Research

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Author for correspondence:

Thomas J. Bridges

e-mail: T.Bridges@surrey.ac.uk

Nonlinear modulation near the Lighthill instability threshold in $2 + 1$ Whitham theory

Thomas J. Bridges¹ and Daniel J. Ratliff²

¹Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK

²Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK

TJB, 0000-0003-0855-0747

The dispersionless Whitham modulation equations in $2 + 1$ (two space dimensions and time) are reviewed and the instabilities identified. The modulation theory is then reformulated, near the Lighthill instability threshold, with a slow phase, moving frame and different scalings. The resulting nonlinear phase modulation equation near the Lighthill surfaces is a geometric form of the $2+1$ two-way Boussinesq equation. This equation is universal in the same sense as Whitham theory. Moreover, it is dispersive, and it has a wide range of interesting multi-periodic, quasi-periodic and multi-pulse localized solutions. For illustration the theory is applied to a complex nonlinear $2 + 1$ Klein–Gordon equation which has two Lighthill surfaces in the manifold of periodic travelling waves.

This article is part of the theme issue ‘Stability of nonlinear waves and patterns and related topics’.

1. Introduction

Modulation of periodic travelling waves, in classical Whitham modulation theory in $2 + 1$ (two space dimensions and time), results in a closed system of first-order nonlinear PDEs

$$\left. \begin{aligned} q_T - \Omega_X &= 0, \\ p_T - \Omega_Y &= 0, \\ p_X - q_Y &= 0 \end{aligned} \right\} \quad (1.1)$$

and

$$\mathcal{A}_T + \mathcal{B}_X + \mathcal{C}_Y = 0.$$

The fourth equation is conservation of wave action with $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ functions of (Ω, q, p) , the modulation

frequency and wavenumbers, with $T = \varepsilon t$, $X = \varepsilon x$ and $Y = \varepsilon y$ slow time and space variables. These modulation equations were first derived by Hayes [1], and other derivations are given in [2, §14.7], [3, ch. 8] and [4, ch. 11].

The wavenumbers and frequency are expressed in terms of the phase via

$$q = \phi_X, \quad p = \phi_Y \quad \text{and} \quad \Omega = \phi_T. \quad (1.2)$$

Substitution into (1.1), and linearizing about the basic state, represented by (ω, k, ℓ) , gives the linear second-order PDE

$$\begin{aligned} & \mathcal{A}_\omega \phi_{TT} + \mathcal{A}_k \phi_{TX} + \mathcal{A}_\ell \phi_{TY} + \mathcal{B}_\omega \phi_{XT} + \mathcal{B}_k \phi_{XX} \\ & + \mathcal{B}_\ell \phi_{XY} + \mathcal{C}_\omega \phi_{YT} + \mathcal{C}_k \phi_{YX} + \mathcal{C}_\ell \phi_{YY} = 0, \end{aligned} \quad (1.3)$$

with the coefficients now functions of the basic state only. Rotate the (T, X, Y) coordinates to diagonalize (1.3),

$$\mathcal{A}_\omega \phi_{\tau\tau} + \frac{\Delta_L}{\mathcal{A}_\omega} \phi_{\hat{\xi}\hat{\xi}} + \frac{\Delta_M}{\Delta_L} \phi_{\hat{\eta}\hat{\eta}} = 0, \quad (1.4)$$

assuming $\mathcal{A}_\omega \neq 0$, $\Delta_L \neq 0$ and $\Delta_M \neq 0$, where τ , $\hat{\xi}$ and $\hat{\eta}$ are the rotated space–time coordinates. Expressions for the coordinates and coefficients are given in §2. The basic state is considered stable when this phase modulation equation is hyperbolic in time ($\Delta_L < 0$ and $\mathcal{A}_\omega \Delta_M > 0$) and unstable otherwise. This equation is discussed further in §2.

The interest in this paper is in a nonlinear theory for the transition of the primary instability which occurs when Δ_L , the *Lighthill determinant*, changes sign, where

$$\Delta_L := \det \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix}. \quad (1.5)$$

The significance of this determinant in the Whitham theory was first pointed out by Lighthill [5] in the $1 + 1$ case. The instability generated by $\Delta_L > 0$ is in the x -direction, making it the preferred direction. Hence y will be labelled the *transverse direction*.

Setting $\Delta_L = 0$ defines a surface in the three-space (ω, k, ℓ) , which will be called the *Lighthill surface*. Clearly $\Delta_L = 0$ is a problem for the coefficient of $\phi_{\hat{\eta}\hat{\eta}}$ in (1.4). The singularity is easily removed by working in the original coordinates as in (1.3). However, we find that the additional condition

$$\Delta_Y := \det \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{C}_\omega & \mathcal{C}_k \end{bmatrix} = 0 \quad (1.6)$$

arises naturally in the theory, which also removes the singularity in the rotated coordinates (1.4). Details of this are given in §4. For systems that are symmetric in the y -direction, this condition is always satisfied in the case $\ell = 0$, that is, a basic state travelling purely in the x -direction.

In addition to the two assumptions $\Delta_L = 0$ and $\Delta_Y = 0$, our strategy for developing a nonlinear modulation theory near $\Delta_L = 0$ surfaces is to slow down the time scale, go into a moving frame and slow down the phase, wavenumber and frequency modulation. The generic Whitham modulation is replaced by the mapping $\theta \mapsto \theta + \varepsilon \phi$,

$$k \mapsto k + \varepsilon^2 q(X, Y, T, \varepsilon), \quad \ell \mapsto \ell + \varepsilon^3 p(X, Y, T, \varepsilon) \quad (1.7)$$

and

$$\omega \mapsto \omega - c_1 \varepsilon^2 q(X, Y, T, \varepsilon) + \varepsilon^3 \Omega(X, Y, T, \varepsilon) - c_2 \varepsilon^3 p(X, Y, T, \varepsilon), \quad (1.8)$$

with the compatibility conditions (1.2), and

$$X = \varepsilon(x - c_1 t), \quad Y = \varepsilon^2(y - c_2 t), \quad T = \varepsilon^2 t, \quad (1.9)$$

with

$$c_1 = \frac{\mathcal{A}_k}{\mathcal{A}_\omega} \quad \text{and} \quad c_2 = \frac{\mathcal{A}_\ell}{\mathcal{A}_\omega}. \quad (1.10)$$

The speeds (c_1, c_2) can be interpreted as a form of nonlinear group velocity. This proposed modulation is not obvious but it is an ansatz and so the proof of its validity is that it satisfies

the governing equations exactly up to fifth order if and only if the modulation functions satisfy a new nonlinear version of the 2 + 1 Whitham equations with dispersion given in (1.11) below. The theoretical justification is given in §4.

Substitution of the above modulation ansatz into the governing equations, which are the Euler–Lagrange equations based on a general abstract Lagrangian, then leads, at fifth order in ε , via a solvability condition, to the new modulation equations replacing (1.1)

$$\left. \begin{aligned} q_T - \Omega_X &= 0, \\ p_T - \Omega_Y &= 0, \\ p_X - q_Y &= 0 \\ \mathcal{A}_\omega \Omega_T + \kappa q q_X + \mathcal{K} q_{XXX} + \frac{\Delta_T}{\mathcal{A}_\omega} p_Y &= 0, \end{aligned} \right\} \quad (1.11)$$

and where Δ_T is the analogue of the Lighthill determinant in the Y -direction, defined by

$$\Delta_T := \det \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_\ell \\ \mathcal{C}_\omega & \mathcal{C}_\ell \end{bmatrix}. \quad (1.12)$$

Differentiating the fourth equation in (1.11) with respect to X and using the first and third equations shows that it is a variant of the 2 + 1 two-way Boussinesq equation, but with coefficients that are universal in the same sense that the Whitham equations are universal (that they follow from the abstract properties of the Lagrangian),

$$\mathcal{A}_\omega q_{TT} + \kappa (q q_X)_X + \mathcal{K} q_{XXX} + \frac{\Delta_T}{\mathcal{A}_\omega} q_{YY} = 0. \quad (1.13)$$

The second coefficient, κ , is a curvature function determined from the second derivatives of $\mathcal{A}(\omega, k, \ell)$ and $\mathcal{B}(\omega, k, \ell)$, and the coefficient of dispersion $\mathcal{K}$ is determined by a Jordan chain argument. For non-degeneracy of this equation, it is assumed that $\mathcal{A}_\omega$, κ , $\mathcal{K}$ and Δ_T are non-zero for parameter values on the Lighthill surface. If one of the coefficients is zero, the strategy would be to re-scale with a new modulation ansatz generating a different modulation equation.

In summary, at generic points (ω, k, ℓ) on a branch of periodic travelling waves, the dispersionless Whitham equations (1.1) are operational. On surfaces where $\Delta_L = 0$, and sub-curves where $\Delta_Y = 0$ in addition, the operational equations are (1.11) with the principal outcome being that there is a dramatic change in the nature of the solution set. The generic equations (1.1) are dispersionless, whereas the equations (1.13) have dispersion and a two-way Boussinesq equation, and the latter is known to have a wide range of complex solutions [6–9].

The paper is organized as follows. First, the linear generic Whitham modulation equations (1.1) are studied in §2 and instabilities identified. The details of the nonlinear modulation leading to (1.11) are carried out in §4. The theory is applied to a nonlinear wave equation based on the Klein–Gordon equation in §3 and §5. Some indication of potential generalizations is given in §6.

2. Instabilities in 2 + 1 Whitham theory

The starting point for the analysis of instabilities in 2 + 1 is the phase version of the Whitham modulation equations linearized about the basic state (1.3) simplified using the identities $\mathcal{A} := \mathcal{L}_\omega$, $\mathcal{B} := \mathcal{L}_k$ and $\mathcal{C} := \mathcal{L}_\ell$,

$$\mathcal{L}_{\omega\omega} \phi_{TT} + \mathcal{L}_{kk} \phi_{XX} + \mathcal{L}_{\ell\ell} \phi_{YY} + 2\mathcal{L}_{\omega k} \phi_{XT} + 2\mathcal{L}_{\omega\ell} \phi_{YT} + 2\mathcal{L}_{k\ell} \phi_{XY} = 0. \quad (2.1)$$

Here, $\mathcal{L}$ is the Lagrangian averaged over one period of the periodic travelling wave. This is typical in the Whitham modulation theory [2, §14.3 and §14.7]. A definition in the context of this paper is given §4.

Equation (2.1) is derived by starting with a periodic travelling wave solution, say $\hat{Z}(\theta, k, \ell, \omega)$, with $\theta = kx + \ell y + \omega t$, which is 2π -periodic in θ . This solution satisfies a PDE which is the

Euler–Lagrange equation for some given Lagrangian functional. To obtain the *linear* phase equation this solution is modulated with the ansatz

$$Z(x, y, t) = \hat{Z}(\theta + \phi, k + \varepsilon q, \ell + \varepsilon p, \omega + \varepsilon \Omega) + \varepsilon^2 W(\theta + \phi, X, Y, T, \varepsilon),$$

with $q = \phi_X$, $p = \phi_Y$ and $\Omega = \phi_T$, where the slow space and time scales are $T = \varepsilon t$, $X = \varepsilon x$, $Y = \varepsilon y$. Substitution into the Euler–Lagrange equation and applying solvability at second order gives the phase modulation equation (2.1) [4, ch. 11].

Starting with (2.1), introduce coordinates that will diagonalize the coefficients

$$\tau = T, \quad \xi = X - c_1 T \quad \text{and} \quad \eta = Y - c_2 T,$$

with (c_1, c_2) defined in (1.10). The derivatives transform to

$$\begin{aligned} \frac{\partial}{\partial T} &= \frac{\partial}{\partial \tau} - c_1 \frac{\partial}{\partial \xi} - c_2 \frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial X} &= \frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial \eta}. \end{aligned}$$

In these new coordinates, the PDE (2.1) transforms to

$$\mathcal{L}_{\omega\omega} \phi_{\tau\tau} + a \phi_{\xi\xi} + 2b \phi_{\xi\eta} + c \phi_{\eta\eta} = 0, \quad (2.2)$$

where

$$\begin{aligned} a &= \frac{\Delta_L}{\mathcal{L}_{\omega\omega}} = \frac{1}{\mathcal{L}_{\omega\omega}} \begin{vmatrix} \mathcal{L}_{\omega\omega} & \mathcal{L}_{\omega k} \\ \mathcal{L}_{k\omega} & \mathcal{L}_{kk} \end{vmatrix}, \\ b &= \frac{\Delta_Y}{\mathcal{L}_{\omega\omega}} = \frac{1}{\mathcal{L}_{\omega\omega}} \begin{vmatrix} \mathcal{L}_{\omega\omega} & \mathcal{L}_{\omega k} \\ \mathcal{L}_{\ell\omega} & \mathcal{L}_{k\ell} \end{vmatrix} \\ \text{and} \quad c &= \frac{\Delta_T}{\mathcal{L}_{\omega\omega}} = \frac{1}{\mathcal{L}_{\omega\omega}} \begin{vmatrix} \mathcal{L}_{\omega\omega} & \mathcal{L}_{\omega\ell} \\ \mathcal{L}_{\ell\omega} & \mathcal{L}_{\ell\ell} \end{vmatrix}. \end{aligned}$$

Now diagonalize the a, b, c terms. Define new coordinates

$$\hat{\tau} = \tau, \quad \hat{\xi} = \xi \quad \text{and} \quad \hat{\eta} = \eta - \beta \xi, \quad \text{with } \beta = \frac{b}{a} = \frac{\Delta_Y}{\Delta_L}, \quad (2.3)$$

when $\Delta_L \neq 0$. Then the transformed derivatives are

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \hat{\tau}}, \quad \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \hat{\xi}} - \beta \frac{\partial}{\partial \hat{\eta}} \quad \text{and} \quad \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \hat{\eta}}.$$

Substitution into the phase equation (2.2) results in

$$\mathcal{L}_{\omega\omega} \frac{\partial^2 \phi}{\partial \hat{\tau}^2} + \frac{1}{\mathcal{L}_{\omega\omega}} \Delta_L \frac{\partial^2 \phi}{\partial \hat{\xi}^2} + \frac{1}{\Delta_L} \Delta_M \frac{\partial^2 \phi}{\partial \hat{\eta}^2} = 0, \quad (2.4)$$

where

$$\Delta_M = \det \begin{bmatrix} \mathcal{L}_{\omega\omega} & \mathcal{L}_{\omega k} & \mathcal{L}_{\omega\ell} \\ \mathcal{L}_{k\omega} & \mathcal{L}_{kk} & \mathcal{L}_{k\ell} \\ \mathcal{L}_{\ell\omega} & \mathcal{L}_{\ell k} & \mathcal{L}_{\ell\ell} \end{bmatrix}. \quad (2.5)$$

The phase equation (2.4) is now in a tidy form for classifying instabilities. First assume

$$\mathcal{L}_{\omega\omega} \neq 0.$$

It is then immediate that if $\Delta_L > 0$ then the PDE is elliptic signalling instability in the $\hat{\xi}$ -direction. Now suppose

$$\mathcal{L}_{\omega\omega} \neq 0 \quad \text{and} \quad \Delta_L < 0,$$

then the PDE is elliptic when

$$\mathcal{L}_{\omega\omega} \Delta_M < 0.$$

When this condition is satisfied the basic state is unstable to perturbations in the $\hat{\eta}$ -direction. The PDE (2.4) is hyperbolic when

$$\mathcal{L}_{\omega\omega} \neq 0, \quad \Delta_L < 0 \quad \text{and} \quad \mathcal{L}_{\omega\omega} \Delta_M > 0.$$

(a) Generic phase equations on Lighthill surfaces

Before developing a modulation theory near Lighthill surfaces in §4, the question of the singularity in the phase equation (2.4) when $\Delta_L = 0$ is considered. The easy way to avoid the singularity is to work with the (ξ, η, τ) coordinates and not shift to $\hat{\eta}$ in (2.3). However, there is a natural way to eliminate this singularity and maintain rotated coordinates (the rotated coordinates give a tidy form to the final two-way Boussinesq equation).

Reformulate Δ_M as

$$\mathcal{A}_\omega \Delta_M = (\mathcal{A}_\omega \mathcal{B}_k - \mathcal{A}_k \mathcal{B}_\omega)(\mathcal{A}_\omega \mathcal{C}_\ell - \mathcal{A}_\ell \mathcal{C}_\omega) - (\mathcal{A}_\omega \mathcal{C}_k - \mathcal{A}_k \mathcal{C}_\omega)^2 = \Delta_T \Delta_L - \Delta_Y^2,$$

with Δ_T defined in (1.12). Hence

$$\frac{\Delta_M}{\Delta_L} = \frac{\Delta_T}{\mathcal{A}_\omega} - \frac{\Delta_Y^2}{\mathcal{A}_\omega \Delta_L}.$$

In order to take the limit $\Delta_L \rightarrow 0$, it will also be required to take $\Delta_Y \rightarrow 0$. This argument is informal, but in the modulation theory the requirement that $\Delta_Y = 0$ will arise naturally.

With the condition $\Delta_L = 0$, the matrix associated with this determinant will have a zero eigenvalue with attendant eigenvector ζ satisfying

$$\begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is assumed that the zero eigenvalue of this matrix is simple. Hence the assumptions associated with $\Delta_L = 0$ are

$$\mathcal{A}_\omega \neq 0, \quad \Delta_L = 0 \quad \text{and} \quad \mathcal{A}_\omega + \mathcal{B}_k \neq 0. \quad (2.6)$$

The canonical choice for the eigenvector ζ is

$$\zeta = \begin{pmatrix} -c_1 \\ 1 \end{pmatrix}. \quad (2.7)$$

The matrix associated with Δ_Y will also have a zero eigenvalue and eigenvector but they are not needed for the leading-order theory.

There are three subsets of (ω, k, ℓ) -space of interest in studying Lighthill singularities. There is the existence set,

$$U := \{(\omega, k, \ell) \mid \text{basic state exists}\},$$

the Lighthill surface,

$$S^L := \{(\omega, k, \ell) \mid \Delta_L = 0\},$$

and the symmetry surface,

$$S^Y := \{(\omega, k, \ell) \mid \Delta_Y = 0\}.$$

The non-trivial and generic intersection of the sets S^L and S^Y results in a curve in (ω, k, ℓ) -space. For the nonlinear modulation on Lighthill surfaces it is assumed that such a curve exists, that is,

$$S^L \cap S^Y \cap U \quad \text{is a non-empty and smooth subset of } U. \quad (2.8)$$

3. Example: 2 + 1 nonlinear wave equation

Consider the nonlinear wave equation, a 2 + 1 complex Klein–Gordon (CKG) equation,

$$g^{11}\Psi_{tt} + g^{22}\Psi_{xx} + g^{33}\Psi_{yy} - \Psi + |\Psi|^2\Psi = 0, \quad (3.1)$$

for the complex-valued function $\Psi(x, y, t)$, where the coefficients g^{ij} are arbitrary non-zero constants. The coefficients include the case where the evolution equation is ill-posed (e.g. $g^{11} = g^{22} = g^{33} = 1$), although the canonical choice of interest here is

$$g^{11} = -1, \quad g^{22} = +1 \quad \text{and} \quad g^{33} = +1. \quad (3.2)$$

The nonlinear wave equation (3.1) is generated by the Lagrangian

$$\mathcal{L}(\Psi, \bar{\Psi}) = \frac{1}{2} \int_{t_1}^{t_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} [g^{11}|\Psi_t|^2 + g^{22}|\Psi_x|^2 + g^{33}|\Psi_y|^2 + |\Psi|^2 - \frac{1}{2}|\Psi|^4] dx dy dt \quad (3.3)$$

on the cube $[x_1, x_2] \times [y_1, y_2] \times [t_1, t_2] \subset \mathbb{R}^3$. The variation $\delta\mathcal{L}/\delta\bar{\Psi} = 0$, with fixed endpoints, generates (3.1), and $\delta\mathcal{L}/\delta\Psi = 0$ generates the conjugate of (3.1).

The nonlinear wave equation (3.1) has a family of exact periodic travelling wave solutions

$$\Psi(x, t) = \Psi_0 e^{i\theta}, \quad \theta = kx + \ell y + \omega t + \theta_0 \quad (3.4)$$

and substitution into (3.1) gives the nonlinear dispersion relation, relating amplitude to the frequency and wavenumber

$$|\Psi_0|^2 = 1 + g^{11}\omega^2 + g^{22}k^2 + g^{33}\ell^2. \quad (3.5)$$

Solutions exist in the set

$$U = \{(\omega, k, \ell) \in \mathbb{R}^3 : 1 + g^{11}\omega^2 + g^{22}k^2 + g^{33}\ell^2 > 0, k \neq 0\}. \quad (3.6)$$

It is assumed that this set is not empty. It is submanifold in the four-dimensional half-space with coordinates (ω, k, ℓ, r) , where $r = |\Psi_0| > 0$. In the canonical metric (3.2), the projection onto (ω, k, ℓ) -space is the hyperboloid

$$\{(\omega, k, \ell, r) \in \mathbb{R}^4 \mid r^2 + \omega^2 - k^2 - \ell^2 = 1\}.$$

The conservation law which represents conservation of wave action is due to an S^1 -symmetry: $e^{is}\Psi$ is a solution of (3.1) whenever Ψ is a solution for any $s \in \mathbb{R}$. The conservation law is

$$A_t + B_x + C_y = 0, \quad \text{with } A = g^{11} \text{Im}(\bar{\Psi}\Psi_t), \quad B = g^{22} \text{Im}(\bar{\Psi}\Psi_x) \quad \text{and} \quad C = g^{33} \text{Im}(\bar{\Psi}\Psi_y).$$

The components of the conservation law are in (math) roman font when considered as functions of x, y, t , and are in calligraphic font when evaluated on the basic state and are then functions of ω, k, ℓ .

Evaluate the components of the conservation law on the family of periodic travelling waves

$$\left. \begin{aligned} \mathcal{A}(\omega, k, \ell) &= \omega g^{11} |\Psi_0|^2 = \omega g^{11} (1 + g^{11}\omega^2 + g^{22}k^2 + g^{33}\ell^2), \\ \mathcal{B}(\omega, k, \ell) &= k g^{22} |\Psi_0|^2 = k g^{22} (1 + g^{11}\omega^2 + g^{22}k^2 + g^{33}\ell^2) \\ \text{and} \quad \mathcal{C}(\omega, k, \ell) &= \ell g^{33} |\Psi_0|^2 = \ell g^{33} (1 + g^{11}\omega^2 + g^{22}k^2 + g^{33}\ell^2). \end{aligned} \right\} \quad (3.7)$$

These expressions can also be obtained by substituting (3.4) into (3.3), averaging over θ , and using the definitions $\mathcal{A} = \mathcal{L}_\omega$, $\mathcal{B} = \mathcal{L}_k$ and $\mathcal{C} = \mathcal{L}_\ell$.

The three key diagnostics in the second-order phase equation (2.4) are computed to be

$$\left. \begin{aligned} \mathcal{A}_\omega &= g^{11}(1 + 3g^{11}\omega^2 + g^{22}k^2), \\ \Delta_L &= g^{11}g^{22}|\Psi_0|^2(1 + 3g^{11}\omega^2 + 3g^{22}k^2) \\ \text{and} \quad \Delta_M &= g^{33}|\Psi_0|^2\Delta_L. \end{aligned} \right\} \quad (3.8)$$

In these expressions, ℓ has been set to zero, after the derivatives are taken, as this is the case needed in the nonlinear modulation. This is to ensure Δ_Y is zero, since for this system

$$\Delta_Y = 2g^{11}g^{22}g^{33}\ell k|\Psi_0|^2.$$

The Lighthill determinant Δ_L changes sign when $1 + 3g^{11}\omega^2 + 3g^{22}k^2 = 0$ and this set is non-empty for the canonical choice (3.2).

The phase equation (2.4) is hyperbolic in the time direction when $\Delta_L < 0$ and $\mathcal{A}_\omega\Delta_M > 0$ or

$$g^{11}g^{22}(1 + 3g^{11}\omega^2 + 3g^{22}k^2) < 0 \quad \text{and} \quad g^{11}g^{33}(1 + 3g^{11}\omega^2 + g^{22}k^2) < 0. \quad (3.9)$$

This set is non-empty in the canonical case (3.2). The interest in this paper is the case where Δ_L changes sign. First, a general theory will be developed for abstract Lagrangian PDEs, and then it is applied to this example.

4. Phase modulation near Lighthill surfaces

It was seen in §2 that the 2 + 1 Whitham equations have the ability to transition from hyperbolic to elliptic with the transition point presenting a degeneracy of the modulation system (1.1). At such points, one must obtain a rebalance through a rescaling of the slow variables to obtain a new PDE governing the evolution of ϕ . It transpires that, on a Lighthill surface, the linear Whitham equations morph into a nonlinear dispersive PDE, namely a geometric version of the 2 + 1 two-way Boussinesq equation. It is called a ‘geometric version’ to flag up that the dependent variable is a wavenumber and the coefficients are determined by the geometry of Lighthill surfaces.

This section presents the details required to derive this new set of modulation PDEs on Lighthill surfaces. The starting point for the theory is a general Lagrangian, which can be put into the abstract form

$$\mathcal{L}(Z) = \int_{t_1}^{t_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left[\frac{1}{2} \langle \mathbf{M}Z_t, Z \rangle + \frac{1}{2} \langle \mathbf{J}Z_x, Z \rangle + \frac{1}{2} \langle \mathbf{K}Z_y, Z \rangle - S(Z) \right] dx dy dt, \quad (4.1)$$

where $\mathbf{M}$, $\mathbf{J}$ and $\mathbf{K}$ are constant skew-symmetric matrices, Z is $\mathbb{R}^{2n}$ -valued, $\langle \cdot, \cdot \rangle$ is a standard Euclidean inner product on $\mathbb{R}^{2n}$, and $S(Z)$ is a smooth scalar-valued function (e.g. [4,10–12]). The Euler–Lagrange equation takes the canonical form

$$\mathbf{M}Z_t + \mathbf{J}Z_x + \mathbf{K}Z_y = \nabla S(Z), \quad Z \in \mathbb{R}^{2n}, \quad (4.2)$$

where ∇S is the gradient with respect to the standard inner product on $\mathbb{R}^{2n}$.

Assume the existence of a periodic travelling wave,

$$Z(x, y, t) = \hat{Z}(\theta, \omega, k, \ell) \quad \text{with } \theta = kx + \ell y + \omega t + \theta_0,$$

where k, ℓ are wavenumbers, ω is the frequency and $\hat{Z}$ is a 2π -periodic function of θ , satisfying the ODE

$$(\omega \mathbf{M} + k \mathbf{J} + \ell \mathbf{K}) \hat{Z}_\theta = \nabla S(\hat{Z}). \quad (4.3)$$

The connection between the averaged Lagrangian in Whitham modulation theory and the Lagrangian (4.1) is established by substituting $\hat{Z}$ into (4.1) and averaging over θ (which is discussed below (4.4)). However, there is a critical advantage here over classical Whitham theory in that the Lagrangian is partitioned according to a multi-symplectic structure.

(a) Multi-symplecticity

For the purposes of this paper, a *multi-symplectic structure* is a collection of skew-symmetric matrices, here $\mathbf{M}$, $\mathbf{J}$ and $\mathbf{K}$, associated with the t , x and y directions, respectively, and together they generate the symmetric linear operator

$$\mathbf{M}\partial_t + \mathbf{J}\partial_x + \mathbf{K}\partial_y,$$

which is central to the theory. The multi-symplectic structure gives a tidy formulation of the governing equation as in (4.2) and (4.3). Another interpretation is as a partition of the Lagrangian function, obtained by Legendre transform or otherwise, into components which separate the t , x and y derivatives.

Of equal importance is an explicit connection between the components of the conservation law and the multi-symplectic structure. Wave action evaluated on this family of travelling waves is

$$\mathcal{A}(\omega, k, \ell) = \mathcal{L}_\omega = \frac{1}{2} \langle \langle \mathbf{M}\hat{Z}_\theta, \hat{Z} \rangle \rangle, \quad (4.4)$$

where $\mathcal{L}(\omega, k, \ell) = \overline{\mathcal{L}(\hat{Z})}$, and $\langle \langle \cdot, \cdot \rangle \rangle$ is an inner product including averaging over θ ,

$$\langle \langle \cdot, \cdot \rangle \rangle = \overline{\langle \cdot, \cdot \rangle} := \frac{1}{2\pi} \int_0^{2\pi} \langle \cdot, \cdot \rangle d\theta.$$

Similarly,

$$\mathcal{B}(\omega, k, \ell) = \mathcal{L}_k = \frac{1}{2} \langle \langle \mathbf{J}\hat{Z}_\theta, \hat{Z} \rangle \rangle \quad \text{and} \quad \mathcal{C}(\omega, k, \ell) = \mathcal{L}_\ell = \frac{1}{2} \langle \langle \mathbf{K}\hat{Z}_\theta, \hat{Z} \rangle \rangle. \quad (4.5)$$

Derivatives of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ can then be related explicitly to the solutions and properties of the governing equation (4.3).

(b) Properties of the linearization

Properties of the linearization are now derived as they are needed at various stages in the theory. Linearization of (4.3) about $\hat{Z}$ generates the linear operator $\mathbf{L}$,

$$\mathbf{L}V = D^2S(\hat{Z})V - (\omega\mathbf{M} + k\mathbf{J} + \ell\mathbf{K})V_\theta, \quad (4.6)$$

where $D^2S(\hat{Z})$ is the Hessian of S evaluated at $\hat{Z}$. Differentiating (4.3), with respect to θ , k , ω and ℓ , generates the equations

$$\mathbf{L}\hat{Z}_\theta = 0, \quad \mathbf{L}\hat{Z}_k = \mathbf{J}\hat{Z}_\theta, \quad \mathbf{L}\hat{Z}_\omega = \mathbf{M}\hat{Z}_\theta \quad \text{and} \quad \mathbf{L}\hat{Z}_\ell = \mathbf{K}\hat{Z}_\theta. \quad (4.7)$$

The first equation shows that $\hat{Z}_\theta$ is in the kernel of $\mathbf{L}$. It is assumed that the kernel is no larger. Then since $\mathbf{L}$ is a symmetric operator

$$\mathbf{L}V = F \text{ is solvable if and only if } \langle \langle \hat{Z}_\theta, F \rangle \rangle = 0.$$

Combinations of the equations (4.7) will be needed in the theory,

$$\mathbf{L}(\hat{Z}_k - c_1\hat{Z}_\omega) = (\mathbf{J} - c_1\mathbf{M})\hat{Z}_\theta \quad \text{and} \quad \mathbf{L}(\hat{Z}_\ell - c_2\hat{Z}_\omega) = (\mathbf{K} - c_2\mathbf{M})\hat{Z}_\theta. \quad (4.8)$$

When combined with $\mathbf{L}\hat{Z}_\theta = 0$, these two equations can be interpreted as twisted symplectic Jordan chains of length two. They are symplectic chains since $\mathbf{L}$ is symmetric and $\mathbf{J} - v\mathbf{M}$ is skew-symmetric for any $v \in \mathbb{R}$, and they are twisted since they combine the x -direction structure matrix

$\mathbf{J}$ with the t -direction structure matrix $\mathbf{M}$. It is the first Jordan chain in (4.8) that is of primary interest on Lighthill surfaces. Define

$$\Gamma_1 = \hat{Z}_\theta \quad \text{and} \quad \Gamma_2 = \hat{Z}_k - c_1 \hat{Z}_\omega.$$

Then

$$\mathbf{L}\Gamma_1 = 0 \quad \text{and} \quad \mathbf{L}\Gamma_2 = (\mathbf{J} - c_1 \mathbf{M})\Gamma_1.$$

We claim that this chain has length four, that is, there exist Γ_3 and Γ_4 satisfying

$$\left. \begin{aligned} \mathbf{L}\Gamma_3 &= (\mathbf{J} - c_1 \mathbf{M})\Gamma_2 \\ \mathbf{L}\Gamma_4 &= (\mathbf{J} - c_1 \mathbf{M})\Gamma_3, \end{aligned} \right\} \quad (4.9)$$

and

if and only if $\Delta_L = 0$, assuming $\mathcal{A}_\omega \neq 0$. To prove this role of $\Delta_L = 0$ it is necessary to connect it with solvability of the two equations in (4.9). The solvability condition for the Γ_3 equation is

$$\begin{aligned} 0 &= \langle \hat{Z}_\theta, (\mathbf{J} - c_1 \mathbf{M})\Gamma_2 \rangle = \langle \hat{Z}_\theta, (\mathbf{J} - c_1 \mathbf{M})(\hat{Z}_k - c_1 \hat{Z}_\omega) \rangle \\ &= \langle \hat{Z}_\theta, \mathbf{J}(\hat{Z}_k - c_1 \hat{Z}_\omega) \rangle - c_1 \langle \hat{Z}_\theta, \mathbf{M}(\hat{Z}_k - c_1 \hat{Z}_\omega) \rangle \\ &= -(\mathcal{B}_k - c_1 \mathcal{B}_\omega - c_1 \mathcal{A}_k + c_1^2 \mathcal{A}_\omega) \\ &= -\frac{\mathcal{A}_\omega \mathcal{B}_k - \mathcal{A}_k \mathcal{B}_\omega}{\mathcal{A}_\omega} \equiv -\frac{\Delta_L}{\mathcal{A}_\omega}. \end{aligned}$$

Similarly solvability of the Γ_4 equation is ensured since

$$\begin{aligned} 0 &= \langle \hat{Z}_\theta, (\mathbf{J} - c_1 \mathbf{M})\Gamma_3 \rangle = -\langle (\mathbf{J} - c_1 \mathbf{M})\hat{Z}_\theta, \Gamma_3 \rangle \\ &= -\langle \mathbf{L}\Gamma_2, \Gamma_3 \rangle \\ &= -\langle \Gamma_2, \mathbf{L}\Gamma_3 \rangle \\ &= -\langle \Gamma_2, (\mathbf{J} - c_1 \mathbf{M})\Gamma_2 \rangle = 0, \end{aligned}$$

due to the skew-symmetry of $\mathbf{J} - c_1 \mathbf{M}$. The symplectic Jordan chain $(\Gamma_1, \dots, \Gamma_4)$ plays a central role in the resolution of the third- and fourth-order terms, and is the mechanism for the emergence of dispersion in (1.11) and (1.13).

(c) Rescaling and modulation near Lighthill surfaces

Phase modulation of the nonlinear system near Lighthill surfaces starts with an ansatz, then proceeds to substitute, expand and analyse the asymptotics. Here the ansatz chosen is based on the argument (1.7) and (1.8). It is a $2 + 1$ generalization of the modulation ansatz in [10],

$$Z(x, y, t) = \hat{Z}(\theta + \varepsilon\phi, k + \varepsilon^2q, \ell + \varepsilon^3p, \omega - \varepsilon^2c_1q - \varepsilon^3c_2p + \varepsilon^3\Omega) + W(\theta, X, Y, T, \varepsilon), \quad (4.10)$$

where ϕ, q, p, Ω all depend on X, Y, T, ε . The relationship between the phase, wavenumbers and frequency is the same as (1.2), and the slow time and space variables are as defined in (1.9) with the moving frame speeds as defined in (1.10).

Expand $\varepsilon^3W = \varepsilon^3W_3 + \varepsilon^4W_4 + \varepsilon^5W_5 + \dots$, substitute the ansatz (4.10) into the Euler–Lagrange equation (4.2), expand everything in powers of ε and solve order by order. There are similarities to the analysis in [10] so we can be brief, highlighting only the key new features due to the $2 + 1$ setting. Zeroth-order terms just recover the governing equation for $\hat{Z}$ in (4.3). The first-order terms recover the tangent equation $\phi \mathbf{L}\hat{Z}_\theta = 0$, which is satisfied for all ϕ .

(i) Second- and third-order equations

At second order, the equations can be reduced to

$$q \mathbf{L}(\hat{Z}_k - c_1 \hat{Z}_\omega) - \phi_X (\mathbf{J} - c_1 \mathbf{M})\hat{Z}_\theta = 0,$$

which is satisfied exactly when $q = \phi_X$ is introduced, and the first equation of (4.8) is used.

At third order,

$$\mathbf{L}W_3 = q_X(\mathbf{J} - c_1\mathbf{M})(\hat{Z}_k - c_1\hat{Z}_\omega) + p\mathbf{L}(\hat{Z}_\ell - c_2\hat{Z}_\omega) - \phi_Y(\mathbf{K} - c_2\mathbf{M})\hat{Z}_\theta + (\Omega - \phi_T)\mathbf{L}\hat{Z}_\omega.$$

Substituting $\phi_T = \Omega$ and $p = \phi_Y$ and using the second equation of (4.8) leaves only the q_X term,

$$\mathbf{L}W_3 = q_X(\mathbf{J} - c_1\mathbf{M})(\hat{Z}_k - c_1\hat{Z}_\omega). \quad (4.11)$$

The Jordan chain (4.9) ensures that this equation is solvable for any q_X and the third-order solution is

$$W_3 = q_X \Gamma_3,$$

modulo the kernel of $\mathbf{L}$. Additional kernel elements are neglected as it is confirmed *a posteriori* that they do not enter the leading-order result.

(ii) Fourth order

At fourth order, new features due to the 2 + 1 setting arise. The fourth-order terms simplify to

$$\begin{aligned} \mathbf{L}W_4 = & q_{XX}(\mathbf{J} - c_1\mathbf{M})\Gamma_3 + \phi q_X((\mathbf{J} - c_1\mathbf{M})(\hat{Z}_{\theta k} - c_1\hat{Z}_{\omega\theta}) - \mathbf{D}^3 S(\hat{Z})(\hat{Z}_\theta, \Gamma_3)) \\ & + p_X(\mathbf{J} - c_1\mathbf{M})(\hat{Z}_\ell - c_2\hat{Z}_\omega) + q_Y(\mathbf{K} - c_2\mathbf{M})(\hat{Z}_k - c_1\hat{Z}_\omega) \\ & + q_T(\mathbf{M}\hat{Z}_k - c_1\mathbf{M}\hat{Z}_\omega) + \Omega_X(\mathbf{J}\hat{Z}_\omega - c_2\mathbf{M}\hat{Z}_\omega). \end{aligned} \quad (4.12)$$

The ϕq_X term can be absorbed into the linear operator $\mathbf{L}$ since

$$\mathbf{L}(\Gamma_3)_\theta = (\mathbf{J} - c_1\mathbf{M})(\hat{Z}_{\theta k} - c_1\hat{Z}_{\omega\theta}) - \mathbf{D}^3 S(\hat{Z})(\hat{Z}_\theta, \Gamma_3).$$

The term with q_{XX} is automatically solvable using the second equation of (4.9). Now assess the solvability of the remaining terms in (4.12). The last two terms are solvable since

$$\begin{aligned} & q_T \langle \hat{Z}_\theta, \mathbf{M}\hat{Z}_k - c_1\mathbf{M}\hat{Z}_\omega \rangle + \Omega_X \langle \hat{Z}_\theta, \mathbf{J}\hat{Z}_\omega - c_1\mathbf{M}\hat{Z}_\omega \rangle \\ & = -q_T(\mathcal{A}_k - c_1\mathcal{A}_\omega) - \Omega_X(\mathcal{B}_\omega - c_1\mathcal{A}_\omega) = -(q_T + \Omega_X)(\mathcal{A}_k - \mathcal{A}_\omega) = 0. \end{aligned}$$

This solvability condition ensures the existence of χ so that

$$\mathbf{L}\chi = \mathbf{M}\hat{Z}_k - c_1\mathbf{M}\hat{Z}_\omega + \mathbf{J}\hat{Z}_\omega - c_2\mathbf{M}\hat{Z}_\omega,$$

since $q_T = \Omega_X$. The p_X and q_Y terms in (4.12) are solvable precisely when

$$\begin{aligned} 0 = & p_X \langle \hat{Z}_\theta, (\mathbf{J} - c_1\mathbf{M})(\hat{Z}_\ell - c_2\hat{Z}_\omega) \rangle + q_Y \langle \hat{Z}_\theta, (\mathbf{K} - c_2\mathbf{M})(\hat{Z}_k - c_1\hat{Z}_\omega) \rangle \\ = & -p_X(\mathcal{B}_\ell - c_1\mathcal{A}_\ell - c_2\mathcal{B}_\omega + c_1c_2\mathcal{A}_\omega) - q_Y(\mathcal{C}_k - c_2\mathcal{A}_k - c_1\mathcal{C}_\omega + c_1c_2\mathcal{A}_\omega) \\ = & -(p_X + q_Y) \frac{\mathcal{A}_\omega \mathcal{C}_k - \mathcal{A}_k \mathcal{A}_\ell}{\mathcal{A}_\omega} \\ = & -(p_X + q_Y) \frac{\Delta_Y}{\mathcal{A}_\omega}. \end{aligned}$$

This condition generates the assumption (1.6), and ensures the existence of σ ,

$$\mathbf{L}\sigma = (\mathbf{J} - c_1\mathbf{M})(\hat{Z}_\ell - c_2\hat{Z}_\omega) + (\mathbf{K} - c_2\mathbf{M})(\hat{Z}_k - c_1\hat{Z}_\omega).$$

Hence, the general fourth-order solution, modulo elements in the kernel of $\mathbf{L}$, is

$$W_4 = \phi q_X(\Gamma_3)_\theta + q_{XX}\Gamma_4 + q_T\chi + q_Y\sigma.$$

After simplification, the fifth-order terms are

$$\begin{aligned}
 \mathbf{L}\tilde{W}_5 = & \Omega_T \mathbf{M} \hat{Z}_\omega + q_{XT} (\mathbf{M} \Gamma_3 + (\mathbf{J} - c_1 \mathbf{M}) \chi) \\
 & + qq_X ((\mathbf{J} - c_1 \mathbf{M}) (\Gamma_3)_\theta + (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_{kk} - 2c_1 \hat{Z}_{\omega k} + c_1^2 \hat{Z}_{\omega\omega}) \\
 & - D^3 S(\hat{Z})(\hat{Z}_k - c \hat{Z}_\omega, \Gamma_3)) + q_{XXX} (\mathbf{J} - c \mathbf{M}) \Gamma_4 \\
 & + p_T (\mathbf{M} \hat{Z}_\ell - c_2 \mathbf{M} \hat{Z}_\omega) + \Omega_Y (\mathbf{K} \hat{Z}_\omega - \mathbf{M} \hat{Z}_\omega) \\
 & + p_Y (\mathbf{K} - c_2 \mathbf{M}) (\hat{Z}_\ell - c_2 \hat{Z}_\omega) + q_{XY} ((\mathbf{J} - c_1 \mathbf{M}) \sigma + (\mathbf{K} - c_2 \mathbf{M}) \Gamma_3). \quad (4.13)
 \end{aligned}$$

The tilde indicates that W_5 incorporates all solvable terms at this order, and their explicit expression is not required at this stage since the analysis does not go beyond the fifth order.

Now proceed to assess solvability of (4.13). The first term gives

$$\Omega_T \langle \hat{Z}_\theta, \mathbf{M} \hat{Z}_\omega \rangle = -\mathcal{A}_\omega \Omega_T.$$

The coefficient of the q_{XT} term vanishes, since

$$\begin{aligned}
 \langle \hat{Z}_\theta, \mathbf{M} \Gamma_3 + (\mathbf{J} - c_1 \mathbf{M}) \chi \rangle &= -\langle \hat{Z}_\omega, (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_k - c_1 \hat{Z}_\omega) \rangle \\
 &\quad - \langle \hat{Z}_k - c_1 \hat{Z}_\omega, \mathbf{M} (\hat{Z}_k - c_1 \hat{Z}_\omega) + (\mathbf{J} - c_1 \mathbf{M}) \hat{Z}_\omega \rangle \\
 &= (1 - 1) \langle \hat{Z}_\omega, (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_k - c_1 \hat{Z}_\omega) \rangle = 0.
 \end{aligned}$$

Similarly, the coefficient of the q_{XY} term vanishes

$$\begin{aligned}
 &\langle \hat{Z}_\theta, (\mathbf{J} - c_1 \mathbf{M}) \sigma + (\mathbf{K} - c_2 \mathbf{M}) \Gamma_3 \rangle \\
 &= -\langle \hat{Z}_k - c_1 \hat{Z}_\omega, (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_\ell - c_2 \hat{Z}_\omega) + (\mathbf{K} - c_2 \mathbf{M}) (\hat{Z}_k - c_1 \hat{Z}_\omega) \rangle \\
 &\quad - \langle \hat{Z}_\ell - c_2 \hat{Z}_\omega, (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_k - c_1 \hat{Z}_\omega) \rangle \\
 &= (1 - 1) \langle \hat{Z}_k - c_1 \hat{Z}_\omega, (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_\ell - c_2 \hat{Z}_\omega) \rangle = 0.
 \end{aligned}$$

The coefficients of the p_T and Ω_Y terms also vanish by the choice of c_2 ,

$$\begin{aligned}
 r_T \langle \hat{Z}_\theta, \mathbf{M} \hat{Z}_\ell - c_2 \mathbf{M} \hat{Z}_\omega \rangle + \Omega_Y \langle \hat{Z}_\theta, \mathbf{K} \hat{Z}_\omega - \mathbf{M} \hat{Z}_\omega \rangle \\
 = -r_T (\mathcal{A}_\ell - c_2 \mathcal{A}_\omega) - \Omega_Y (\mathcal{C}_\omega - c_2 \mathcal{A}_\omega) \\
 = -(r_T + \Omega_Y) (\mathcal{A}_\ell - \mathcal{A}_\omega) = 0.
 \end{aligned}$$

The nonlinearity has the coefficient

$$\begin{aligned}
 &\langle \hat{Z}_\theta, (\mathbf{J} - c_1 \mathbf{M}) (\Gamma_3)_\theta + (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_{kk} - 2c_1 \hat{Z}_{\omega k} + c_1^2 \hat{Z}_{\omega\omega}) - D^3 S(\hat{Z})(\hat{Z}_k - c_1 \hat{Z}_\omega, \Gamma_3) \rangle \\
 &= \langle \Gamma_3, (\mathbf{J} - c_1 \mathbf{M}) \hat{Z}_{\theta\theta} - D^3 S(\hat{Z})(\hat{Z}_k - c_1 \hat{Z}_\omega, \hat{Z}_\theta) \rangle + \langle \hat{Z}_\theta, (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_{kk} - 2c_1 \hat{Z}_{\omega k} + c_1^2 \hat{Z}_{\omega\omega}) \rangle \\
 &= \langle (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_k - c_1 \hat{Z}_\omega), \hat{Z}_{\theta k} - c_1 \hat{Z}_{\theta\omega} \rangle + \langle \hat{Z}_\theta, (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_{kk} - 2c_1 \hat{Z}_{\omega k} + c_1^2 \hat{Z}_{\omega\omega}) \rangle \\
 &= (\partial_k - c_1 \partial_\omega) (\langle \hat{Z}_\theta, (\mathbf{J} - c_1 \mathbf{M}) (\hat{Z}_k - c_1 \hat{Z}_\omega) \rangle) = -(\partial_k - c_1 \partial_\omega)^2 (\mathcal{B} - c_1 \mathcal{A}).
 \end{aligned}$$

From this calculation, define

$$\kappa = (\partial_k - c_1 \partial_\omega)^2 (\mathcal{B} - c_1 \mathcal{A}),$$

where the ∂_k and ∂_ω derivatives are taken with c_1 fixed. The p_Y term generates the coefficient

$$\langle \hat{Z}_\theta, \mathbf{K} \hat{Z}_\ell - c_2 (\mathbf{K} \hat{Z}_\omega + \mathbf{M} \hat{Z}_\ell) + c_2^2 \mathbf{M} \hat{Z}_\omega \rangle = -\mathcal{C}_\ell + c_2 (\mathcal{A}_\ell + \mathcal{C}_\omega) - c_2^2 \mathcal{A}_\omega = -\frac{\Delta_T}{\mathcal{A}_\omega}.$$

The last term is familiar from the 1 + 1 case in [10] and it is defined as

$$\langle \hat{Z}_\theta, (\mathbf{J} - c \mathbf{M}) \Gamma_4 \rangle = -\mathcal{H}.$$

To summarize, solvability of the fifth-order equation (4.13) requires Ω, q, p to satisfy

$$\mathcal{A}_\omega \Omega_T + \kappa q q_X + \mathcal{K} q_{XXX} + \frac{\Delta_T}{\mathcal{A}_\omega} p_Y = 0,$$

which upon differentiation with respect to X gives

$$\mathcal{A}_\omega q_{TT} + \left(\frac{1}{2} \kappa q^2 + \mathcal{K} q_{XX} \right)_{XX} + \frac{\Delta_T}{\mathcal{A}_\omega} q_{YY} = 0. \quad (4.14)$$

For non-degeneracy of this equation, it is assumed that

$$\mathcal{A}_\omega \neq 0, \quad \kappa \neq 0, \quad \mathcal{K} \neq 0 \quad \text{and} \quad \Delta_T \neq 0. \quad (4.15)$$

The two-way Boussinesq equation (4.14) can be unfolded to the case where $\Delta_L = \mu \varepsilon^2$, where μ is of order 1 and $\text{sign}(\mu) = \text{sign}(\Delta_L)$, in order to investigate behaviour close to the Lighthill surface. The unfolded equation is

$$\mathcal{A}_\omega q_{TT} + \frac{\mu}{\mathcal{A}_\omega} q_{XX} + \left(\frac{1}{2} \kappa q^2 + \mathcal{K} q_{XX} \right)_{XX} + \frac{\Delta_T}{\mathcal{A}_\omega} q_{YY} = 0. \quad (4.16)$$

(iv) The curvature κ

An expression for the coefficient κ can be derived which emphasizes that it is geometrically a curvature function. Define

$$f(s) = \mathcal{B}(\omega - sc_1, k + s) - c_1 \mathcal{A}(\omega - sc_1, k + s).$$

Then

$$f'(s) = -c_1 \mathcal{B}_\omega + \mathcal{B}_k - c_1(-c_1 \mathcal{A}_\omega + \mathcal{A}_k)$$

and

$$f''(s) = \mathcal{B}_{kk} - c_1 \mathcal{A}_{kk} - 2c_1 \mathcal{B}_{\omega k} + 2c_1^2 \mathcal{A}_{\omega k} + c_1^2 (\mathcal{B}_{\omega\omega} - c_1 \mathcal{A}_{\omega\omega}).$$

Setting $s = 0$, we get that $\kappa = f''(0)$. Similarly, setting $s = 0$ in the first derivative,

$$f'(0) = \left\langle \begin{pmatrix} -c_1 \\ 1 \end{pmatrix}, \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix} \begin{pmatrix} -c_1 \\ 1 \end{pmatrix} \right\rangle = 0,$$

since $(-c_1 \ 1)^T$ is in the kernel of the Lighthill matrix (2.7). In terms of the canonical eigenvector (2.7), the function $f(s)$ can be interpreted as

$$f(s) = \langle \zeta, \mathbf{f}(\mathbf{w}) \rangle,$$

where here $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^2$, and

$$\mathbf{w} := \begin{pmatrix} \omega \\ k \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{w}) := \begin{pmatrix} \mathcal{A}(\omega, k) \\ \mathcal{B}(\omega, k) \end{pmatrix}$$

and so

$$\kappa := f''(0) = \frac{d^2}{ds^2} \langle \zeta, \mathbf{f}(\mathbf{w} + s\zeta) \rangle \Big|_{s=0}.$$

Hence κ is the curvature of the path through the shifted wave action

$$s \mapsto \mathcal{B}(\omega - sc_1, k + s) - c_1 \mathcal{A}(\omega - sc_1, k + s) := f(s).$$

5. Example: complex Klein–Gordon on Lighthill surfaces

Continuing the example of the nonlinear wave equation (3.1) in §3, consider now the case where the family of periodic travelling waves has the singularity $\Delta_L = 0$.

The matrix in the Lighthill determinant is

$$\begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{B}_\omega & \mathcal{B}_k \end{bmatrix} = \begin{bmatrix} g^{11}(1 + 3g^{11}\omega^2 + g^{22}k^2 + g^{33}\ell^2) & 2g^{11}g^{22}\omega k \\ 2g^{11}g^{22}\omega k & g^{22}(1 + g^{11}\omega^2 + 3g^{22}k^2 + g^{33}\ell^2) \end{bmatrix}.$$

Setting the determinant to zero gives

$$\Delta_L = g^{11}g^{22}|\Psi_0|^2(1 + 3g^{11}\omega^2 + 3g^{22}k^2 + g^{33}\ell^2).$$

The other two key determinants are

$$\Delta_Y = \det \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ \mathcal{C}_\omega & \mathcal{C}_k \end{bmatrix} = 2g^{11}g^{22}g^{33}\ell k |\Psi_0|^2$$

and

$$\Delta_T = \det \begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_\ell \\ \mathcal{C}_\omega & \mathcal{C}_\ell \end{bmatrix} = g^{11}g^{33}|\Psi_0|^2(g^{11} + (3g^{11} + g^{22})k^2 + \ell(g^{11} + g^{33})\ell^2).$$

Since $g^{11}g^{22}g^{33}|\Psi_0|^2 \neq 0$, the only way that the condition $\Delta_Y = 0$ can be satisfied is by taking $k\ell = 0$. Assume $k \neq 0$ (or the wave trivializes in the x -direction), then

$$\Delta_Y = 0 \quad \Leftrightarrow \quad \ell = 0,$$

giving $S^Y = \{(\omega, k, \ell) \mid \ell = 0\}$. With $\ell = 0$ the set where $\Delta_L = 0$ reduces to

$$S^L = \{(\omega, k, \ell) : 1 + 3g^{11}\omega^2 + 3g^{22}k^2 = 0\}.$$

Now restrict attention to the canonical metric (3.2). Then the set where the two-way Boussinesq equation can emerge is

$$S^L \cap S^Y \cap U = \{(\omega, k, \ell) \in U : 1 - 3\omega^2 + 3k^2 = 0 \text{ and } \ell = 0\}, \quad (5.1)$$

with U defined in (3.6). This set is a hyperbola in the (ω, k) plane.

Now compute the coefficients in the two-way Boussinesq equation (4.14). The coefficient κ is computed as follows:

$$\kappa = (\mathcal{B}_{kk} - c_1\mathcal{A}_{kk}) - 2c_1(\mathcal{B}_{\omega k} - c_1\mathcal{A}_{\omega k}) + c_1^2(\mathcal{B}_{\omega\omega} - c_1\mathcal{A}_{\omega\omega}).$$

Computing

$$c_1 = \frac{\mathcal{A}_k}{\mathcal{A}_\omega} \Big|_{S^L \cap S^Y} = -\frac{\omega}{k}$$

and

$$\begin{aligned} [\mathcal{B}_{\omega\omega} - c_1\mathcal{A}_{\omega\omega}]|_{S^L \cap S^Y} &= 4k + \frac{2}{k}, \\ [\mathcal{B}_{\omega k} - c_1\mathcal{A}_{\omega k}]|_{S^L \cap S^Y} &= -4\omega, \\ [\mathcal{B}_{kk} - c_1\mathcal{A}_{kk}]|_{S^L \cap S^Y} &= 4k - \frac{3}{2k}. \end{aligned}$$

Combining gives

$$\kappa = \frac{2}{3k^3}.$$

Similarly, $\mathcal{A}_\omega|_{S^L \cap S^Y} = 2k^2$ and the coefficient of q_{YY} is

$$\frac{\Delta_T}{\mathcal{A}_\omega} \Big|_{S^L \cap S^Y} = \frac{(2/3)(1 + 2k^2)}{2k^2} = \frac{1 + 2k^2}{3k^2}.$$

It remains to compute the coefficient of dispersion. Normally, the multi-symplectification of (3.3) to the form (4.1) would be required to set up the Jordan chain. However, since the Lighthill surface has been reduced to the case of $\ell = 0$, the coefficient of dispersion is the same as the $1 + 1$ case treated in [10],

$$\mathcal{K} = -\frac{1}{18k^4}.$$

Hence, the emergent two-way Boussinesq equation for $(\omega, k, \ell) \in S^L \cap S^Y$ is, therefore,

$$2k^2 q_{TT} + \frac{2}{3k^3} (qq_X)_X - \frac{1}{18k^4} q_{XXXX} + \frac{1+2k^2}{3k^2} q_{YY} = 0. \quad (5.2)$$

The importance of the assumption $k \neq 0$ in U (3.6) is evident here. The resulting Boussinesq equation is linearly ill-posed in the X -direction since the coefficients of q_{TT} and q_{XXXX} have opposite sign, and it is linearly ill-posed in the Y -direction since the coefficients of q_{TT} and q_{YY} have the same sign.

Unfolding (letting Δ_L be of order ε^2) and scaling the coefficients leads to the following canonical form for the emergent $2 + 1$ two-way Boussinesq equation

$$u_{\tau\tau} + s_1 u_{\xi\xi} + \left(\frac{1}{2}u^2\right)_{\xi\xi} - u_{\xi\xi\xi\xi} + u_{\eta\eta} = 0, \quad s_1 = \pm 1,$$

where $s_1 = -1$ ($s_1 = +1$) on the stable (unstable) side of the Lighthill curve (5.1).

To summarize, the CKG equation (3.1) has a family of exact periodic travelling waves. Modulation of these travelling waves in the neighbourhood of the Lighthill surfaces (5.1) leads to a reduction to the $2 + 1$ two-way Boussinesq equation (5.2). The reduced equation contains a range of bounded periodic, quasi-periodic and localized solutions, but it also portends more dramatic behaviour in the original CKG equation in that it is linearly ill-posed in both the X - and Y -directions, and so general initial data may be dramatically unstable.

6. Concluding remarks

In this paper, it has been shown that the classical $2 + 1$ Whitham modulation equations break down when the Lighthill determinant vanishes. This vanishing determinant is called a co-dimension 1 singularity as it requires variation of only one parameter to find such singularities. Another way to view the co-dimension 1 nature is to note that setting $\Delta_L = 0$ in the three-dimensional space with coordinates (ω, k, ℓ) generates a surface with one-dimensional normal space.

On the other hand, it was found that the two-way Boussinesq equation required the additional assumption that $\Delta_Y = 0$, suggesting that the emergence of a two-way Boussinesq equation is co-dimension 2 in the $2 + 1$ case. However, in *symmetric* systems it remains co-dimension 1. A family of periodic travelling waves is transverse symmetric when

$$\mathcal{A}(\omega, k, -\ell) = \mathcal{A}(\omega, k, \ell), \quad \mathcal{B}(\omega, k, -\ell) = \mathcal{B}(\omega, k, \ell) \quad \text{and} \quad \mathcal{C}(\omega, k, -\ell) = -\mathcal{C}(\omega, k, \ell).$$

With this property it is immediate that $\Delta_Y = 0$ on $\ell = 0$. The transverse symmetry property follows when the governing PDE has a reflection symmetry in the y -direction.

In this paper it was assumed that the basic state had only one phase. There will be many more examples of this singularity in the multi-phase case. Some progress in analysing singularities in the multi-phase case is in [12,13], but a theory generalizing Lighthill surfaces to the multi-phase case is an open problem.

In the emergent equation (4.14), the non-degeneracy conditions are (4.15). When these conditions fail, then new scaling is required and higher order terms appear. This programme has

been carried out for breakdown of the Korteweg–de Vries equation in Ratliff [13]. For example, when $\mathcal{A}_\omega = 0$ it is expected that a higher order time derivative will emerge, when $\kappa = 0$ a higher order nonlinearity is expected, and when $\mathcal{K} = 0$ a sixth-order derivative is expected.

In this paper X has been a preferred direction. However, an analogous theory can be developed with Y or T as the preferred direction. A modulation equation of the form (4.14) will emerge but with X , T and Y permuted and different expressions for the coefficients.

The singularity $\Delta_M = 0$ with $\Delta_L \Delta_Y \neq 0$ can be treated in the same way as in this paper and it is expected to generate an analogous $2 + 1$ two-way Boussinesq equation but with the fourth derivative and nonlinearity in the $\hat{\eta}$ direction and with different coefficients.

The interest in reducing a general PDE to the $2 + 1$ two-way Boussinesq equation is that this reduced equation can be analysed in great detail. It has a range of multi-pulse, multi-periodic, solitonic and more complex solutions. Some results on the solutions of two-way Boussinesq equations in general can be found in [6–9].

The theory in this paper is asymptotically valid, in the sense that the ansatz satisfies the abstract governing equations exactly up to fifth order. Rigorous validity is generally approached by comparing solutions of the full problem with projected solutions of the reduced problem in an appropriate norm for an asymptotically large time interval. For conservative PDEs there are very few results of this type. A canonical example of a successful validity proof of the reduction from one conservative PDE to another is the reduction of defocusing a nonlinear Schrödinger equation to the Korteweg–de Vries equation in [14]. However, a validity theory for reduction from an abstract Lagrangian, or even a reduction from the particular wave equation in §3, to the $2 + 1$ two-way Boussinesq equation as in this paper is an open problem.

Data accessibility. Details of relevant data associated with the publishing and accessibility can be found on the University of Surrey publications repository: <http://epubs.surrey.ac.uk>.

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