnormal conditions were used. Harring et al. (2012) pointed out that previous studies (Klein & Moosbrugger, 2000; Klein & Muthén, 2007; Marsh et al., 2004; Wall & Amemiya, 2001) had only investigated the impact of mild nonnormality on nonlinear latent structural model estimation. As discussed previously, the multivariate assumption applies not only to the latent exogenous variables but to the error terms for the observed indicators as well. Therefore, three conditions were chosen for latent exogenous variables: (a) skewness = 3, kurtosis = 20 (severe nonnormality); (b)

skewness = 2, kurtosis = 7 (moderate nonnormality); and (c) bivariate normality. Two conditions were chosen for error terms for the observed indicators: (a) normality and (b) skewness = 3, kurtosis = 20 (severe nonnormality). These conditions were fully crossed to adjudicate both independent effects as well as their joint influence. The Fleishman-Vale-Maurelli procedure (Fleishman, 1978; Vale & Maurelli, 1983) was used to generate combined levels of skewness and kurtosis for the nonnormal distributions of the observed indicator variables and latent exogenous predictors.

Data-Generating Model

The latent portion of the model follows the regression structure in Equation 1 with population data generated from

$$\eta = 10 + 2\xi_1 + 3\xi_2 + 0.127\xi_1\xi_2 + \zeta,$$

where, as Figure 1 implies, three observed variables were generated for each of the three latent variables. Observed indicators of the endogenous latent variable, y_1 , y_2 , and y_3 were not centered and entered into the model directly. For the uncentered approach, the indicators for the exogenous predictors (x_1 , x_2 , x_3 and x_4 , x_5 , x_6) were generated with means of 0. The endogenous indicator errors ε_1 , ε_2 , and ε_3 were generated with their variances chosen so that the reliability of each indicator corresponded to those levels under investigation (see, e.g., Bollen, 1989). For the centered approaches, the indicators for the exogenous predictors were mean centered with three product terms created using a matched-pair strategy on the basis of the centered indicators. The raw scores $x_1, \dots, x_6$ were used to derive product terms for the uncentered approach. The exogenous indicator errors, δ_i were generated under the distributional conditions previously described, again with the variances of the errors chosen so that the reliability of each indicator adhered to the study levels detailed above. The regression disturbance ζ was generated to have a mean of zero and variance of 10. Because the means of the exogenous variables ξ_1 and ξ_2 were arbitrary, they were generated as

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \sim N \left(\begin{bmatrix} 8 \\ 12 \end{bmatrix}, \begin{bmatrix} 16 & \\ 6 & 25 \end{bmatrix} \right)$$

and followed the distributional assumptions previously outlined. Following Marsh et al. (2004), the unstandardized loadings relating each indicator to its latent variable were set to 1 (for scaling purposes), 0.7, and 0.7.

Study Outcomes

Four models were fit to the 18 sets of population data respectively: the unconstrained model proposed by Marsh et al. (2004, 2006) with error variance-covariance structure constrained to be diagonal and is referred to as the uncorrelated error (UE) models,

centered and uncentered; the correlated error (CE) models with the six error covariances specified in the model, centered and uncentered (see mathematical derivation of latent covariances between exogenous predictors and interaction term in Appendix B).

Mplus 6.12 (Muthén & Muthén, 2010) was used to estimate each of the models. Because there are 18 sets of population data, 36 models were run (An Mplus input file can be found in Appendix C). The estimate for the population regression coefficient for the interaction effect, $\hat{\gamma}_3$, was the focus of the study, although the following conventional structural equation modeling fit indices were also mentioned: comparative fit index (CFI), root mean square error of approximation (RMSEA), and standardized root mean square residual (SRMR). It should be noted that the discrepancy between values in Table 1 and Table 3 for the exogenous predictors is a by-product of using a mean centered versus uncentered model. A proof is available from the authors.

Results and Discussion

Table 1 reports the results of the model fit indices and the model parameter estimates when the indicators $x_1, \dots, x_6$ were mean centered (with Table 2 defining NN-1 through NN-6). It can be seen that UE and CE performed equally well in terms of parameter estimates as well as the model fit indices across all the conditions, with the CFI always being 1 and the SRMR and RMSEA being extremely close to 0. Most parameters were recovered very well. All the estimates for the regression coefficient for the interaction effect were close to the population parameter 0.127, ranging from 0.122 to 0.130, and the average estimate of the interaction regression coefficient for UE models was 0.1271 and 0.1266 for CE models across the 18 conditions. Considering the random generation of the population data, this fluctuation seemed quite reasonable, and the estimates were very stable across all the conditions.

When the reliability was high (0.95), the UE and CE models produced identical parameter estimates for the regression coefficients for the interaction across all degrees of nonnormality. The estimates for the regression coefficients for the first-order effects were also same for the UE and CE models across almost all the conditions, except for severely nonnormal latent variables plus severely nonnormal errors. When the reliability of the indicators was medium (relatively good), similar or identical estimates for the regression coefficients could also be obtained for the UE and CE models across all the conditions. The same was also true of the coefficients for the first-order effects. When the reliability was low, there was small variation among the parameter estimates, but this variation we deemed acceptable. It can then be concluded for the models examined here that the degree of nonnormality and the indicator reliability actually did not influence the parameter estimates to an appreciable degree when the indicators were centered.

Table 3 presents the results of the model fit indices and the model parameter estimates when the indicators $x_1, \dots, x_6$ were uncentered. It can be seen that CE models performed perfectly well in terms of fit indices across all the conditions, with CFI always being 1 and the SRMR and RMSEA being close to 0. All the estimates for the regression coefficient for the interaction effect were close to the population parameter

Table 1. Results for the Mean-Centered Approach.

	$\hat{\gamma}_1$		$\hat{\gamma}_2$		$\hat{\gamma}_3$		CFI		SRMR		RMSEA	
	UE	CE	UE	CE	UE	CE	UE	CE	UE	CE	UE	CE
Reliability = 0.45												
NN-1	3.507	3.507	3.998	3.999	0.127	0.126	1.000	1.000	0.001	0.001	0.000	0.001
NN-2	3.527	3.527	4.011	4.011	0.126	0.128	1.000	1.000	0.001	0.001	0.000	0.000
NN-3	3.554	3.553	4.018	4.018	0.129	0.124	1.000	1.000	0.001	0.001	0.000	0.000
NN-4	3.563	3.565	3.915	3.913	0.122	0.123	1.000	1.000	0.004	0.002	0.000	0.004
NN-5	3.546	3.545	4.072	4.077	0.137	0.135	1.000	1.000	0.004	0.002	0.000	0.005
NN-6	3.576	3.575	4.084	4.085	0.124	0.122	1.000	1.000	0.003	0.002	0.002	0.002
Reliability = 0.75												
NN-1	3.522	3.522	4.016	4.017	0.127	0.127	1.000	1.000	0.001	0.001	0.000	0.001
NN-2	3.524	3.524	4.023	4.023	0.125	0.125	1.000	1.000	0.001	0.000	0.000	0.000
NN-3	3.527	3.527	4.008	4.008	0.127	0.127	1.000	1.000	0.001	0.000	0.000	0.000
NN-4	3.538	3.536	4.040	4.040	0.130	0.130	1.000	1.000	0.003	0.002	0.000	0.003
NN-5	3.527	3.529	4.016	4.016	0.127	0.126	1.000	1.000	0.002	0.002	0.000	0.004
NN-6	3.500	3.499	4.015	4.016	0.124	0.124	1.000	1.000	0.001	0.001	0.002	0.000
Reliability = 0.95												
NN-1	3.525	3.525	4.017	4.017	0.127	0.127	1.000	1.000	0.000	0.000	0.001	0.001
NN-2	3.520	3.520	4.016	4.016	0.127	0.127	1.000	1.000	0.000	0.000	0.001	0.001
NN-3	3.523	3.523	4.015	4.015	0.127	0.127	1.000	1.000	0.001	0.000	0.000	0.000
NN-4	3.509	3.510	4.034	4.034	0.126	0.126	1.000	1.000	0.002	0.001	0.006	0.005
NN-5	3.526	3.526	3.998	3.998	0.125	0.125	1.000	1.000	0.001	0.001	0.003	0.002
NN-6	3.534	3.534	4.028	4.028	0.130	0.130	1.000	1.000	0.002	0.001	0.002	0.002

Note. CE = correlated error model; CFI = comparative fit index; RMSEA = root mean square error of approximation; SRMR = standardized root mean square residual; UE = uncorrelated error model.

Table 2. Definitions of NN-1 Through NN-6.

		Latent Exogenous Variables		
		Severe Nonnormality	Moderate Nonnormality	Normal
Error	Normal	NN-1	NN-2	NN-3
	Severe nonnormality	NN-4	NN-5	NN-6

0.127, ranging from 0.122 to 0.130, and the estimates of the interaction regression coefficient averaged across the 18 conditions was 0.1263. Considering the random generation of the population data, this fluctuation was reasonable, and the estimates were very stable across all the conditions. The reliability of the indicators and the degree of nonnormality actually did not exert an influence on the estimation when error covariances were taken into account.

For UE models, however, in which the error covariances were not specified, when the reliability of the indicators was high (0.95), the latent parameter estimates deviated from their respective population values of 3, 2, and 0.127 to a large extent across all the conditions. At the same time, the estimates for the interaction parameter showed a consistent downward bias, but the estimates for the first-order regression coefficient tended to show an upward bias. The model fit indices did not support a good fit of the UE models, as the CFI was much lower than the common benchmark value 0.95, and the RMSEA was also much higher than the common benchmark value of 0.05. The SRMR, though, still performed relatively well.

When the reliability of the indicators was low (0.45) and medium (0.75), 12 conditions were not able to converge. When the reliability was 0.75, the estimates for the interaction coefficient were consistently lower than the population value, while they were consistently higher when the reliability of the indicators was low. The estimates for the regression coefficients for the first-order effects deviated from the true values to an even larger degree. The model fit indices also showed that UE models fit poorly because the error covariances were not properly specified. This also demonstrated that when there was model misspecification (in this case, the error covariance structures was missing), the reliability exerted a strong impact on the estimation of the model and would tend to cause convergence issues as the reliability of the indicators decreased. CE models, however, were stable and minimally affected by suboptimal reliability of the indicators. This result, therefore, illustrates the contention made previously that the unreliability of the indicators poses fewer problems to the fully latent model estimation, but this should be confined to the conditions in which the model, including error structure, was properly specified.

Summary and Conclusions

Both the mathematical derivation and empirical population analysis demonstrate that if the multivariate normality assumption is violated, the error covariances between

Table 3. Results for the Uncentered Approach.

	$\hat{\gamma}_1$			$\hat{\gamma}_2$			$\hat{\gamma}_3$			CFI		SRMR		RMSEA	
	UE	CE		UE	CE		UE	CE		UE	CE	UE	CE	UE	CE
Reliability = 0.45															
NN-1 ^a	0.975	1.980		2.376	2.989		0.154	0.127		0.662	1.000	0.063	0.001	0.241	0.001
NN-2 ^a	0.966	2.000		2.368	3.002		0.156	0.127		0.663	1.000	0.065	0.001	0.241	0.001
NN-3 ^a	0.830	2.082		2.272	3.039		0.169	0.122		0.647	1.000	0.070	0.001	0.241	0.000
NN-4 ^a	1.000	2.030		2.311	2.911		0.154	0.125		0.665	1.000	0.063	0.002	0.241	0.000
NN-5 ^a	0.941	2.068		2.397	3.082		0.161	0.125		0.666	1.000	0.065	0.002	0.239	0.002
NN-6 ^a	0.834	2.088		2.265	3.070		0.172	0.125		0.653	1.000	0.070	0.002	0.238	0.000
Reliability = 0.75															
NN-1 ^a	-4.701	1.985		-1.550	2.999		0.553	0.127		0.790	1.000	0.027	0.001	0.262	0.001
NN-2 ^a	0.158	2.017		1.781	3.000		0.241	0.126		0.790	1.000	0.028	0.002	0.262	0.002
NN-3 ^a	0.491	1.978		2.034	2.997		0.231	0.128		0.782	1.000	0.032	0.000	0.262	0.000
NN-4 ^a	-1.695	1.991		0.563	3.034		0.356	0.129		0.793	1.000	0.027	0.003	0.261	0.004
NN-5 ^a	-0.013	2.017		1.644	3.000		0.253	0.126		0.789	1.000	0.028	0.002	0.263	0.002
NN-6 ^a	0.450	2.016		1.997	3.008		0.233	0.124		0.780	1.000	0.032	0.002	0.262	0.002
Reliability = 0.95															
NN-1	2.099	2.005		3.073	3.005		0.119	0.127		0.883	1.000	0.005	0.000	0.270	0.001
NN-2	2.178	1.999		3.125	3.000		0.112	0.127		0.883	1.000	0.005	0.000	0.270	0.001
NN-3	2.316	1.997		3.213	2.999		0.099	0.127		0.880	1.000	0.006	0.000	0.270	0.000
NN-4	2.099	2.013		3.095	3.032		0.118	0.126		0.884	1.000	0.005	0.001	0.270	0.005
NN-5	2.238	2.025		3.147	2.998		0.108	0.125		0.883	1.000	0.006	0.001	0.270	0.000
NN-6	2.280	1.974		3.195	2.989		0.103	0.130		0.882	1.000	0.006	0.001	0.268	0.000

Note. CE = correlated error model; CFI = comparative fit index; RMSEA = root mean square error of approximation; SRMR = standardized root mean square residual; UE = uncorrelated error model.

a. The UE model encountered a convergence problem, yielding a nonpositive definite latent variable covariance matrix.

the product indicators and the first-order indicators based on the derived product indicators should be included in the model when the uncentered unconstrained model is used to estimate the model. With these error covariances being improperly constrained to be zero, the estimates of the regression coefficient for the interaction effect tend to show downward biases. At the same time, the model estimation is vulnerable to the unreliability of the indicators of the latent variables. It was shown that when the reliability of the indicators was extremely high, the model estimation performed reasonably well, but as the reliability of the indicators decreased, the chance of nonconvergence increased, and the model estimates tended to deviate from their population true values to a large extent. In contrast, when these error covariances were properly specified in the model, the model performed equally well in terms of parameter estimates and model fit indices, no matter the severity of nonnormality.

When the indicators for the first-order latent variables are mean centered before entering the model and the product terms are created on the basis of the deviation scores, whether or not these error covariances are specified seems to yield little effect on the parameter estimates and the model fit for the conditions examined. The mathematical derivation has shown that when the latent variables or error terms are not normally distributed, the error covariances between the indicators for the first-order latent variables and the indicators for the latent interaction term are not zero. But when the indicators for the first order latent variables are mean-centered, these covariances appear negligible (with corresponding correlations close to zero) and thus have little influence on the parameter estimation.

Previous studies (see, e.g., Algina & Moulder, 2001) indicated that the uncentered approach would encounter convergence problems. They derived the error variances and error covariances under multivariate normality and further pointed out that error variances and covariances were complicated functions of other parameters, thus causing the convergence problem. To avoid convergence problems, they recommended using the centered approach instead. The results from this study showed that the uncentered unconstrained model indeed suffered from this convergence problem. But the CE models proposed here were able to accommodate this uncentered condition and performed well in terms of both parameter estimates and the fit indices. When the indicators were centered, both UE and CE models performed well. This actually shows that the reason for the nonconvergence problem is not caused by whether the indicators are centered or not but by the fact that the previous studies have left out these error covariances, thus misspecifying the model in the first place. Under multivariate normality, the structure of these error covariances was very simple, as can be seen in the previous derivation. But if multivariate normality is violated, these error covariances have very complicated structures, including the third moments. In the uncentered approach, this misspecification of the model has a direct impact on the parameter estimation. But when the indicators are centered, the third moment left (see the previous derivation) in these covariances may be not large enough to affect the estimation result.

Even so, it is strongly recommended that CE models be used, because without including these error covariances, the model itself is misspecified. As a matter of fact, under the CE model specification, the choice between the uncentered and centered approaches becomes essentially arbitrary.

So far, the product-indicator method remains one of the most popular methods for fitting latent interaction models, because it is theoretically easier to understand by applied researchers and is easily implementable in standard structural equation modeling packages (EQS, LISREL, Mplus, etc.). Although insights and advances have been made in terms of the understanding of the mean structure and centering in product-indicator methods, how to specify the error structure has remained clouded or ignored. This study serves as a start to explore the proper specification of the error structures, and it reveals the importance of the proper specification of the error covariances for the unconstrained model.

Of course, in this study we focused only on the unconstrained model, and it is not yet known whether this misspecification will also affect the approaches built on the unconstrained model, such as the double-mean-centering approach (Lin et al., 2010) and the residual-centering approach (Little et al., 2006). Therefore, more studies are called for to study further the impact of the misspecified error structure within latent nonlinear models.

Appendix A

When estimating the interaction model such as the one specified in Equation 1, Kenny and Judd (1984) imposed several types of constraints upon the model: (a) constraints on the factor loadings; (b) constraining the variance of the latent interaction; (c) based on a normality assumption, constraining the covariance between the latent exogenous variables and their interaction to be zero; (d) constraining the errors of each of the indicators constituting the latent interaction; and (e) constraining the mean of the latent interaction.

Constraint of Factor Loadings

If exogenous latent variables ξ_1 and ξ_2 each have three indicators, x_1, x_2, x_3 and x_4, x_5, x_6 , as in the measurement model represented in Figure 1, then the loadings of the indicators on the interaction term were constrained to be equal to the product of the loadings for the two indicators that created the interaction indicator

$$\lambda_7 = \lambda_1 \lambda_4 = 1, \lambda_8 = \lambda_2 \lambda_5, \lambda_9 = \lambda_3 \lambda_6.$$

Constraint of Variance of the Interaction Term

Assuming that $\xi_1, \xi_2, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6$, and ζ are in mean-deviation form, multivariate normal, and uncorrelated (except that ξ_1 and ξ_2 are allowed to correlate), the variance of the interaction latent variable $\xi_1 \xi_2$ can also be constrained as

$$\begin{aligned}
\phi_{33} &= \text{var}(\xi_1 \xi_2) \\
&= \text{var}(\xi_1) + \text{var}(\xi_2) + [\text{cov}(\xi_1, \xi_2)]^2 \\
&= \phi_{11} \phi_{22} + \phi_{21}^2.
\end{aligned}$$

Normality Constraint

On the basis of the normality assumption, the covariance between the interaction term and each of the first-order terms were set to zero (i.e., $\phi_{31} = \phi_{32} = 0$).

Constraints of Error Structures

The error structure of product indicators are constrained to be the functions of error structures of the indicators of the exogenous latent variables:

$$\begin{aligned}
\delta_7 &= \xi_1 \delta_4 + \delta_1 \xi_2 + \delta_1 \delta_4 \\
\theta_{\delta_7} &= \phi_{11} \theta_{\delta_4} + \phi_{22} \theta_{\delta_1} + \theta_{\delta_1} \theta_{\delta_4} \\
\delta_8 &= \lambda_{x_2} \xi_1 \delta_5 + \lambda_{x_5} \delta_2 \xi_2 + \delta_2 \delta_5 \\
\theta_{\delta_8} &= \lambda_{x_2}^2 \phi_{11} \theta_{\delta_5} + \lambda_{x_5}^2 \phi_{22} \theta_{\delta_2} + \theta_{\delta_2} \theta_{\delta_5} \\
\delta_9 &= \lambda_{x_3} \xi_1 \delta_6 + \lambda_{x_6} \delta_3 \xi_2 + \delta_3 \delta_6 \\
\theta_{\delta_9} &= \lambda_{x_3}^2 \phi_{11} \theta_{\delta_6} + \lambda_{x_6}^2 \phi_{22} \theta_{\delta_3} + \theta_{\delta_3} \theta_{\delta_6}.
\end{aligned}$$

Constraint of the Mean of the Interaction Term

On the basis of the Kenny-Judd model, Jöreskog and Yang (1996) added another constraint, which was to constrain the mean of interaction term,

$$\kappa_3 = E(\xi_1 \xi_2) = E[(\xi_1 - \mu_{\xi_1})(\xi_2 - \mu_{\xi_2})] = \text{cov}(\xi_1, \xi_2) = \phi_{21},$$

where ξ_1 and ξ_2 are assumed to have a mean of zero.

Appendix B

From Bohrnstedt and Goldberger (1969), the variance of a product of two random variables, in this case, the interaction term, $\xi_1 \xi_2$, can be specified as

$$\begin{aligned}
\text{Var}(\xi_1 \xi_2) &= \kappa_1^2 \phi_{22} + \kappa_2^2 \phi_{11} + E[(\Delta \xi_1)^2 (\Delta \xi_2)^2] \\
&\quad + 2\kappa_1 E[(\Delta \xi_1)(\Delta \xi_2)^2] + 2\kappa_2 E[(\Delta \xi_1)^2 (\Delta \xi_2)] \\
&\quad + 2\kappa_1 \kappa_2 \phi_{21} - \phi_{21}^2,
\end{aligned}$$

where $\Delta \xi_t = \xi_t - E[\xi_t]$, for $t = 1, 2$. Under bivariate normality, the third moments disappear, and $E[(\Delta \xi_1)^2 (\Delta \xi_2)^2] = \phi_{11} \phi_{22} + 2\phi_{21}^2$. As a consequence,

$$Var(\xi_1\xi_2) = \kappa_1^2\phi_{22} + \kappa_2^2\phi_{11} + 2\kappa_1\kappa_2\phi_{21} + \phi_{11}\phi_{22} - \phi_{21}^2. \quad (B1)$$

In practice, if the exogenous predictors have (or are assumed to have) zero means, then the formula above reduces to

$$Var(\xi_1\xi_2) = \phi_{11}\phi_{22} - \phi_{21}^2. \quad (B2)$$

If, on the other hand, the exogenous predictors do not have a zero mean vector, the variance of the interaction term is denoted by the more complicated expression in Equation B1.

The covariance between each exogenous predictor and the product term can be specified as

$$Cov(\xi_1, \xi_1\xi_2) = \kappa_1\phi_{21} + \kappa_2\phi_{11} + E[\Delta\xi_1\Delta\xi_1\Delta\xi_2], \quad (B3)$$

$$Cov(\xi_2, \xi_1\xi_2) = \kappa_2\phi_{21} + \kappa_1\phi_{22} + E[\Delta\xi_1\Delta\xi_2\Delta\xi_2]. \quad (B4)$$

Under multivariate normality, the last term in the expressions (the third moment) is zero. However, unless the factors have a zero mean vector, even under normality, the covariances do not necessarily disappear. Of course, most past research (Algina & Moulder, 2001; Jöreskog & Yang, 1996; Marsh et al., 2004; Wall & Amemiya, 2001) assumes this, so that

$$Cov(\xi_1, \xi_1\xi_2) = 0 \quad Cov(\xi_2, \xi_1\xi_2) = 0.$$

Appendix C

TITLE:

Population analysis for mean-centered model

DATA:

FILE IS C:\ data5c.dat;

NOBSERVATIONS=500000;

DEFINE:

x1z1 = x1*z1;

x2z2 = x2*z2;

x3z3 = x3*z3;

VARIABLE:

NAMES ARE x1 x2 x3 z1 z2 z3 y1 y2 y3;

USEVAR ARE x1 x2 x3 z1 z2 z3 y1 y2 y3 x1z1 x2z2 x3z3;

CENTERING IS GRANDMEAN(x1-x3 z1-z3);

ANALYSIS:

ESTIMATOR=ML;

ITERATIONS=10000;

MODEL:

[x1-x3@0];

```

[z1-z3@0];
[x1z1@0];
x1-x3*; z1-z3*; x1z1-x3z3*; y1-y3*;
!For the "Correlated error" model, error covariances are specified
!For the "Uncorrelated error" model, these error covariances would not be specified
!x1 WITH x1z1; x2 WITH x2z2; x3 WITH x3z3;
!z1 WITH x1z1; z2 WITH x2z2; z3 WITH x3z3;
X BY x1@1 x2*.7 x3*.7;
Z BY z1@1 z2*.7 z3*.7;
Y BY y1@1 y2*.7 y3*.7;
XZ BY x1z1@1 x2z2*.49 x3z3*.49;
Y ON X* Z* XZ*; [Y@0];
X*; Z*; XZ*; Y*;
Z WITH X*; X WITH XZ*; Z WITH XZ*;
[X@0]; [Z@0]; [XZ*];
OUTPUT;
SAMP STAND TECH4;

```

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Note

1. Under multivariate normality of the latent predictors, the covariances between the first-order terms and the interaction are constrained to be zero (Algina & Moulder, 2001).

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