Published in final edited form as:

J Behav Decis Mak. 2018 January ; 31(1): 100–114. doi:10.1002/bdm.2037.

The role of independence and stationarity in probabilistic models of binary choice

Michel Regenwetter and

Department of Psychology, University of Illinois at Urbana-Champaign, 603, E. Daniel Street, Champaign, IL 61820, USA

Clinton P. Davis-Stober

Department of Psychological Sciences, University of Missouri at Columbia Columbia, MO 65211, USA

Abstract

After more than 50 years of probabilistic choice modeling in Economics, Marketing, Political Science, Psychology, and related disciplines, theoretical and computational advances give scholars access to a sophisticated array of modeling and inference resources. We review some important, but perhaps often overlooked, properties of major classes of probabilistic choice models. For within-respondent applications, we discuss which models require repeated choices by an individual to be independent and response probabilities to be stationary. We show how some model classes, but not others, are invariant over variable preferences, variable utilities, or variable choice probabilities. These models, but not others, accommodate pooling of responses or averaging of choice proportions within participant when underlying parameters vary across observations. These, but not others, permit pooling/averaging across respondents in the presence of individual differences. We also review the role of independence and stationarity in statistical inference, including for probabilistic choice models that, themselves, do not require those properties.

Keywords

Probabilistic choice models; iid assumptions; modeling heterogeneity; statistical testing

When selecting a probabilistic choice model to use in their work, scholars may want to consider which model class makes what assumptions and which model class warrants or precludes what types of data manipulations. In this conceptual and theoretical synthesis, we summarize, for several prominent probabilistic choice models, whether or not they assume mutually independent (in the sense of probability theory) responses and/or stationary response probabilities.¹ We also unpack whether various models are invariant under varying preferences, varying utility functions, or varying response probabilities, either because of individual differences or because of within-subject heterogeneity in behavior. We discuss what it means to average or pool binary choice data across respondents and/or across time

¹It is beyond the scope of this paper to discuss tests of these properties. There is a small literature on this topic, but little consensus about the appropriate methodology (see, e.g., Birnbaum 2012, Cha et al. 2013, Dai 2017, Smith and Batchelder 2008).

points. To that end, we spell out, for some prominent statistical tests of probabilistic choice models, whether they, in turn, make additional independence and/or stationarity assumptions even when the models themselves do not require such properties to hold. We concentrate on those models, statistical tests, and data generating processes whose basic observational unit - one observation - is one person's selection of one choice alternative among two, at one moment in time.

Motivation and Background

Consider the following two questions posed in an experiment. We use these questions for various illustrations throughout the paper.

QUESTION 1: Which do you prefer, Gamble A that offers a 28% chance of winning \$31.43, or Gamble B that offers a 32% chance of winning \$27.50?

QUESTION 2: Do you prefer Gamble C that offers a 36% chance of winning \$2444, or Gamble D that offers a 40% chance of winning \$22?

These gambles have very similar expected values, namely $EV(A) = \$8.8004$, $EV(B) = \$8.8$, $EV(C) = \$8.7984$, $EV(D) = \$8.8$. According to algebraic utility theory, a decision maker's response to such a question is determined by his or her preference: For example, an expected value maximizer prefers A to B because A has a higher expected value than B. Such a decision maker likewise prefers D to C.

If a person made choices perfectly consistent with expected value theory, then this person would always choose gamble A when presented with gamble pair (A,B); and s/he would always choose gamble D when presented with gamble pair (C,D). There would be neither uncertainty in individual choices nor variability in choice behavior across multiple choice occasions. If expected value theory were to hold then there would also be no variability across people. However, in situations like this, different people make different choices and most individuals fluctuate in their choices if some questions are repeated. It is also common, for 'difficult' choices, that decision makers waiver before making a choice. It has been a famous challenge to formally model individual differences and other sources of choice variability, such as response errors. There is an extensive literature across several disciplines about how to expand algebraic utility theory into probabilistic choice models (see, e.g., Ballinger and Wilcox 1997, Harless and Camerer 1994, Hey and Orme 1994, Loomes and Sugden 1995, Luce 1959, Luce and Suppes 1965, Luce and Narens 1994, Luce 1997, Marley and Regenwetter 2017, Rieskamp 2008, Wilcox 2008). This challenge is sometimes referred to as *Luce's challenge* to honor R. Duncan Luce's life long contributions to tackling this problem (e.g. Luce 1959, Luce and Suppes 1965, Luce and Narens 1994, Luce 1997). The key goal has been to replace or augment algebraic models by probabilistic ones. Much of that literature has emphasized that a probabilistic choice model is not a mere ad-on to an algebraic theory like expected value theory, cumulative prospect theory or, say, hyperbolic discounting theory, because different probabilistic choice models, grafted onto one and the same algebraic core theory, make rather different predictions about behavior. While highly relevant to that line of inquiry, our considerations here are at a more basic, that is, fundamental and conceptual, level.

Scholars employ probabilistic choice models in two major ways: One major approach is often referred to as stochastic (or probabilistic) specification. Here, scholars usually aim to accomplish two goals. 1) They aim to move from algebra, such as utility maximization (e.g., Kahneman and Tversky 1979, Tversky and Kahneman 1992) or a heuristic decision rule (such as, e.g., Gigerenzer and Goldstein 1999, Goldstein and Gigerenzer 2002), to choice probability (see, e.g., Blavatsky and Pogrebna 2010, Harless and Camerer 1994, Hey and Orme 1994, Nilsson et al. 2011, Rieskamp 2008, Stott 2006, Hilbig and Moshagen 2014). The purpose of this step is to model either a decision maker's uncertainty about what to choose or the scholar's uncertainty about what the decision maker, who satisfies a given theory would, in fact, choose. 2) They aim to move from modeling the probability of one observation to modeling the joint probability of a collection of observations. This probability provides the underpinnings of the likelihood function that underlies many statistical analyses, such as goodness of fit, parameter estimation and out of sample prediction. Rather than transform a deterministic theory into a probabilistic one, the second major approach uses stochastic process models to capture inherently probabilistic constructs, such as deliberation (see, e.g. Brown and Heathcote 2008, Busemeyer and Townsend 1993, Ratcliff and Rouder 1998, Tsetsos et al. 2016a) or persuasion over time (see, e.g. Falmagne 1997, Regenwetter et al. 1999, Falmagne and Ovchinnikov 2002). The strength of this approach lies in its ability to reach beyond probabilistic choice and generate additional predictions, such as reaction times or joint response probabilities for panel data. The first approach oftentimes leaves independence and stationarity assumptions implicit, e.g., because it is very natural to assume that, say, the utility of a choice option, is constant within a given person over the course of an experiment. In contrast, stochastic process models usually spell out how choices at different time points (and/or of different respondents) are related to each other. While our discussion is relevant to both approaches, we cast the paper in terms of the first approach.

In the example above, the first step is to develop a model for the probability P_{AB} that A is chosen over B in gamble pair (A,B); and for the probability P_{CD} that C is chosen over D in gamble pair (C,D). In its most general form, for the two pairs of gambles above, this binary choice model has two free parameters P_{AB} and P_{CD} (since $P_{BA} = 1 - P_{AB}$ and $P_{DC} = 1 - P_{CD}$). The probability P_{AB} is the probability for one observation, as is the probability P_{CD} . If we were to consider 100 potential questions, modeling each observation separately and all observations jointly, a comprehensive model of *binary choice probabilities* could have 100 free parameters, one for each of the 100 potential questions we may ask.² What about the joint probability of a collection of observations? This is what we need for likelihood-based statistical analysis. As we see later, models of combinations of binary choice probabilities and statistical tests of binary choice models on data sets, as used in the literature, usually reduce the number of parameters dramatically with the help of independence and stationarity assumptions.

For most of the paper, we concentrate on what different models predict about various behaviors and their probabilities. That discussion is about models and their conceptual and

²There could even be additional parameters modeling relationships among responses.

mathematical properties, not about inference from empirical observations. Throughout that discussion, we only mention data and statistical inference in passing. The last section of the paper then changes perspective to discuss data and statistical inference. Because the paper is intended as a nontechnical theoretical and conceptual synthesis, we omit technical mathematical details and we often forego mathematical generality by concentrating on simple examples that are representative of general facts.

Independent choices and stationary choice probabilities

The first part of this paper reviews which probabilistic choice models have the property or the requirement that binary choices be independent and identically distributed (iid). We will usually refer more concretely to *stationary* probabilities *instead of identically distributed choices* when possible, for ease of reading. For most of this discussion, we refer to models of an individual. We define *independence*³ of two binary choices in Table 1. To be concrete, we consider two questions that were asked at two distinct times t_1 and t_2 . A decision maker makes independent binary choices regarding gamble pair (K,L) at time t_1 and gamble pair (X,Y) at time t_2 if the probability of choosing K and X is the product $P_{KL} \times P_{XY}$ (upper left quadrant of the table), the probability of choosing K and Y is the product $P_{KL} \times P_{YX}$ (lower left quadrant of the table), etc. Binary choices are *identically distributed* (at different points in time or for different respondents) if the pairwise choice probability for each pair of choice alternatives is constant across repeated observations (over time or across participants). Since we emphasize models of the individual, as we mentioned above, the following terminology is more self-explanatory. The binary choice probabilities are *stationary* if the probability of choosing one alternative over another is the same no matter whether the question is raised at t_1 or t_2 . In Table 1, if gamble pair (K,L) presented at t_1 is the same gamble pair as (X,Y) presented at t_2 , then stationarity means that $P_{KL} = P_{XY}$ and nonstationarity means that $P_{KL} \neq P_{XY}$. In Table 1, binary choices can satisfy or violate any combination of independence and stationarity, because the products inside the table may or may not hold, and the column and row marginal probabilities P_{KL} , P_{XY} may or may not be equal when $(K,L) = (X,Y)$.

We now review well-known models of binary choice probabilities. In each case, we consider whether the given model implies independence and/or stationarity. Tables 2-4 below show the same kind of information as Table 1 above, but in a more compact format.

The Luce Model (Bradley-Terry model, Softmax Rule)

We start with a famous model where the binary choice probability for a pair of gambles is defined axiomatically as a function of the “strength of preference” derived from an algebraic model. For the latter, we use expected value theory for simplicity. *Luce's choice model* for binary choice probabilities (Luce, 1959), which is also often referred to as *the choice axiom*, translates numerical utilities into binary choice probabilities. While there are several variants of the Luce model (see, e.g., Cattelan 2012, Luce 1977, for reviews), we focus on one variant without much loss of generality: For an expected value maximizer, a common version of the Luce model states that

³These concepts are somewhat technical. Nonetheless, we keep formal notation to a minimum, for ease of reading. Where the distinction becomes crucial, we employ the more precise notation $P_{1,KL}$, $P_{2,XY}$; etc., to keep track of the time of observation.

$$\begin{aligned} P_{AB} &= \frac{e^{EV(A)}}{e^{EV(A)} + e^{EV(B)}} = \frac{e^{8.8004}}{e^{8.8004} + e^{8.8}} = .5001, \\ P_{CD} &= \frac{e^{EV(C)}}{e^{EV(C)} + e^{EV(D)}} = \frac{e^{8.7984}}{e^{8.7984} + e^{8.8}} = .4996. \end{aligned} \quad (1)$$

This is a model of binary choice probabilities with no free parameters at all, because the binary choice probabilities are uniquely and fully determined by the expected values of the gambles. Because all the expected values are essentially equal, these choice probabilities are essentially $\frac{1}{2}$. In a Luce model, nothing is said explicitly about repeated binary choices and joint probabilities of all choices forming a data set. Nonetheless, it is clear that binary choice probabilities as defined by Expression 1 among the same pair of gambles are implicitly assumed to be stationary over repeated trials: Since the expected values of A and B do not change over repeated presentations of lotteries A and B, the probability P_{AB} is fixed across repeated choices among A and B.

Expression 1 does not, however, require repeated choices to be independent of each other. Using slightly simplified notation compared to Table 1, Table 2 shows an example in which the binary choice probabilities satisfy Luce's choice model for binary choice probabilities with expected value as utility function as stated in Expression 1, but the binary choices are not independent. Since the Luce model (1) does not specify joint probabilities of multiple binary choices, the interior cells of the table are free to take any values as long as the binary choice probabilities along the margins of the table satisfy the Luce model (1). For instance, the probability that the decision maker chooses A in QUESTION 1 and C in QUESTION 2 could be .49, as shown in the upper left quadrant of Table 2. If the decision maker chose independently, this probability would need to be the product of the binary choice probabilities, i.e., $.4996 \times .5001 = P_{AB} \times P_{CD} = .2498$. The binary choice probabilities along the margins of the table satisfy Luce's choice model for an expected value maximizer, but binary choices in this table are not independent.

In sum, the Luce model, as stated in Eq. 1, does not imply (i.e., does not require) independent binary choices but, if the utility function is fixed, as is the case with EV, then it implies stationary binary choice probabilities. On the other hand, it is straightforward that the Luce model readily violates stationarity for most core theories. Consider, for example, cumulative prospect theory, in which the overall utility $u(K)$ of a lottery K depends on parameters governing the probability weighting function and the utility function for money. If these parameter values are not fixed, then the Luce model

$$P_{KL} = \frac{e^{u(K)}}{e^{u(K)} + e^{u(L)}} \Bigg| \text{Strength of preference in favour of } K$$

may generate different choice probabilities among the same two choice alternatives K, L at different times. For example, if the value of the probability weighting parameter(s)

governing the person's preferences differs between t_1 and t_2 , then the values $u(K)$ and $u(L)$ at t_1 and t_2 may differ. Consequently, the choice probabilities need not be stationary.

As we see later, it is standard in applications and statistical tests of the Luce model (e.g. Abdellaoui et al. 2011, Camerer and Ho 1994, Stott 2006) to make an auxiliary, and oftentimes implicit, iid assumption. According to that assumption, the choice between A and B in QUESTION 1 and the choice between C and D in QUESTION 2 are independent of each other (i.e., the scenario in Table 2 is disallowed), and the utility values $u(K)$, $u(L)$ of any options K , L , are constant across time points t_1 and t_2 . This assumption is not part of the Luce model but rather an auxiliary assumption added for theoretical or statistical tractability: Theoretically, a model with a fixed utility for each option is more parsimonious. Statistically, such a theory reduces degrees of freedom, it strongly simplifies parameter estimation and hypothesis testing, and hence it strongly simplifies prediction of out-of-sample behavior.

This means that a poor statistical fit or predictive performance of a Luce model on a set of data can have at least three, very different, causes: i) the core theory may be invalid, ii) the Luce model may not be an appropriate representation of the heterogeneity in the data, or iii) the independence and stationarity assumptions, which are not part of the Luce model itself, may be inappropriate for the data at hand, even when the core theory and the Luce model are valid. For example, a decision maker may satisfy cumulative prospect theory and the Luce model may accurately represent the uncertainty in executing his preferences in choice, but the probability weighting parameter(s) and the utility for money might not be identical at all times.

The constant error model

A second, quite different model of binary choice probabilities, is the *constant error* model⁴ (see, e.g., Bröder and Schiffer 2003, Carbone and Hey 2000, Glöckner 2009). It says that, in any binary choice, there is some chance ϵ that a decision maker mistakenly chooses an option that s/he does not actually prefer. Hence, the basic formulation of the constant error model has one single free parameter, ϵ . Here, the binary choice probabilities for an expected value maximizer in QUESTION 1 and QUESTION 2 are $P_{AB} = 1 - \epsilon$, $P_{CD} = \epsilon$. It is natural to assume that the error rate is bounded by $\epsilon \leq \frac{1}{2}$, or even $\epsilon < \frac{1}{2}$: A person is more likely to choose what s/he prefers than what s/he does not prefer. Some scholars who use constant error models consider models with estimated error rates much higher than 25% as poor fits and models with estimated error rates far below 15% as cases of potential overfitting (e.g., Camerer 1989, Harless and Camerer 1994, Starmer and Sugden 1989).

Notice that, according to the constant error model, especially with a small error rate, the binary choice probabilities for the two questions are very different from those in the Luce model above. Even for the same algebraic core theory, these two models make very different predictions about behavior. Nonetheless, like Luce's model, the constant error model does not imply (i.e., does not require) independence, but thanks to the fixed value of ϵ , if the

⁴We do not discuss the various “true-and-error models” of M. Birnbaum here because their basic unit of observation is not one pairwise choice among one pair of options, rather it is a pattern of pairwise choices. They are also not models of an individual decision maker in that they typically model heterogeneity across repeated choices and across decision makers jointly.

preferences remain fixed then the model yields stationary binary choice probabilities. We give an example of dependent binary choice in Table 3, for an expected value maximizer: For example, the probability of choosing B in QUESTION 1 and D in QUESTION 2 is $\frac{\epsilon}{2}$, which differs from $P_{AB} \times P_{DC} = (1 - \epsilon) \times (1 - \epsilon)$ whenever $\epsilon < \frac{1}{2}$. Overall, like the Luce model with a fixed utility function, the constant error model with a fixed preference state (here “A preferred to B and D preferred to C”) implies stationary choice probabilities but not independence. Like the Luce model, a constant error specification of cumulative prospect theory or of any other theory with free parameters, readily violates stationarity unless the algebraic core structure and its parameter values are invariant across observations. Like the Luce model, the constant error model may fit or predict data poorly because i) the core theory is inadequate, or ii) the constant error specification does not capture the heterogeneity at hand, or iii) the independence and stationarity assumptions inherent in the statistical analysis, that are not actually part of the constant error model itself, are not warranted, even when both the core theory and the constant error model are valid.⁵

We proceed to another classic model, the “weak utility” model, whose nested submodels include the Luce model, and, in the case of transitive preferences, also the constant error model.

The weak utility model and weak stochastic transitivity

When the utility function u is the expected value, the *weak utility model* (Luce and Suppes 1965) states that

$$\underbrace{P_{KL} \geq \frac{1}{2} \Leftrightarrow u(K) \geq u(L)}_{\text{weak utility model}} \Leftrightarrow \underbrace{EV(K) > EV(L)}_{\text{for } u = EV} \text{ for any distinct } K, L. \quad (2)$$

This is also called a *modal choice* model in that the preferred alternative is more likely to be chosen than not (Regenwetter et al. 2014). If we abstract away from theories like expected value, the weak utility model on the left in its general form (with the utility function u a free ‘parameter’), in turn, is mathematically equivalent to *weak stochastic transitivity* (Block and Marschak 1960, Iverson and Falmagne 1985, Luce and Suppes 1965, Marschak 1960, Rieskamp et al. 2006, Tversky 1969), according to which, for all distinct K, L, M ,

$$\text{if we have } \left[P_{KL} \geq \frac{1}{2} \text{ and } P_{LM} \geq \frac{1}{2} \right] \text{ then we also have } P_{KM} \geq \frac{1}{2}. \quad (3)$$

Since this property has to hold for every triple of three distinct options, weak stochastic transitivity comprises a rather complicated set of constraints on binary choice probabilities.

⁵For a paper that replaces constant error rates by theoretically motivated order-constraints on error rates, see Hilbig and Moshagen (2014).

The weak utility model can violate independence, as we have already documented with both the dependent Luce model in Table 2 and the dependent constant error model in Table 3. The binary choice probabilities according to the weak utility model also need not be stationary. Even with a fixed utility function, say, expected value, the binary choice probabilities can vary, as long as they stay within the constraints set by Expression 2. More generally, a decision maker could satisfy the equivalence in Expression 2 when asked QUESTION 1 and she could satisfy the weak utility model with a different function u on the right hand side of Expression 2 in QUESTION 2. Hence, Expression 2 could yield $P_{AB} > \frac{1}{2}$, $P_{DC} > \frac{1}{2}$ in QUESTION 1, and $P_{BA} > \frac{1}{2}$, $P_{CD} > \frac{1}{2}$ in QUESTION 2. In all: Weak stochastic transitivity implies neither independent choices nor stationary choice probabilities.

Parametric random utility (discrete choice) models

A major competitor to the constant error model are parametric random utility models such as Thurstone models in psychology and probit and logit models in econometrics. These models assume that there are noise or error random variables $\mathbf{E}_K, \mathbf{E}_L, \dots$ such that, in the case of expected value maximization,

$$P_{KL} = \text{prob}(\underbrace{EV(K) + \mathbf{E}_K}_{\text{noisy utility of } K} > \underbrace{EV(L) + \mathbf{E}_L}_{\text{noisy utility of } L}). \quad (4)$$

Often, but not always, the joint distribution used for the noise random variables implies that they are independent and identically distributed (within one binary choice). When the noise random variables satisfy a multivariate normal distribution, they are called *Thurstone* or *probit* models, when they satisfy a multivariate extreme value distribution, they are called *logit* models (Ballinger and Wilcox 1997, Blavatsky and Pogrebna 2010, Conte et al. 2011, Fu et al. 2000, Hausman and Wise 1978, Liu and Mahmassani 2000, Marley 1992, Maydeu-Olivares and Hernández 2007, McFadden and Train 2000, McFadden 1991, McFadden and Richter 1970, Manski and McFadden 1981, Papatla and Krishnamurthi 1992, Stott 2006, Strauss 1981, Thurstone 1927b,a, Wilcox 2008). The Thurstonian/probit modeling approach includes models whose multivariate normal distribution (within a trial) are either independent or correlated (see, e.g., Tsai and Böckenholt 2006, Thurstone 1927a,b). Discrete choice models of the form given in Expression 4 with iid (within each binary choice) Gumbel (i.e., double-exponential) distributions have been shown to satisfy a version of Luce's choice model for binary choice (Yellott 1977). The result was extended to incorporate interdependent random utilities (Strauss 1981). Similar to the Luce model, while discrete choice models do not technically require stationarity, scholars often assume stationarity for the sake of theoretical simplicity and statistical tractability.

Some scholars (e.g., Carbone and Hey 2000, Hey and Orme 1994, Hey and Carbone 1995, Hey 2005, 1995) modified Eq. 4 to create a model, according to which there are noise random variables $\mathbf{E}_{KL}$ with expected value zero, such that, in the case of expected value,

$$P_{KL} = \text{prob}\left(\underbrace{EV(K) - EV(L)}_{\text{algebraic strength of preference}} > \underbrace{E_{KL}}_{\text{random noise}} \right). \quad (5)$$

These models, as well as the logit and probit models (4), capture the idea that the expected value maximizer mistakenly picks the option with lower expected value if the noise term is sufficiently large to ‘overwhelm’ the true strength of preference. Models like those in Eqs. 4 and 5 vary a great deal by their distributional assumptions for the error terms (see, e.g. Blavatsky and Pogrebna 2010, Buschena and Zilberman 2000, Stott 2006). A major contrast to the constant error model is that Eqs. 4 and 5 specifically allow different gamble pairs to have different binary choice probabilities, with the probability of choosing the inferior option being very low/high for a pair of gambles that have very different/similar expected values, as we have already seen in Luce’s model. Maybe more importantly still, the models in Eqs. 4 and 5 oftentimes explicitly imply that certain choices will be similar or identical to the flip of a fair coin, whereas constant error models with an error rate around 50% are generally not considered useful because they are considered to fit poorly.

Generally, the models in Eqs. 4 and 5 imply neither independence nor stationarity across multiple observations, such as the two answers to QUESTION 1 and QUESTION 2. For example, as we saw in Table 2, the Luce model does not require the binary choice among K and L at t_1 to be independent from the choice between X and Y at time t_2 . Similarly, in a simple Thurstone model where the joint distribution of E_K, E_L is a product of standard normals (that is, where error terms are independent within a binary choice), for example, the binary choice among K and L at time points t_1 and t_2 may nonetheless be correlated, because the two draws from E_K may be correlated, as may be the two draws from E_L . Expression 5 will give the same binary choice probability at t_1 and t_2 for this Thurstone model because the expected values are fixed. But that changes when we replace expected value theory by cumulative prospect theory. Here the binary choice probabilities readily violate stationarity when the probability weighting or utility parameters are not constant across observations. Nonetheless, statistical analyses of these models usually depend on auxiliary independence and stationarity assumptions to get simple and tractable test statistics (see, e.g., Blavatsky and Pogrebna, 2010, or Stott, 2006).

The general (distribution-free) random utility model

If we abstract away from the formulation in Eq. 5, we reach another prominent model of binary choice probabilities the general *random utility model* for binary choice probabilities (Block and Marschak 1960, Becker et al. 1963). Here, each gamble’s utility is a random variable, say, the utility of K is a random variable U_K . A common restriction⁶ on the joint distribution of these random variables is that the random utilities of any two options differ from each other with probability one, i.e., $P(U_K = U_L) = 0$ whenever $K \neq L$. Because it makes no additional distributional assumptions whatsoever, this model is extraordinarily

⁶Most scholars do not view this as restrictive since continuous joint distributions typically satisfy it.

general and sometimes called a *distribution-free* random utility model. The general random utility model states that

$$P_{KL} = \frac{\text{prob}(\mathbf{U}_K > \mathbf{U}_L)}{\text{probability that } K \text{ has higher utility than } L} \quad (6)$$

This model, stated in several different forms, has been the subject of intense mathematical study across several disciplines (Bolotashvili et al. 1999, Fiorini 2001a,b, Fishburn 1992, 2001, Fishburn and Falmagne 1989, Gilboa 1990, Grötschel et al. 1985, Koppen 1995, Loomes and Sugden 1998, 1995, Marley 1991, Marschak 1960, Niederée and Heyer 1997, Suck 1992, Ziegler 1995). For example, when considering all possible pairs $X = Y$ among up to five choice options, the choice probabilities P_{XY} are known to satisfy the model in Eq. 6 if and only if the following collection of constraints holds:

$$P_{KL} + P_{LM} - P_{KM} \leq 1, \text{ for any distinct } K, L, M. \quad (7)$$

In its general form of Eq. 6, the model allows the utility random variables to be interdependent across stimuli and across multiple observations. Very little is known about the special case where the random utilities are independent across choice options (within a pair-wise choice), but unconstrained otherwise (Suck 2002). For our discussion here note that the general random utility model per se does not state whether repeated observations are iid or not (as we will see in more detail below).

Linear order model, random preference models and random function models of binary choice probabilities

A closely related model of binary choice probabilities states that, when asked whether s/he prefers K or L , the decision maker engages in a cognitive process that implicitly or explicitly rank orders options from best to worst. This process is not necessarily deterministic. Each possible preference ranking $\succ$ has some probability $P_{\succ}$, with the possibility that one single ranking might have all probability mass concentrated on it. According to the *linear order model* (Becker et al. 1963, Block and Marschak 1960), which has also been dubbed the *random preference model* (Loomes and Sugden 1995), the probability that the respondent chooses K over L is the probability that s/he has a linear order (a ranking without ties) in which K is preferred to L (denoted $K \succ L$). Formally,

$$P_{KL} = \frac{\sum_{\text{those linear orders } \succ \text{ in which } K \succ L} P_{\succ}}{\text{Probability that } K \text{ is ranked before } L} \quad (8)$$

There are important nested submodels and variants of this model to consider: For example, a random preference specification of cumulative prospect theory (Regenwetter et al. 2014)

allows positive probability only for those linear orders $\succ$ that satisfy cumulative prospect theory. This is the same as assuming that the probability weighting and utility parameters in cumulative prospect theory are probabilistic with an unknown joint distribution. Similarly, the linear orders $\succ$ in Equation 8 could be replaced by heuristics from a heuristic toolbox or preference relations satisfying some other core theory. In such a case, the decision maker who faces a binary choice between K and L makes a choice consistent with a heuristic or other preference relation that is probabilistically selected according to an unknown probability distribution (see Guo and Regenwetter 2014, for an example).

Last, but not least, a *random function model* of binary choice probabilities (Regenwetter and Marley 2001) states that a decision maker faced with the choice between two gambles will carry out a cognitive process that, implicitly or explicitly, maps gambles into utilities. This process, with some probability, corresponds to some mathematical mapping (or “utility function”) that values i more highly than j . Writing $\mathcal{F}$ for some collection of such functions,

$$P_{KL} = \frac{\text{prob}(\{f \in \mathcal{F} \text{ s.t. } f(K) > f(L)\})}{\text{Probability of a function that assigns } K \text{ a higher utility than } L} \quad (9)$$

For this to work with binary choice probabilities, $\mathcal{F}$ must contain exclusively 1-to-1 mappings. While the original model considered the collection of *all* 1-to-1 mappings, recent developments have considered the case when $\mathcal{F}$ is the collection of all functions that result from varying the weighting and utility parameters in cumulative prospect theory, or some other such theory (see, e.g., Regenwetter et al. 2014).⁷

It has long been known (see, e.g., Regenwetter and Marley 2001) that binary choice probabilities satisfy Eq. 6 if and only if they satisfy Eq. 8 (allowing *all linear orders*) if and only if they satisfy Eq. 9 (allowing *all 1-to-1 functions*). It is also important to note that any model satisfying Eq. 4 or 5 must satisfy Eq. 6, i.e., a rejection of the general random utility model (6) on a given set of data would imply a rejection of a very large class of heavily used discrete choice models.

We now consider independence and stationarity in random preference and corresponding random function models using an example based on expected utility theory. This is a stylized example for the purpose of keeping our illustration simple: Table 4 shows two collections of binary choice probabilities of a hypothetical expected utility maximizer who waivers in his risk attitude. When asked Question 1, his utility for winning $\$x$ is of the form $u(x) = x^2$ with probability 0.90, and $u(x) = x$ otherwise. Table 6 in the Appendix shows the utilities of gambles A, B, C, and D in both cases. Because the decision maker waivers in his utility function, he has preference ranking ABCD with probability 0.90 and preference ranking DCBA with probability 0.10, when asked QUESTION 1. The corresponding binary choice probabilities for QUESTION 1, obtained from the random preference model (8) and the corresponding random function model (9) are included in Tables 4 and 6 (with the

⁷We omit the mathematical details of assuring that the model uses a well-defined probability space.

probability, P_{AB} , in response to QUESTION 1, underlined). By construction, these choice probabilities are induced by a probability distribution over two utility functions, and, likewise, by a probability distribution over two rankings. Hence they satisfy the triangle inequalities (7). When asked QUESTION 2, our hypothetical respondent, not only waives again, but his probabilities of different risk attitudes depend on his response to QUESTION 1. If he chose option A in QUESTION 1, then he changes to the utility function $u(x) = x^{1.2}$ with probability $\frac{1}{2}$ and $u(x) = x^{0.999}$ otherwise. On the other hand, if he chose B in QUESTION 1 then the hypothetical decision maker uses function $u(x) = x^{1.2}$ with probability 0.90 and $u(x) = x$ otherwise. The Appendix provides detailed explanations for this construction, including four illustrative tables that summarize key information. It is clear from the top panel of Table 4 that the binary choice probabilities when the decision maker answers QUESTION 1 differ from those when the decision maker answers QUESTION 2, hence the choice probabilities are not stationary. The bottom panel shows that the binary choices are also not independent. The joint probabilities are not products of the marginals: For example, the joint probability that the decision maker chooses B in QUESTION 1 and D in QUESTION 2 is 0.01. If choices were independent, that joint probability would have to be almost five times higher, namely $.1 \times .46 = .046$. Hence, the binary choices of this hypothetical decision maker are neither independent nor are the choice probabilities stationary. Nonetheless, by construction, the choice probabilities satisfy the random function and the corresponding random preference model (and hence, also the triangle inequalities) at the time of each of QUESTIONS 1 & 2.

This stylized construction is representative of the general case: Random preference and random function models need not generate independent choices and their binary choice probabilities need not be stationary. This also applies to random preference models in which the permissible preference states are heuristics in a toolbox: The decision maker's responses to QUESTION 1 need not be independent on two different occasions, i.e., the probability with which any given heuristic is used in the second response may depend on what the decision maker chose in the first response. Likewise, the probability distribution over heuristics in a toolbox need not be the same at all times.

Nonstationary choice probabilities

Most of our discussion so far aimed at modeling an individual. We have seen that most probabilistic choice models readily violate stationarity: Oftentimes, the underlying core theory does not require the individual to have the same preference, utility, or strength of preference at all times (e.g., a Luce specification of expected utility theory, per se, does not require an individual's risk attitude to be the same at all times). Sometimes, even one and the same preference, utility function, or strength of preference may permit a whole range of choice probabilities for a given pair of choice alternatives (e.g., the weak utility model permits a range of binary choice probabilities for any given utility function and any given pair of choice alternatives). As we move from modeling one person to multiple respondents we should not assume that everybody shares the same utility function, shares the same preference or heuristic, or chooses A in QUESTION 1 with the same probability. If we assume that every individual acts deterministically and that we randomly draw individuals

from the population using a stationary sampling procedure, then it follows that choice probabilities are stationary. On the other hand, prior research on individuals strongly suggests that we should not treat individual decision makers as through they made their choices deterministically with a fixed underlying algebraic theory and fixed parameters.

As we have seen earlier, nonstationarity within a single individual often arises if, for example, the parameters in a core theory, such as, say, probability weights in cumulative prospect theory, differ between one occasion and another. Likewise, looking across a population of individuals, the choice probabilities are nonstationary (i.e., not identically distributed) when different members of the population differ in their choice probabilities. For example, even if everybody satisfies cumulative prospect theory, individual differences in probability weighting and utility parameters can cause nonidentical choice probabilities, even if all individuals are furthermore consistent with the same probability model, such as, say, the Luce model. We now consider consequences of violating stationarity.

Consequences of Nonstationarity

We investigate what the various models predict about the binary choice probabilities, either of a given individual at a randomly selected moment in time or of a randomly selected member of a population. Treating the selection of the time of observation or the selection of a member from a population as probabilistic creates surprising challenges. For example, as we see below, if a person satisfies a Luce model at all times, but not always with the same core utility values, then the predicted probability of the person's choice at a randomly selected time point violates the Luce model. Likewise, if every member of a heterogeneous population satisfies a Luce model, then the probability that a randomly selected respondent chooses A in QUESTION 1 violates the Luce model.

The Luce Model (Bradley-Terry model, Softmax Rule)

Considering the gambles A, B, C, D of our earlier examples, if a person satisfies one utility function u_1 for gambles at time t_1 and a different utility function u_2 for gambles at time t_2 , there does generally not exist a utility function u such that

$$\frac{\frac{e^{u_1(A)}}{e^{u_1(A)} + e^{u_1(B)}} + \frac{e^{u_2(A)}}{e^{u_2(A)} + e^{u_2(B)}}}{2} = \frac{e^{u(A)}}{e^{u(A)} + e^{u(B)}} \quad (10)$$

$$\text{and } \frac{\frac{e^{u_1(C)}}{e^{u_1(C)} + e^{u_1(D)}} + \frac{e^{u_2(C)}}{e^{u_2(C)} + e^{u_2(D)}}}{2} = \frac{e^{u(C)}}{e^{u(C)} + e^{u(D)}} \quad (11)$$

This means that the Luce model is *not invariant under nonstationarity*. Suppose that every member of a population satisfies a Luce choice specification of cumulative prospect theory,

but suppose that there are individual differences in probability weights and/or utility parameters. The probability P_{XY} that a randomly selected person chooses X in the pair (X, Y) will generally violate the Luce model specification of cumulative prospect theory. The difficulty of satisfying Eqs. 10-11 illustrates this problem for the case of flipping a coin between two decision makers, for example.

The Constant Error model, the Weak Utility model, and Weak Stochastic Transitivity

Next, we unpack the ramifications of a nonstationary constant error model. It is convenient to discuss this case together with the weak utility model, and weak stochastic transitivity. Consider three choice options, say, gambles A, B, C, and three linear orders (rankings from best to worst), say, ABC, BCA, and CAB, either held by one person at three observation times labeled t_1 , t_2 , and t_3 , or held by three participants labeled t_1 , t_2 , and t_3 . Suppose that the error probability in a constant error model equals $\epsilon = 0.10$. Writing $P_{i,KL}$ for the binary choice probabilities, either at time period t_i within respondent, or for respondent t_i , the constant error model states that

$$P_{1,AB} = 1 - P_{2,AB} = P_{3,AB} = .9, \quad (12)$$

$$P_{1,BC} = P_{2,BC} = 1 - P_{3,BC} = .9, \quad (13)$$

$$P_{1,AC} = 1 - P_{2,AC} = 1 - P_{3,AC} = .9. \quad (14)$$

At each i , these binary choice probabilities satisfy weak stochastic transitivity. The average (i.e., pooled) binary choice probabilities are

$$P_{AB} = \frac{P_{1,AB} + P_{2,AB} + P_{3,AB}}{3} = \frac{1.9}{3} = 0.633 > \frac{1}{2}, \quad (15)$$

$$P_{BC} = \frac{P_{1,BC} + P_{2,BC} + P_{3,BC}}{3} = \frac{1.9}{3} = 0.633 > \frac{1}{2}, \quad (16)$$

$$P_{AC} = \frac{P_{1,AC} + P_{2,AC} + P_{3,AC}}{3} = \frac{1.1}{3} = 0.367 < \frac{1}{2}. \quad (17)$$

This configuration of binary choice probabilities violates weak stochastic transitivity.⁸ The only constant error model with error rate below 50% that can accommodate these probabilities is a constant error model where $\epsilon = \frac{1.1}{3}$ and where the preference is the intransitive cycle: A preferred to B preferred to C preferred to A. In the case that t_1, t_2, t_3 denote three participants, this is similar to the classical Condorcet voting paradox (Condorcet 1785) in which “rational” voters generate a collective “irrational” preference.

Notice, in passing, that statistical tests of the constant error model, the weak utility model, and weak stochastic transitivity could lead to major artifacts if ‘true’ preference states vary either across observations or across participants (Regenwetter et al. 2011). For example, Davis-Stober et al. (2016) discuss a case in which they showed that different presentation formats of stimuli created such an artifact: When pooling the data across presentation formats, weak stochastic transitivity was violated (Tsetsos et al. 2016a), even though it held separately in data sets segregated by presentation format (see also Tsetsos et al. 2016b, for a reply).

While, on the one hand, the weak utility model is not invariant under nonstationarity, on the other hand, it also accommodates extreme heterogeneity of certain kinds: The constraints on binary choice probabilities in Expression 2 are unaffected under all monotonic transformations of the function u . In other words, the weak utility model is invariant under varying utility functions that are all monotonically increasing transformations of each other.

Parametric random utility (discrete choice) models

Parametric random utility and discrete choice models are not invariant under nonstationarity. The average of two or more parametric random utility models is typically not a random utility model with the same core theory and/or the same error distribution. Suppose that u_1 and u_2 are two functions that satisfy the same core theory but assign different values to (at least some of the) choice options. Specifically, suppose that

$$P_{1,KL} = \text{prob}(u_1(K) + \mathbf{E}_{1,K} > u_1(L) + \mathbf{E}_{1,L}) \text{ and} \\ P_{2,KL} = \text{prob}(u_2(K) + \mathbf{E}_{2,K} > u_2(L) + \mathbf{E}_{2,L}),$$

with $P_{1,KL} \neq P_{2,KL}$. For example, assume u_1 and u_2 are two different functions obtained by varying the probability weighting and utility parameters in cumulative prospect theory, and let the random variables $\mathbf{E}_{1,L}, \mathbf{E}_{2,L}, \mathbf{E}_{1,K}, \mathbf{E}_{2,K} \dots$ be Gumbels. Then, we can typically not find a function u , also satisfying cumulative prospect theory, together with Gumbel random variables $\mathbf{E}_K, \mathbf{E}_L, \dots$, such that $P_{KL} = \frac{P_{1,KL} + P_{2,KL}}{2}$ is of the form

$$P_{KL} = \text{Prob}(u(K) + \mathbf{E}_K > u(L) + \mathbf{E}_L).$$

⁸Formally, because weak stochastic transitivity is a non-convex union of convex polytopes, a convex combination of points satisfying weak stochastic transitivity need not satisfy weak stochastic transitivity.

We have effectively seen this already in the discussion of Luce's choice axiom. A notable exception to parametric random utility models' lack of invariance under nonstationarity is when u_1, u_2 are such that $u = \frac{u_1 + u_2}{2}$ satisfies the same core theory as u_1 and u_2 , and when the error terms are independent standard normal distributions.

General random utility model, random preference models, and random function models

We complete this section with the discussion of the general random utility model (6), the linear order (random preference) model (8), and the random function model (9), as well as their nested submodels that permit only certain linear orders, or only certain utility functions. We first illustrate why these models are invariant under nonstationary distributions by considering the triangle inequalities (7) as a simple example. Suppose, either, that an individual satisfies the triangle inequalities at times t_1, t_2, t_3 , or that individuals t_1, t_2, t_3 all satisfy the inequalities:

$$P_{1,KL} + P_{1,LM} - P_{1,KM} \leq 1, \forall \text{distinct } K, L, M. \quad (18)$$

$$\text{and } P_{2,KL} + P_{2,LM} - P_{2,KM} \leq 1, \forall \text{distinct } K, L, M, \quad (19)$$

$$\text{and } P_{3,KL} + P_{3,LM} - P_{3,KM} \leq 1, \forall \text{distinct } K, L, M. \quad (20)$$

Then the pooled binary choice probabilities also satisfy the triangle inequalities, that is,

$$\frac{P_{1,KL} + P_{2,KL} + P_{3,KL}}{3} + \frac{P_{1,LM} + P_{2,LM} + P_{3,LM}}{3} - \frac{P_{1,KM} + P_{2,KM} + P_{3,KM}}{3} \leq 1, \forall \text{distinct } K, L, M$$

(21)

If the binary choice probabilities satisfy the triangle inequalities but differ across time points or vary across people, then the averaged (overall) binary choice probabilities also still satisfy the triangle inequalities.⁹

⁹Formally, the linear ordering model forms a convex polytope. Convex polytopes are closed under convex combination, hence any weighted average of points in the linear ordering polytope is again a point in the polytope. The same applies to random preference models in general, including those that use a toolbox of heuristics as core theory.

In general, any probability distribution over random preference models yields another random preference model. For example, a coin flip between two random preference models

$$P_{KL}^* = \sum_{\text{all preference states } \succ \text{ in which } K \succ L} P_{\succ}^*, \text{ and } P_{KL}^\diamond = \sum_{\text{all preference states } \succ \text{ in which } K \succ L} P_{\succ}^\diamond,$$

gives another (pooled) random preference model

$$P_{KL} = \sum_{\text{all preference states } \succ \text{ in which } K \succ L} P_{\succ},$$

by setting $P_{\succ} = \frac{P_{\succ}^* + P_{\succ}^\diamond}{2}$, for each preference state $\succ$. Similarly, a probability distribution over general random utility models (6) or random function models (9) yields a general random utility or random function model.

Table 5 summarizes the results we have presented so far, including, for each model class, whether it implies stationarity and whether it implies independence. For all cases in which a model does not imply stationarity, the table also reports whether the model is invariant under nonstationary choice probabilities. We are now ready to proceed from the properties of models and of their predicted choice probabilities to statistical inference from data.

The Role of Independence and Stationarity in Statistical Inference

How does one carry out a statistical test to evaluate a model of binary choice probabilities? We focus on likelihood based (classical and Bayesian) methods. Regardless of which among the above binary choice models we aim to test, we need to select an “unconstrained” model of the data generating process, in order to define a test statistic. In a choice task, such a model spells out the joint probability of any potential set of observations. Because this model is almost universally the same, it is routine to omit its explicit statement in scholarly publications. However, this model is very important here because it relies on its own independence and stationarity assumptions. Starting with a general formulation, we show how a series of simplifying assumptions gives rise to the standard model that provides the foundation for the commonly used log-likelihood ratio test statistic (G^2) for binary choice data.

For expository simplicity, suppose for now that a study will ask a participant four questions, namely QUESTION 1 at time t_1 , followed by QUESTION 2 at time t_2 , then QUESTION 1 at a later time t_3 , and, last, QUESTION 2 at time t_4 . A model of the data generating process spells out, for each possible data set, the probability of that data set, as a function of some parameters. A standard first simplification in designing such a model is to disregard the magnitude of the inter-stimulus time intervals and only track the order of the observations. As an example, consider the observation stream consisting of the choice of A in QUESTION 1 at time t_1 , C in QUESTION 2 at time t_2 , B in QUESTION 1 at time t_3 , and C in QUESTION 2 at time t_4 , which we denote by $A \rightarrow C \rightarrow B \rightarrow C$. Writing $\mathcal{P}_{XY|h}$ for the

probability of choosing X among (X, Y) given a history h of prior choices, the probability of this observation stream is given by

$$\begin{aligned}
 & \underbrace{\mathcal{P}_{AB}}_{\text{of choosing } A \text{ with no prior history}} \times \underbrace{\mathcal{P}_{CD|A}}_{\text{Prob. of choosing } C \text{ after having chosen } A} \times \underbrace{\mathcal{P}_{BA|A \rightarrow C}}_{\text{Prob. of choosing } B \text{ after having chosen first } A \text{ then } C} \times \underbrace{\mathcal{P}_{CD|A \rightarrow C \rightarrow B}}_{\text{Prob. of choosing } C \text{ after having chosen first } A \text{ then } C \text{ then } B} \\
 & = \mathcal{P}_{AB} \times \mathcal{P}_{CD|A} \times (1 - \mathcal{P}_{AB|A \rightarrow C}) \times \mathcal{P}_{CD|A \rightarrow C \rightarrow B}.
 \end{aligned}
 \tag{22}$$

Even for just four questions, like in this example, this model of the data generating process permits a distinct probability parameter for each possible response following each possible prior history. This means that we need one parameter $\mathcal{P}_{AB}$ for the response at t_1 , two parameters, $\mathcal{P}_{CD|A}$ and $\mathcal{P}_{CD|B}$, for the response at time t_2 because there are two possible histories of a prior choice leading up to t_2 , four parameters for the response at time t_3 because there are four possible histories of prior choices leading up to t_3 , and eight parameters for the response at time t_4 . This general model of the data generating process may have as many as $1 + 2 + 4 + 8 = 15$ free parameters to account for just four binary observations. Note that, had we assumed that the magnitude of the inter-stimulus time intervals mattered, then we might need even many more parameters.

Clearly, it is not possible to estimate 15 or more parameters from four individual choices. As we mentioned earlier, most statistical tests make auxiliary assumptions aimed at reducing the number of free parameters in the generating model to a manageable level. We can reduce the number of parameters dramatically if we assume that the prior history never matters. More concretely, the choice of K over L is independent of prior choices. This means that $\mathcal{P}_{CD|A} = \mathcal{P}_{CD|B}$, which we can relabel as $\mathcal{P}_{2,CD}$ to indicate that this binary choice may only depend the fact that this is the choice at time t_2 . Likewise, independence implies, for example, that

$$\mathcal{P}_{CD|A \rightarrow C \rightarrow B} = \mathcal{P}_{CD|B \rightarrow C \rightarrow B} = \mathcal{P}_{CD|A \rightarrow D \rightarrow B} = \dots = \mathcal{P}_{CD|B \rightarrow C \rightarrow A},$$

which we denote as $\mathcal{P}_{4,CD}$ to denote that this binary choice only depends on the time t_4 , and not on earlier choices. Thanks to independence, we have reduced the number of parameters to just four. Using these parameters, the probability of the observation $A \rightarrow C \rightarrow B \rightarrow C$ reduces from the form in Eqn. 22 to

$$\begin{aligned}
 &= \underbrace{\mathcal{P}_{1,AB}}_{\text{Probability at time } t_1 \text{ of choosing } A \text{ over } B} \times \underbrace{\mathcal{P}_{2,CD}}_{\text{Probability at time } t_2 \text{ of choosing } C \text{ over } D} \times \underbrace{(1 - \mathcal{P}_{3,AB})}_{\text{Probability at time } t_3 \text{ of choosing } B \text{ over } A} \times \underbrace{\mathcal{P}_{4,CD}}_{\text{Probability at time } t_4 \text{ of choosing } C \text{ over } D} \\
 &\quad \mathcal{P}(A \rightarrow C \rightarrow B \rightarrow C)
 \end{aligned}
 \tag{23}$$

While well-defined, this probability model still causes the challenge of having to estimate four parameters from just four binary choices. If we add stationarity, however, then we obtain a much simpler model with just two parameters, namely

$$\mathcal{P}(A \rightarrow C \rightarrow B \rightarrow C) = \mathcal{P}_{AB} \times \mathcal{P}_{CD} \times (1 - \mathcal{P}_{AB}) \times \mathcal{P}_{CD}.$$

Finally, we can state the standard model for the general case of more than four pairwise choices. Writing N_{KL} for the number of times that K was chosen over L , the general *unconstrained model* is a product of binomial random variables, namely

$$\begin{aligned}
 \text{Probability of choice responses} &= \binom{N_{AB} + N_{BA}}{N_{AB}} \mathcal{P}_{AB}^{N_{AB}} (1 - \mathcal{P}_{AB})^{N_{BA}} \binom{N_{CD} + N_{DC}}{N_{CD}} \mathcal{P}_{CD}^{N_{CD}} (1 - \mathcal{P}_{CD})^{N_{DC}}.
 \end{aligned}
 \tag{24}$$

By disregarding the magnitude of the inter-stimulus time intervals, by assuming independent choices, and by assuming stationary response probabilities, this model requires only two parameters. Now, we are able to use statistical estimation, because we can estimate the two parameters $\mathcal{P}_{AB}$, $\mathcal{P}_{CD}$ no matter how many times we ask QUESTION 1 and QUESTION 2 to the decision maker, as long as we ask each question at least once. Substituting the binary choice probabilities $\mathcal{P}_{KL}$ of a Luce, constant error, or random utility model for $\mathcal{P}_{KL}$ for all pairs (K, L) in Expression 24 yields a nested submodel of the unconstrained model (and therefore satisfies independence and stationarity).

As discussed in earlier sections, we are not restricted to interpreting this probability model solely as a model of individual choice over distinct time points. One could equivalently model a population of decision makers, who are sampled one at a time and each given a single choice whose outcome is recorded. For this case, independence simply means that each sampled decision maker's response is independent from the others. If the decision makers are sampled uniformly with replacement then stationarity follows if the average

choice probability for each pair of options is stationary. That follows from, but does not require, the assumption that all individuals in the population have stationary choice probabilities. If we treat the sampling of all participants with replacement as instantaneous, then these properties apply naturally.

All in all, independence and stationarity assumptions allow us to construct simple, parsimonious probability models for collections of observations. For most experimental applications, the choice probabilities are unknown and must be estimated from the choice responses. A key question of inference is whether or not the choice probabilities generating the experimental data are consistent with a binary choice model of interest. A classical approach to this problem is to use a *likelihood ratio test*, which we briefly describe.

In a likelihood ratio test, one considers the likelihood of the data, given model constraints (e.g., Luce, constant error, random utility, etc.), in the numerator of the test statistic, and the likelihood of the data, given the unconstrained model of the data generating process, in the denominator. As we have reviewed above, the standard unconstrained model of the data generating process makes the simplifying assumptions that observed binary choices are independent and assumes that repeated binary choices among the same objects are governed by the same choice probabilities. Because the constrained model is treated as a nested submodel of the unconstrained model in the denominator, it inherits the latter's independence and stationarity properties, regardless whether the substantive model (Luce, constant error, random utility, etc.) requires them when considered by itself. As a consequence, a standard likelihood ratio test is cast in terms of the ratio of these two likelihood functions:

$$\begin{aligned} & \text{Likelihood ratio of } \frac{\text{constrained model}}{\text{to unconstrained model}} \\ &= \frac{\prod_{\substack{K,L \\ K \neq L}} \binom{N_{KL} + N_{LK}}{N_{KL}} P_{KL}^{N_{KL}} (1 - P_{KL})^{N_{LK}}}{\prod_{\substack{K,L \\ K \neq L}} \binom{N_{KL} + N_{LK}}{N_{KL}} \mathcal{P}_{KL}^{N_{KL}} (1 - \mathcal{P}_{KL})^{N_{LK}}} \quad (25) \end{aligned}$$

Writing $Q_{KL} = \frac{N_{KL}}{N_{KL} + N_{LK}}$ for the proportion of times K was chosen of L , it is standard to estimate the unconstrained binary choice probabilities $\mathcal{P}_{KL}$ via their maximum likelihood values, Q_{KL} .

When certain requirements are met (see, e.g., Hays 1988, Hogg and Craig 1995), then $-2 \log$ of the likelihood ratio in Eq. 25, the $\mathcal{G}^2$ -statistic, will have an asymptotic Chi-squared (χ^2) distribution that can be used to check goodness-of-fit and carry out significance tests in “large” samples. Regenwetter et al. (2011) critiqued the prior literature for the fact that weak stochastic transitivity (see also Iverson and Falmagne 1985, for an important earlier paper) and the triangle inequalities violate those technical requirements, hence the usual χ^2 test for

the G^2 statistic, as it has been used in some prior papers, is not appropriate. The appropriate *order-constrained* inference methods are somewhat more technical (Silvapulle and Sen 2005, Davis-Stober 2009) but can be implemented via easy-to-use software (Regenwetter et al. 2014). Using an identical likelihood construction, Bayesian methods can be applied by adding an additional assumption on the binary choice probabilities. Rather than consider the parameters as fixed, unknown values, we can conceptualize them as random variables themselves with a “prior” distribution representing the researcher's uncertainty regarding their values. Under similar independence and stationarity assumptions, this setup enables the calculation of Bayes factors (Kass and Raftery 1995), a measure of statistical evidence, between competing models (see Davis-Stober et al. 2015, Cavagnaro and Davis-Stober 2014, for details).

It is important to emphasize that not all binary choice models imply independence or stationarity, although researchers often introduce those properties through the use of the likelihood ratio (25), to facilitate statistical testing. Thus, it is important to keep in mind that a violation of either assumption implies a violation of the “unconstrained model” while not necessarily implying a violation of a given “constrained” model, such as, say, a random preference model. For that case, for example, we have seen in Appendix A and in Table 4 that the linear order model itself does not use or require the iid property¹⁰ that is used in the likelihood ratio (25).

It is also important to emphasize that many algebraic theories are mute about whether their algebraic parameters are constant within person over the course of an experiment or not, and about whether different people may have different algebraic parameter values or not. Nonetheless, scholars often introduce stationarity by assuming implicitly that the parameters of the core theory are constant. Whether algebraic parameters may vary or not has implications for model selection and complexity measures. Classical measures of complexity, based on counting parameters, such as AIC and BIC, do not adequately capture variability. A model with one free parameter in the algebraic core and no response error can have vastly different flexibility from a model with a fixed algebraic core and one free parameter for response error. For a full discussion of adequate complexity measures as they apply to a variety of probabilistic choice models, see Davis-Stober and Brown (2011).

Conclusions

We have reviewed, for several major classes of probabilistic models of binary choice, whether or not they imply stationary choice probabilities and whether or not they imply independent choices. We have also reviewed the consequences of violating stationarity and shown how standard quantitative tests of such models use auxiliary iid assumptions to formulate tractable test statistics. If binary choice probabilities are not stationary, then this potentially invalidates all the models that explicitly require stationarity and that are not invariant under violations of stationarity, such as Luce's choice model with a fixed utility function, constant error models with a fixed preference relation, and a number of heavily

¹⁰Indeed, Regenwetter et al. (2011) used an iid assumption only in their choice of a test statistic, namely in the formulation of the likelihood ratio in Eq. 25.

used parametric random utility (discrete choice) models. However, if binary choices are not iid, this does not necessarily invalidate models that make no iid assumptions, even if their statistical tests introduced auxiliary iid assumptions (e.g., via Eq. 25). Rather, non-iid data will place a question mark on the statistical tests in question, just as non-iid sampling places a question mark on, say, an ANOVA or t-test, rather than the scientific hypotheses per se, say $\mu_1 = \mu_2$. When an assumption underlying a statistical test is invalidated, then one has to ask how robust the test is to such a violation. We leave a discussion of those statistical matters for future work.

Acknowledgments

We thank Michael Birnbaum, Dan Cavagnaro, Jason Dana, Ying Guo, Marc Jekel, A.A.J. Marley, Anna Popova, and Chris Zwilling, for critical comments on earlier drafts. Several other scholars have contributed in many discussions at various conferences. This work was supported by NSF grants SES 08-20009, SES 10-62045, SES 14-59699 (PI: M. Regenwetter), SES 14-59866 (PI: C. Davis-Stober), a National Institutes of Health grant (K25AA024182, PI: C. Davis-Stober), and by a University of Illinois, Arnold O. Beckman, Research Award (PI: M. Regenwetter). Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of their colleagues, funding agencies, or universities.

Appendix: Example of a non-iid random preference model

Consider a hypothetical expected utility maximizer who waivers in his risk attitude. When asked QUESTION 1, his utility for winning $\$x$ is of the form $u(x) = x^2$ with probability 0.90, and $u(x) = x$ otherwise. Table 6 shows the utilities of gambles A, B, C, and D in both cases. Preference ranking ABCD has probability 0.90 and preference ranking DCBA has probability 0.10. Note that, while the decision maker only chooses among A and B in that question (the choice probability is underlined in Table 6), the theory nonetheless defines preferences among all options and choice probabilities for all pairs. Now, suppose that, when asked QUESTION 2, the decision maker's risk attitude depends on his response to Question 1. If he chose option A in QUESTION 1, then he changes to $u(x) = x^{1.2}$ with probability $\frac{1}{2}$ and $u(x) = x^{0.999}$ otherwise. If he chose B in QUESTION 1 then he uses function $u(x) = x^{1.2}$ with probability 0.90 and $u(x) = x$ otherwise. The conditional probability given a choice of A in QUESTION 1, that drives the response for QUESTION 2, $P_{CD} = .5$, is underlined. But if he chose B in QUESTION 1 then he uses function $u(x) = x^{1.2}$ with probability 0.90 and $u(x) = x$ otherwise. This means that, in the latter case, while changing utility functions, he uses the same preference rankings with the same probabilities. The conditional probability given a choice of B in QUESTION 1, that drives the response for QUESTION 2, $P_{CD} = .9$, is underlined. In all, if he chose A then the binary choice probabilities change, and if he chose B they do not change.

Table 7 shows the conditional probabilities of the utility functions (and linear orders, i.e., rankings) for QUESTION 2, given the responses to QUESTION 1, as well as the induced conditional binary choice probabilities. Comparing Tables 6 and 7, the choice probabilities are stationary if the decision maker chose B in QUESTION 1, but not stationary otherwise. Table 8 gives the unconditional probabilities of the utility functions, rankings, and binary choices for QUESTION 2. These differ from those for QUESTION 1, hence the binary choice probabilities are nonstationary. These binary choice probabilities for QUESTION 2 are induced by a probability distribution over three utility functions, and, likewise, by a

probability distribution over three rankings. Hence they satisfy the triangle inequalities (7). Table 9 summarizes the key results. The top part shows that these hypothetical binary choice probabilities are stationary conditional on the decision maker choosing option B in QUESTION 1, but not otherwise. Hence, the unconditional binary choice probabilities are not stationary. The bottom panel shows why the binary choices are not independent. The joint probabilities are not products of the marginals: For example, the joint probability that the decision maker chooses B in QUESTION 1 and D in QUESTION 2 is 0.01. If choices were independent, that joint probability would have to be almost five times higher, namely $1 \times .46 = .046$.

This shows that the linear order model (8), the general random utility model (6), and the random function model (9) require neither independence nor stationarity. One of these two properties can hold while the other is violated. Note also, by considering Tables 6 to 8, that binary choices could, in fact, be iid even when the utility functions (or rankings) might be interdependent and when the probability measures on the utility functions might change. Requiring the binary choice probabilities to remain fixed and binary choices to be independent is weaker than requiring the ranking probabilities to remain fixed and the rankings to be independent, or assuming a single constant utility function across observations.

References

- Abdellaoui M, Driouchi A, L'Haridon O. Risk aversion elicitation: reconciling tractability and bias minimization. *Theory and Decision*. 2011; 71:63–80.
- Ballinger T, Wilcox N. Decisions, error and heterogeneity. *Economic Journal*. 1997; 107:1090–1105.
- Becker GM, DeGroot MH, Marschak J. Stochastic models of choice behavior. *Behavioral Science*. 1963; 8:41–55.
- Birnbaum M. A statistical test of independence in choice data with small samples. *Judgment and Decision Making*. 2012; 7:97–109.
- Blavatsky P, Pogrebna G. Models of stochastic choice and decision theories: Why both are important for analyzing decisions. *Journal of Applied Econometrics*. 2010; 25:963–986.
- Block, HD., Marschak, J. Random orderings and stochastic theories of responses. In: Olkin, I, Ghurye, S, Hoeffding, H, Madow, W., Mann, H., editors. *Contributions to Probability and Statistics*. Stanford University Press; Stanford: 1960. p. 97–132.
- Bolotashvili G, Kovalev M, Girlich E. New facets of the linear ordering polytope. *SIAM Journal on Discrete Mathematics*. 1999; 12:326–336.
- Bröder A, Schiffer S. Bayesian strategy assessment in multi-attribute decision making. *Journal of Behavioral Decision Making*. 2003; 16(3):193–213.
- Brown SD, Heathcote A. The simplest complete model of choice response time: Linear ballistic accumulation. *Cognitive psychology*. 2008; 57(3):153–178. [PubMed: 18243170]
- Buschena D, Zilberman D. Generalized expected utility, heteroscedastic error, and path dependence in risky choice. *Journal of Risk and Uncertainty*. 2000; 20:67–88.
- Bussemeyer JR, Townsend JT. Decision field theory: A dynamic-cognitive approach to decision making in an uncertain environment. *Psychological Review*. 1993; 100:432–459. [PubMed: 8356185]
- Camerer C, Ho TH. Violations of the betweenness axiom and nonlinearity in probability. *Journal of Risk and Uncertainty*. 1994; 8:167–196.
- Camerer CF. An experimental test of several generalized utility theories. *Journal of Risk and Uncertainty*. 1989; 2:61–104.
- Carbone E, Hey JD. Which error story is best? *Journal of Risk and Uncertainty*. 2000; 20:161–176.

- Cattelan M. Models for paired comparison data: A review with emphasis on dependent data. *Statistical Science*. 2012;412–433.
- Cavagnaro DR, Davis-Stober CP. Transitive in our preferences, but transitive in different ways: An analysis of choice variability. *Decision*. 2014; 1(2):102.
- Cha YC, Choi M, Guo Y, Regenwetter M, Zwilling C. Reply: Birnbaum's (2012) statistical tests of independence have unknown type-i error rates and do not replicate within participant. *Judgment and Decision Making*. 2013; 8(8):55–73.
- Condorcet, M. Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix (Essai on the application of the probabilistic analysis of majority vote decisions). Imprimerie Royale; Paris: 1785.
- Conte A, Hey J, Moffatt P. Mixture models of choice under risk. *Journal of Econometrics*. 2011; 162:79–88.
- Dai J. Are intertemporal preferences transitive? a bayesian analysis of repeated individual intertemporal choices. *Decision*. 2017; 4(1):1–24.
- Davis-Stober CP. Analysis of multinomial models under inequality constraints: Applications to measurement theory. *Journal of Mathematical Psychology*. 2009; 53:1–13.
- Davis-Stober CP, Brown N. A shift in strategy or "error"? strategy classification over multiple stochastic specifications. *Judgment and Decision Making*. 2011; 6(8):800–813.
- Davis-Stober CP, Brown N, Cavagnaro DR. Individual differences in the algebraic structure of preferences. *Journal of Mathematical Psychology*. 2015; 66:70–82.
- Davis-Stober, CP., Park, S., Brown, N., Regenwetter, M. Reported violations of rationality may be aggregation artifacts. *Proceedings of the National Academy of Sciences of the United States of America*. 2016. Online at www.pnas.org/cgi/doi/10.1073/pnas.1606997113
- Falmagne JC. Stochastic token theory. *Journal of Mathematical Psychology*. 1997; 41:129–143. [PubMed: 9237917]
- Falmagne JC, Ovchinnikov S. Media theory. *Discrete Applied Mathematics*. 2002; 121:103–118.
- Fiorini S. Determining the automorphism group of the linear ordering polytope. *Discrete Applied Mathematics*. 2001a; 112:121–128.
- Fiorini, S. PhD thesis. Université Libre de Bruxelles; 2001b. Polyhedral Combinatorics of Order Polytopes.
- Fishburn PC. Induced binary probabilities and the linear ordering polytope: A status report. *Mathematical Social Sciences*. 1992; 23:67–80.
- Fishburn PC. Signed orders, choice probabilities, and linear polytopes. *Journal of Mathematical Psychology*. 2001; 45:53–80. [PubMed: 11178922]
- Fishburn PC, Falmagne JC. Binary choice probabilities and rankings. *Economic Letters*. 1989; 31:113–117.
- Fu TT, Li LA, Lin YM, Kan K. A limited information estimator for the multivariate ordinal probit model. *Applied Economics*. 2000; 32:1841–1851.
- Gigerenzer, G., Goldstein, DG. the ABC Research Group. Betting on one good reason: The take the best heuristic. In: Gigerenzer, G., Todd, PM., editors. *Simple Heuristics That Make Us Smart*. Oxford University Press; New York: 1999. p. 75-95.
- Gilboa I. A necessary but insufficient condition for the stochastic binary choice problem. *Journal of Mathematical Psychology*. 1990; 34:371–392.
- Glöckner A. Investigating intuitive and deliberate processes statistically: The multiple-measure maximum likelihood strategy classification method. *Judgment and Decision Making*. 2009; 4(3): 186.
- Goldstein DG, Gigerenzer G. Models of ecological rationality: the recognition heuristic. *Psychological review*. 2002; 109(1):75. [PubMed: 11863042]
- Grötschel M, Jünger M, Reinelt G. Facets of the linear ordering polytope. *Mathematical Programming*. 1985; 33:43–60.
- Guo Y, Regenwetter M. Quantitative tests of the Perceived Relative Argument Model: Comment on Loomes (2010). *Psychological Review*. 2014; 121(4):696–705. [PubMed: 25347316]

- Harless DW, Camerer CF. The predictive value of generalized expected utility theories. *Econometrica*. 1994; 62:1251–1289.
- Hausman JA, Wise DA. A conditional probit model for qualitative choice: Discrete decisions recognizing independence and heterogeneous preferences. *Econometrica*. 1978; 46:403–426.
- Hays, W. *Statistics*. 4th. Holt, Rinehart & Winston; New York: 1988.
- Hey JD. Experimental investigations of errors in decision making under risk. *European Economic Review*. 1995; 39:633–640.
- Hey JD. Why we should not be silent about noise. *Experimental Economics*. 2005; 8:325–345.
- Hey JD, Carbone E. Stochastic choice with deterministic preferences: An experimental investigation. *Economics Letters*. 1995; 47:161–167.
- Hey JD, Orme C. Investigating generalizations of expected utility theory using experimental data. *Econometrica*. 1994; 62:1291–1326.
- Hilbig BE, Moshagen M. Generalized outcome-based strategy classification: Comparing deterministic and probabilistic choice models. *Psychonomic bulletin & review*. 2014; 21(6):1431–1443. [PubMed: 24865279]
- Hogg, RV., Craig, AT. *Introduction to Mathematical Statistics*. fifth. Macmillan; New York: 1995.
- Iverson GJ, Falmagne JC. Statistical issues in measurement. *Mathematical Social Sciences*. 1985; 10:131–153.
- Kahneman D, Tversky A. Prospect theory: An analysis of decision under risk. *Econometrica*. 1979; 47:263–291.
- Kass RE, Raftery AE. Bayes factors. *Journal of the American Statistical Association*. 1995; 90:773–795.
- Koppen M. Random utility representation of binary choice probabilities: Critical graphs yielding critical necessary conditions. *Journal of Mathematical Psychology*. 1995; 39:21–39.
- Liu YH, Mahmassani HS. Global maximum likelihood estimation procedure for multinomial probit (MNP) model parameters. *Transportation Research Part B - Methodological*. 2000; 34:419–449.
- Loomes G, Sugden R. Incorporating a stochastic element into decision theories. *European Economic Review*. 1995; 39:641–648.
- Loomes G, Sugden R. Testing different stochastic specifications of risky choice. *Economica*. 1998; 65:581–598.
- Luce, RD. *Individual Choice Behavior: A Theoretical Analysis*. John Wiley; New York: 1959.
- Luce RD. The choice axiom after twenty years. *Journal of Mathematical Psychology*. 1977; 15:215–233.
- Luce RD. Several unresolved conceptual problems of mathematical psychology. *Journal of Mathematical Psychology*. 1997; 41:79–87.
- Luce, RD., Narens, L. Fifteen problems in the representational theory of measurement. In: Humphreys, P., editor. *Patrick Suppes: Scientific Philosopher, volume 2: Philosophy of Physics, Theory Structure, Measurement Theory, Philosophy of Language, and Logic*. Kluwer; Dordrecht: 1994. p. 219–245.
- Luce, RD., Suppes, P. Preference, utility and subjective probability. In: Luce, RD., Bush, RR., Galanter, E., editors. *Handbook of Mathematical Psychology. Vol. III*. Wiley; New York: 1965. p. 249–410.
- Manski, CF., McFadden, D., editors. *Structural Analysis of Discrete Data with Econometric Applications*. MIT Press; Cambridge, Massachusetts: 1981.
- Marley, A., Regenwetter, M. Choice, preference, and utility: Probabilistic and deterministic representations. In: Batchelder, W., Colonius, H., Dzhafarov, E., J, M., editors. *New Handbook of Mathematical Psychology*. Cambridge University Press; 2017.
- Marley AAJ. Context dependent probabilistic choice models based on measures of binary advantage. *Mathematical Social Sciences*. 1991; 21:201–218.
- Marley, AAJ. Developing and characterizing multidimensional Thurstone and Luce models for identification and preference. In: Ashby, FG., editor. *Multidimensional models of perception and cognition Scientific psychology series*. Lawrence Erlbaum; Hillsdale, NJ: 1992. p. 299–333.

- Marschak, J. Binary-choice constraints and random utility indicators. In: Arrow, KJ, Karlin, S., Suppes, P., editors. *Proceedings of the first Stanford symposium on mathematical methods in the social sciences*. Vol. 1959. Stanford University Press; Stanford, Ca: 1960. p. 312–329.
- Maydeu-Olivares A, Hernández A. Identification and small sample estimation of Thurstone's unrestricted model for paired comparisons data. *Multivariate Behavioral Research*. 2007; 42:323–347. [PubMed: 26765490]
- McFadden D. Advances in computation, statistical methods, and testing of discrete choice models. *Marketing Letters*. 1991; 2:215–229.
- McFadden, D., Richter, MK. Revealed stochastic preference Unpublished manuscript, Department of Economics. University of California at Berkeley; 1970.
- McFadden D, Train K. Mixed MNL models for discrete response. *Journal of Applied Econometrics*. 2000; 15:447–470.
- Niederée, R., Heyer, D. Generalized random utility models and the representational theory of measurement: a conceptual link. In: Marley, AAJ., editor. *Choice, Decision and Measurement: Essays in Honor of R Duncan Luce*. Lawrence Erlbaum; Mahwah, NJ: 1997. p. 155–189.
- Nilsson H, Rieskamp J, Wagenmakers E. Hierarchical bayesian parameter estimation for cumulative prospect theory. *Journal of Mathematical Psychology*. 2011; 55:84–93.
- Papatla P, Krishnamurthi L. A probit model of choice dynamics. *Marketing Science*. 1992; 11:189–206.
- Ratcliff R, Rouder JN. Modeling response times for two-choice decisions. *Psychological Science*. 1998; 9(5):347–356.
- Regenwetter M, Dana J, Davis-Stober CP. Transitivity of preferences. *Psychological Review*. 2011; 118:42–56. [PubMed: 21244185]
- Regenwetter M, Davis-Stober CP, Lim SH, Guo Y, Popova A, Zwilling C, Cha YC, Messner W. QTest: quantitative testing of theories of binary choice. *Decision*. 2014; 1(1):2–34.
- Regenwetter M, Falmagne JC, Grofman B. A stochastic model of preference change and its application to 1992 presidential election panel data. *Psychological Review*. 1999; 106:362–384.
- Regenwetter M, Marley AAJ. Random relations, random utilities, and random functions. *Journal of Mathematical Psychology*. 2001; 45:864–912.
- Rieskamp J. The probabilistic nature of preferential choice. *Journal of Experimental Psychology: Learning, Memory, and Cognition*. 2008; 34:1446–1465.
- Rieskamp J, Busemeyer JR, Mellers BA. Extending the bounds of rationality: Evidence and theories of preferential choice. *Journal of Economic Literature*. 2006; 44:631–661.
- Silvapulle, M., Sen, P. *Constrained Statistical Inference: Inequality, Order, and Shape Restrictions*. John Wiley & Sons, New York; New York: 2005.
- Smith B, Batchelder W. Assessing individual differences in categorical data. *Psycho-nomic Bulletin and Review*. 2008; 15:713–731.
- Starmer C, Sugden R. Violations of the independence axiom in common ratio problems: An experimental test of some competing hypotheses. *Annals of Operations Research*. 1989; 19(1):79–102.
- Stott H. Cumulative prospect theory's functional menagerie. *Journal of Risk and Uncertainty*. 2006; 32:101–130.
- Strauss D. Choice by features: An extension of luce's choice model to account for similarities. *British Journal of Mathematical and Statistical Psychology*. 1981; 34(1):50–61.
- Suck R. Geometric and combinatorial properties of the polytope of binary choice probabilities. *Mathematical Social Sciences*. 1992; 23:81–102.
- Suck R. Independent random utility representations. *Mathematical Social Sciences*. 2002; 43(3):371–389.
- Thurstone LL. A law of comparative judgement. *Psychological Review*. 1927a; 34:273–286.
- Thurstone LL. Three psychophysical laws. *Psychophysical Review*. 1927b; 34:424–432.
- Tsai RC, Böckenholt U. Modelling intransitive preferences: A random-effects approach. *Journal of Mathematical Psychology*. 2006; 50:1–14.

- Tsetsos K, Moran R, Moreland J, Chater N, Usher M, Summerfield C. Economic irrationality is optimal during noisy decision making. *Proceedings of the National Academy of Sciences of the United States of America*. 2016a; 113(1):3102–7. [PubMed: 26929353]
- Tsetsos, K., Moran, R., Moreland, JC., Chater, N., Usher, M., Summerfield, C. Reply to davis-stober et al: Violations of rationality in a psychophysical task are not aggregation artifacts *Proceedings of the National Academy of Sciences*; 2016b. p. 201608989
- Tversky A. Intransitivity of preferences. *Psychological Review*. 1969; 76:31–48.
- Tversky A, Kahneman D. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*. 1992; 5:297–323.
- Wilcox, N. Stochastic models for binary discrete choice under risk: A critical primer and econometric comparison. In: Cox, JC., Harrison, GW., editors. *Research in Experimental Economics*. Vol. 12. Bingley, UK: Emerald; 2008. p. 197-292.
- Yellott JIJ. The relationship between Luce's choice axiom, Thurstone's theory of comparative judgement, and the double exponential distribution. *Journal of Mathematical Psychology*. 1977; 15:109–144.
- Ziegler, GM. *Lectures on Polytopes*. Springer-Verlag; Berlin: 1995.

Table 1

Independent binary choices in responses to gamble pair (K,L) at t_1 and gamble pair (X,Y) at t_2 .

		Choice between K and L at time t_1		
		K	L	
Choice between X and Y at time t_2	X	Choose K and X with prob. $P_{KL} \times P_{XY}$	Choose L and X with prob. $P_{LK} \times P_{XY}$	P_{XY}
	Y	Choose K and Y with prob. $P_{KL} \times P_{YX}$	Choose L and Y with prob. $P_{LK} \times P_{YX}$	$P_{YX} = 1 - P_{XY}$
		P_{KL}	$P_{LK} = 1 - P_{KL}$	

Table 2

Luce model of an expected value maximizer with dependent binary choices. Note: This is also a weak utility model with dependent choices.

	A	B	
C	.4900	.0096	$P_{CD} = .4996$
D	.0101	.4903	$P_{DC} = .5004$
$P_{AB} = .5001$		$P_{BA} = .4999$	

Table 3

Constant error model of an expected value maximizer with dependent binary choices when $\epsilon < \frac{1}{2}$. Note: This is also a weak utility model with dependent choices.

	A	B	
C	$\frac{\epsilon}{2}$	$\frac{\epsilon}{2}$	$P_{CD} = \epsilon$
D	$1 - \frac{3\epsilon}{2}$	$\frac{\epsilon}{2}$	$P_{DC} = 1 - \epsilon$
$P_{AB} = 1 - \epsilon$		$P_{BA} = \epsilon$	

Nonstationary (non-identically distributed) and dependent binary choices satisfying the random function model, hence satisfying the triangle inequalities and the linear order model.

Table 4

	P_{AB}	P_{AC}	P_{AD}	P_{BC}	P_{BD}	P_{CD}
QUESTION 1:	.9	.9	.9	.9	.9	.9
QUESTION 2:	.54	.54	.54	.99	.54	.54

A	B
C	.45 .09 $P_{CD} = .54$
D	.45 .01 $P_{DC} = .46$

$P_{AB} = .9$	$P_{BA} = .1$
---------------	---------------

Table 5

Summary of assumptions, special requirements, and properties of various different probabilistic models of binary choice.

Model	Assumptions, Special Requirements & Properties
Luce model with a fixed utility function (including fixed parameters)	Implies stationarity, but not independence (see Table 2).
Luce model in general	Implies neither stationarity nor independence. Not invariant under nonstationarity (see Eqs. 10-11).
Constant error model with a fixed preference	Implies stationarity, but not independence (see Table 3).
Constant error model in general	Implies neither stationarity nor independence. Not invariant under nonstationarity (see Eqs. 12-17).
Weak utility model (modal choice model) with a fixed utility function	Implies neither stationarity nor independence (see Table 3). Invariant under nonstationarity.
Weak utility model (modal choice model) in general	Implies neither stationarity nor independence. Not invariant under nonstationarity. Invariant under monotonic transformations of the utility function
Weak stochastic transitivity	Implies neither stationarity nor independence (see Table 3). Not invariant under nonstationarity (see Eqs. 12-17).
Parametric random utility (discrete choice) model (e.g., logit/probit)	Typically implies neither stationarity nor independence. Uses a variety of distributional assumptions. Typically not invariant under nonstationarity (see, e.g., Luce model).
Linear order model General random utility model Random function model	Implies neither stationarity nor independence (see Table 4). Invariant under nonstationarity (see Eqs. 18-21).

: Likelihood ratio tests routinely make an auxiliary *iid sampling* assumption of independence and stationarity, for mathematical tractability, including to reduce the number of free parameters.

Author Manuscript

Author Manuscript

Author Manuscript

Author Manuscript

Table 6

Non-iid random function model when Question 1 is answered.

Probability	Functional form	Numerical Values	Ranking	P_{AB}	P_{AC}	P_{AD}	P_{BC}	P_{BD}	P_{CD}
QUESTION 1:									
0.9	$u(x) = x^2$	$u(A) = 277$	ABCD	.9	.9	.9	.9	.9	.9
		$u(B) = 242$							
		$u(C) = 215$							
		$u(D) = 194$							
0.1	$u(x) = x$	$u(A) = 1.57$	DCBA	.9	.9	.9	.9	.9	.9
		$u(B) = 1.68$							
		$u(C) = 1.78$							
		$u(D) = 1.88$							

Non-iid random function model when Question 2 is answered, conditional on the response to Question 1.

Table 7

Conditional Probability	Functional form	Numerical Values	Ranking	Conditional Binary Choice Probs.							
				P_{AB}	P_{AC}	P_{AD}	P_{BC}	P_{BD}	P_{CD}		
QUESTION 2 conditional on A chosen in QUESTION 1:											
0.5	$u(x) = x^{1.2}$	$\begin{cases} u(A) = 17.5 \\ u(B) = 17.1 \\ u(C) = 16.7 \\ u(D) = 16.3 \end{cases}$	ABCD								
0.5	$u(x) = x^{0.999}$	$\begin{cases} u(A) = 8.7701 \\ u(B) = 8.7709 \\ u(C) = 8.7703 \\ u(D) = 8.7728 \end{cases}$	DBCA								
QUESTION 2 conditional on B chosen in QUESTION 1:											
0.9	$u(x) = x^{1.2}$	$\begin{cases} u(A) = 17.5 \\ u(B) = 17.1 \\ u(C) = 16.7 \\ u(D) = 16.3 \end{cases}$	ABCD								
0.1	$u(x) = x$	$\begin{cases} u(A) = 1.57 \\ u(B) = 1.68 \\ u(C) = 1.78 \\ u(D) = 1.88 \end{cases}$	DCBA								

Table 8
Non-iid random function model when Question 2 is answered, not conditional on the response to Question 1.

Probability	Functional form	Numerical Values	Ranking	P_{AB}	P_{AC}	P_{AD}	P_{BC}	P_{BD}	P_{CD}
QUESTION 2:									
$0.9 \times 0.5 = .45$	$u(x) = x^{1.2}$	$u(A) = 17.5$	ABCD						
		$u(B) = 17.1$							
		$u(C) = 16.7$							
		$u(D) = 16.3$							
$0.9 \times 0.5 = .45$	$u(x) = x^{0.999}$	$u(A) = 8.7701$	DBCA						
		$u(B) = 8.7709$							
		$u(C) = 8.7703$							
		$u(D) = 8.7728$							
$0.1 \times 0.9 = .09$	$u(x) = x^{1.2}$	$u(A) = 17.5$	ABCD	.54	.54	.54	.99	.54	<u>.54</u>
		$u(B) = 17.1$							
		$u(C) = 16.7$							
		$u(D) = 16.3$							
$0.1 \times 0.1 = .01$	$u(x) = x$	$u(A) = 1.57$	DCBA						
		$u(B) = 1.68$							
		$u(C) = 1.78$							
		$u(D) = 1.88$							

Author Manuscript

Author Manuscript

Author Manuscript

Author Manuscript

Dependent and nonstationary (i.e., non-iid) binary choices satisfying the random function model, hence satisfying the triangle inequalities for each question asked.

Table 9

	P_{AB}	P_{AC}	P_{AD}	P_{BC}	$P_{(BD)}$	P_{CD}
QUESTION 1:						
QUESTION 2 (given chose B in QUESTION 1):	.9	.9	.9	.9	.9	.9
QUESTION 2 (given chose A in QUESTION 1):	.5	.5	.5	1	.5	.5
QUESTION 2 (unconditional):	.54	.54	.54	.99	.54	.54

	A	B
C	$.9 \times .5 = .45$	$.1 \times .9 = .09$
D	$.9 \times .5 = .45$	$.1 \times .1 = .01$
	$P_{AB} = .9$	$P_{BA} = .1$
		$P_{CD} = .54$
		$P_{DC} = .46$