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Relationships between Borda voting and Zermelo ranking

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Abstract

There are many situations wherein a group of individuals (e.g., voters, experts, sports writers) must produce an ordered list of ‘best’ alternatives selected from a given group of alternatives (e.g., candidates, proposals, sports teams). Two long established mechanisms that have been used for this task are ‘Zermelo’s Ranking Method’ (1929) and ‘Borda’s Voting Scheme’ (1781). The main purpose of this paper is to point out that they are, under certain common circumstances, identical. We then show that Zermelo’s Method can be used in situations that Borda’s Method is not designed to handle.

1 Introduction

Let A denote a finite set of ‘alternatives’ (e.g., candidates, proposals, sports teams), and let V denote a finite set of individuals (e.g., voters, experts, sports writers). The individuals are collectively charged to somehow express their opinions (submit some type of ‘ballot’), and, via some mechanism, come up with a ranked list (from ‘most preferred’ to ‘least preferred’) of some (or all) of the alternatives in A . This is a common type of ‘group decision problem’.

Two general types of mechanisms that can be used to ‘solve’ this problem are ‘Voting Methods’ and ‘Methods of Paired Comparisons’.

Voting Methods require a ‘ballot’ from each voter in V in order to determine a group choice from a set A of alternatives (see, e.g., Anderson 1994a; Brams and Fishburn 1991; Fishburn 1977; Levin and Nalebuff 1995; Niemi and Riker 1976; Saari 2000; Straffin 1980). Many of these methods are called “ranked ballot methods” in that the ballots they use are ordinal *rankings* of the alternatives. Ties in these rankings are sometimes allowed, but abstentions

are very rarely allowed. In addition to possibly using different types of ballots, various voting methods differ in how they determine a winner from the voter's ballots, and they can be compared to each other according to whether or not they satisfy selected properties.

Ranked ballots are required to be *transitive* (i.e., if a ranked ballot prefers alternative i to alternative j , and prefers alternative j to alternative k , then it necessarily prefers alternative i to alternative k). For those ranked-ballot voting methods that allow ballots with unranked alternatives, the alternatives that are not explicitly ranked on that ballot are practically always considered to be inferior to all of the alternatives that are explicitly ranked on that ballot.

The French mathematician Jean Charles de Borda (1781) suggested that the alternatives should be ranked according to the number of 'points' that they earn, where points are awarded to an alternative, a , according to the number of alternatives that are collectively ranked below a (or, if allowed, tied with a) by the voters. We describe Borda's voting method more precisely in Sect. 2.

Below, we assume that the alternatives are numbered from 1 through $|A|$, where $|A|$ is the number of alternatives in A . Then Methods of Paired Comparisons (see, e.g., Anderson 1994b or David 1988) require each voter to submit, as a ballot, a *matrix* of size $|A|$ by $|A|$ whose i - j th entry (when $i \neq j$) is intended to represent a voter's opinion about how alternative i compares with alternative j . Typically, a one in entry i - j indicates that the matrix-ballot prefers i to j , a one-half in entry i - j indicates an explicit tie, and a zero in that entry indicates that the matrix-ballot prefers alternative j to i .

Furthermore, unlike voting with ranked ballots, transitivity is not generally assumed with paired comparisons (checking a matrix for transitivity is tedious, although Gass (1998) suggests a complete enumeration scheme for this).

One particularly important Method of Paired Comparisons was introduced by the German set-theorist Ernst Zermelo (1929), who suggested that the alternatives be ranked according to a maximum likelihood function based on the collective pair-wise scores submitted by the voters. We review Zermelo's Method in Sect. 3.

Note that the inputs required for Zermelo's Method (matrices of paired comparisons) are quite different from the inputs required for Borda's Method (ranked ballots). Ranked ballots are simpler to cast than matrix ballots (which can be very large if $|A|$ is large). But because they contain more information than ranked ballots, matrix ballots might more accurately reflect the voter's opinions on the alternatives.

In Sect. 4, we show that the methods of Borda and Zermelo yield identical rankings if the ranked ballots are complete and the matrix ballots are consistent with them.

Throughout, for any matrix S , we let $S(i, j)$ denote the i - j th component of S .

2 Borda's method

A voter's input to Borda's Method (i.e., a 'ranked ballot') consists of an ordered list of a subset B of the alternatives in A , with either a symbol like " $>$ " (is preferred to) or a symbol like " $=$ " (is tied with) separating the alternatives in the list. In this paper, we do not use the symbol " $<$ " when comparing alternatives. Also, no alternative is allowed to be listed twice (or more) on any ranked ballot considered here. Hence, all ranked ballots considered below are transitive, and so any matrix ballot that corresponds to a ranked ballot is also transitive. However, a matrix ballot that does not correspond to a ranked ballot need not be transitive.

On a ranked ballot, only a subset, say B , of the alternatives in A need to be ranked, and any alternative that is not ranked (i.e., is in the set $A-B$) is considered inferior to all alternatives in B on that ballot.

The alternatives are scored by awarding 'points' according to the following rules:

For each ballot, Borda suggested that each alternative ranked on that ballot be given one point for each alternative ranked below it on that ballot.¹ Also, if ties are allowed and alternative a is tied with exactly k other alternatives on a ballot, then a is given $1/2$ of a point for each of those k alternatives, for a total of $k/2$ points for those tied alternatives on that ballot.

The score for alternative a is found by summing (over all ballots) the number of points a earns, and the ranking is determined by these scores.

3 A method of paired comparisons

Methods of Paired Comparisons are schemes designed to establish a ranking (or a partial ranking) of the elements of a set, A , based on comparisons between the elements in some (or all) of the pairs of elements of A . There are a number of such methods (see Anderson 1994b or David 1988 for examples).

One of the most popular of these methods is originally due to Zermelo (1929), and, as noted by Stob (1984), has been rediscovered by others (Bradley and Terry 1952; Ford 1957). Originally it was applied to complete tournaments, but, more recently, it was rediscovered by Jech (1983) and applied to incomplete tournaments (see Stob 1985). While it often is termed the 'Bradley/Terry' method, we refer to it here as Zermelo's Method.

The input to Zermelo's Method is an $|A|$ by $|A|$ matrix R (called a 'result' or 'outcome' matrix). R is used to calculate a 'match' matrix M by setting $M(i, j) = R(i, j) + R(j, i)$ for all i, j . (The diagonals of these matrices have no meaning.)

¹There are numerous other 'point scoring schemes' Voters comparing sports teams usually give their best team n points, their second best team $n-1$ points, their third best team $n-2$ points, and so on down to giving their n th best team 1 point, and giving all other teams no points. However, voters judging individual performances usually give between 0 and n points to each performer. Judges in boxing are usually required to give the winner of each round n points for that round, for some fixed value of n , and give the loser of the round $n-x$ points for some x less than n . See, e.g., Fishburn and Gehrlein (1976); or Saari (2000) for further discussion on point-count schemes). In this note we are addressing Borda's $|A|-1, |A|-2, \dots, 1, 0$ point-count method.

Roughly speaking, the i - j th entry (i, j) of the match matrix M records the number of times alternative i is compared to alternative j (in the sports analogy, how many times team i has played team j), and the result matrix R summarizes the outcomes of these comparisons (in a team sports setting, $R(i, j)$ is the number of wins by team i over team j , plus 1/2 of the number of ties between team i and team j , out of the M_{ij} times that they played each other).

We will assume that R is *irreducible*, which means that, for every ordered pair i, j ($i \neq j$), there is a t and a sequence $R(i = j_1, j_2), R(j_2, j_3), \dots, R(j_{t-1}, j_t = j)$ of positive results connecting that pair. This insures that i and j can be compared. Irreducibility is easy to check on a computer by summing the first $|A| - 1$ powers of R , in which case R is irreducible if and only if all off diagonal entries of this sum are positive. If R is not irreducible, Conner and Grant (1999) present an ad hoc generalization that can, in some cases, compare them.

Zermelo (1929) postulated that each alternative i in A has some fixed ‘strength’ (or ‘rating’)

w_i . These ‘strengths’ should be such that the ratios $\frac{w_i}{w_i + w_j}$ represent probabilities, say p_{ij} , that an individual randomly-selected voter would rank i above j based on the data in M and R . Given a set of such ‘strengths’, then, if there are no ties in the data, the *likelihood* that the resulting probabilities are consistent with the results given by R is

$$L_1(p) = \prod_{i=2}^{|A|} \prod_{j=1}^{i-1} \left(\frac{M(i, j)}{R(i, j)} \right) [p_{ij}]^{R(i, j)} [p_{ji}]^{R(j, i)}$$

or, in terms of the strengths,

$$L_2(w) = \prod_{j=2}^{|A|} \prod_{i=1}^{j-1} \left(\frac{M(i, j)}{R(i, j)} \right) \left[\frac{w_i}{w_i + w_j} \right]^{R(i, j)} \left[\frac{w_j}{w_i + w_j} \right]^{R(j, i)}, \quad (1)$$

where

$$\left(\frac{M(i, j)}{R(i, j)} \right) = \frac{M(i, j)!}{R(i, j)! R(j, i)!}.$$

Factoring out the constant term $\prod_{j=2}^{|A|} \prod_{i=1}^{j-1} \left(\frac{M(i, j)}{R(i, j)} \right)$ this leads naturally to the optimization problem:

$$\max_{w > 0} \prod_{j=2}^{|A|} \prod_{i=1}^{j-1} \left[\frac{w_i}{w_i + w_j} \right]^{R(i, j)} \left[\frac{w_j}{w_i + w_j} \right]^{R(j, i)}. \quad (2)$$

Note that the constraints $w_j > 0$ can be imposed since the fractions $\frac{w_i}{w_i + w_j}$ are invariant with respect to multiplication. Keener (1993) shows that this problem has a unique positive solution when R is irreducible. These strengths are then used to establish a consensus ranking of the alternatives.

To allow ties in the data, write $R(i, j)$ as $W(i, j) + T(i, j)/2$, where $W(i, j)$ is the number of Zermelo ballots that rank i ahead of j , and $T(i, j)$ is the number of Zermelo ballots on which i and j are tied. Then the optimization problem given by (1 and 2) above becomes:

$$\max_{w>0} \prod_{j=2}^{|A|} \prod_{i=1}^{j-1} \left(\frac{M(i, j) - T(i, j)}{W(i, j)} \right) \left[\frac{w_i}{w_i + w_j} \right]^{W(i, j)} \left[\frac{w_j}{w_i + w_j} \right]^{W(j, i)}, \quad (3)$$

which, after factoring out the constant term, is identical to (2).

It is well-known (see, e.g., Anderson 1994b or Jech 1983), or the proof of the theorem in the next section) that Zermelo's probabilities have the property that the *expected* number of

wins by k (plus ties divided by 2) according to these probabilities, $\sum_{i=1, i \neq k}^{|A|} M(i, k) p_{ik}$, equals the *actual* count of the number of wins by k (plus ties divided by 2),

$$\sum_{\nu \in V} \sum_{i=1, i \neq k}^{|A|} \tilde{R}_{\nu}(k, i), \text{ in the data upon which the results are based.}$$

4 A connection between the methods of Zermelo and Borda

One obvious difference between the two methods is that the ballots (inputs) are different. Borda uses (possibly truncated) rankings (with ties allowed) of the alternatives in A , while Zermelo requires a paired comparison matrix with positive entries for many (but possibly not all) pairs of alternatives. Given any ranked ballot from a voter ν , one can always easily construct a unique matrix ballot that corresponds to it by setting $\tilde{R}_{\nu}(i, j) = 1$ whenever i is rated above j on the ranked ballot, setting $\tilde{R}_{\nu}(i, j) = 1/2$ when i and j are tied on the ranked ballot, and setting $\tilde{R}_{\nu}(i, j) = 0$ otherwise.

The opposite is not true, however. Zermelo's ballots are distinctly more general. For example, suppose a voter feels that $a > b$ and $c > d$, but does not have enough information to compare either a or b to either c or d . This is easy to express using a matrix ballot, but it cannot be expressed with a ranked ballot.

While Borda's Method dates back to 1781 and Zermelo's Method to 1929, it seems to have gone unnoticed that, under certain reasonable circumstances, these methods produce the same ranking. Note that the double sum

$$B_k = \sum_{\nu \in V} \sum_{\substack{i=1 \\ i \neq k}}^{|A|} \bar{R}_\nu(k, i)$$

gives the Borda score for alternative k , and the term-by-term matrix sum $R = \sum_{\nu \in V} \tilde{R}_\nu$ gives the corresponding result matrix, which is used as the input matrix for Zermelo's Method.

The main result of this paper follows. (Recall that a 'ranked ballot' for Borda's Method treats all un-ranked alternatives as being tied with each other, and being inferior to all explicitly ranked alternatives.)

Theorem

If ranked ballots are used as the inputs to Borda's Method as described above, and the sum of the corresponding matrix ballots is used as the input to Zermelo's Method, then these methods will necessarily produce the same ranking of the alternatives.

Proof—Taking logs of the objective function of (2) to make the differentiation easier, the optimization problem of Zermelo's Method is equivalent to:

$$\max F(w) = \sum_{i=1}^{|A|-1} \sum_{j=1+1}^{|A|} (R(i, j) [\ln(w_i) - \ln(w_i + w_j)] + R(j, i) [\ln(w_j) - \ln(w_i + w_j)]).$$

At an optimal solution, the derivatives of F must vanish. Therefore, we need to solve

$$\frac{\partial F}{\partial w_k} = \sum_{\substack{i=1 \\ i \neq k}}^{|A|} \left(R(k, i) \left(\frac{1}{w_k} - \frac{1}{w_i + w_k} \right) + R(i, k) \left(-\frac{1}{w_i + w_k} \right) \right) = \sum_{\substack{i=1 \\ i \neq k}}^{|A|} \left(\frac{1}{w_k} R(k, i) - \frac{1}{w_i + w_k} (R(k, i) + R(i, k)) \right) = 0.$$

This occurs when

$$\frac{1}{w_k} \sum_{i=1}^{|A|} R(k, i) = \sum_{\substack{i=1 \\ i \neq k}}^{|A|} \frac{1}{w_i + w_k} (R(i, k) + R(k, i)),$$

so it occurs when

$$\sum_{\substack{i, k \\ i \neq k}} R(k, i) = \sum_{\substack{i, k \\ i \neq k}} \left(\frac{w_k}{w_i + w_k} (R(i, k) + R(k, i)) \right).$$

The left-hand side here is the Borda score B_k for alternative k . This score is a count of the number of voters who ranked k above (plus 1/2 tied with) the other alternatives. On the right-hand side, in the inner pair of parentheses, is the i - k th entry of the match matrix M . Since $w_k/(w_k + w_i)$ is the postulated probability, p_{ki} , that alternative k beats alternative i , the expected number of wins by with these probabilities equals the actual number. Thus, we have

$$\sum_{\substack{i=1 \\ i \neq k}}^{|A|} \left(M(i, k) \frac{w_k}{w_i + w_k} \right) = B_k. \quad (4)$$

Since we do not allow abstentions here, the number of matches involving any pair of alternatives is equal to the number of voters ($|V|$). Thus,

$$|V| \sum_{i \neq k} \left(\frac{w_k}{w_i + w_k} \right) = B_k,$$

or

$$\sum_{i \neq k} \left(\frac{w_k}{w_i + w_k} \right) = B_k / |V|.$$

Now let $b_k = B_k / |V|$ for all relevant k . Then, for any pair k and $\bar{k}$, where $k \neq \bar{k}$,

$$\begin{aligned} b_k - b_{\bar{k}} &= \sum_{i \neq k} \left(\frac{w_k}{w_i + w_k} \right) \\ &\quad - \sum_{i \neq \bar{k}} \left(\frac{w_{\bar{k}}}{w_i + w_{\bar{k}}} \right) \\ &= \left(\sum_{i \neq k, \bar{k}} \left(\frac{w_k}{w_i + w_k} - \frac{w_{\bar{k}}}{w_i + w_{\bar{k}}} \right) \right) \\ &\quad + \left(\frac{w_k}{w_{\bar{k}} + w_k} - \frac{w_{\bar{k}}}{w_k + w_{\bar{k}}} \right) = \left(\sum_{i \neq k, \bar{k}} \left(\frac{w_i(w_k - w_{\bar{k}})}{(w_i + w_k)(w_i + w_{\bar{k}})} \right) \right) \\ &\quad + \left(\frac{w_k - w_{\bar{k}}}{w_{\bar{k}} + w_k} \right) \\ &= \left(\sum_{i \neq k, \bar{k}} \left(\frac{w_i}{(w_i + w_k)(w_i + w_{\bar{k}})} \right) + \frac{1}{w_{\bar{k}} + w_k} \right) (w_k - w_{\bar{k}}) \end{aligned}$$

which implies that $b_k > b_{\bar{k}}$ if and only if $w_k > w_{\bar{k}}$. Thus, we see that Borda ranks k higher than $\bar{k}$ if and only if Zermelo does the same.

5 Two applications

Two applications for this theorem are the following.

First, and more obvious, Borda's method might be being used in certain voting situations, but abstentions are not allowed in those situations because it is not known how to deal with them in the proper manner. Now abstentions can consistently be allowed, if desired, in those voting situations.

Second, and less obvious, the organizers of a voting situation could be quite satisfied using ranked ballots and Borda voting, and might not want to allow abstentions, no matter how consistent the extension to abstentions was. Further, they may also be quite satisfied with giving each alternative $|A| - 1$ points for each ballot that uniquely ranks that alternative in first place, giving $|A| - 2$ points for each unique second place ranking, and so on down to giving 1 point for each unique next-to-last place ranking, giving no points for a unique last-place ranking, and giving $|A| - i - \frac{k}{2}$ points for each $k + 1$ -way tie for i th place through $i + k$ th place. But suppose that they are regularly asked: "Why $|A| - 1$ points for first, $|A| - 2$ for second, and so on? Why not $|A|^2$ points for each first, $(|A| - 1)^2$ for each second, and so on? Or, why not 5 points for each first, 3 points for each second, 1 point for each third, and no points for being ranked below third on a ballot? Or why not any of the infinitely many other ways of assigning points to alternatives based on ranked ballots?"

An answer here is that the assignment of $|A| - 1$ points for each unique first, $|A| - 2$ points for each unique second, and so on, gives the same ranking as the more sophisticated Zermelo's method produces when there are no abstentions, but is easier to compute and understand than Zermelo's method.

6 Some implications

Since the two methods are identical under the circumstances described above, any argument for (or against) one of these methods is also an argument for (or against) the other under those circumstances. However, Zermelo's ballots are strictly more general than Borda's ballots in that Zermelo's Method applies everywhere that Borda's method applies, but not vice-versa.

There is a large literature addressing the properties of Borda's Method and we will briefly address a small portion of it here.

All voting methods are faulty in some judgmental sense, and Borda's Method is no exception. In particular, it may fail to rank an alternative in first place when that alternative is ranked first by a landslide majority of the voters.

Also, it has been criticized as being vulnerable to manipulation. While no voting method is immune from manipulation, some (such as STV) are perceived to be less so in the sense that the problem of determining the preferences of other voters is NP-complete (see Bartholdi and Orlin 1991; Nitzan 1985). To see why Borda's method is criticized in this regard, consider the following property that any given ranked voting method may or may not have.

Strongly Manipulable

Let B_1 be a set of ranked ballots on which alternative z is ranked in last place, behind every other alternative, by every voter. Let W_1 be the set of winning alternatives that a particular voting method selects given those ballots. Now suppose that the selection process were to be re-held with a minority of the voters moving z up one place (ahead of exactly one other alternative) on their ballots, and with no other changes being made. Let B_2 be the set of these (some revised but most unchanged) ballots, and let W_2 be the set of winning alternatives that the voting method selects given the ballots B_2 . Then the voting method in question is said to be *strongly manipulable* if there exist sets of ballots B_1 and B_2 as described such that $W_1 \cap W_2$ is empty.

The following example shows that Borda's Method, and so Zermelo's also (when Zermelo is given matrix ballots that correspond to ranked ballots), are strongly manipulable according to this definition. Consider 3 alternatives— a , b , and z —and 5 voters whose original ballots are $a > b > z$ for 3 voters, and are $b > a > z$ for 2 voters. Then a is the unique winner according to Borda and Zermelo, and to many other voting methods. Now suppose that the two voters who did not rank a first move z up one place, so that the five ballots now are $a > b > z$ for 3 voters and are $b > z > a$ for 2 voters. Then b becomes the unique winner according to Borda and Zermelo, even though a still receives a majority of the first place votes. See Fishburn (1974a,b) for further discussion of this property.

However, Borda's Method does enjoy (at least) four properties that many other voting methods do not. Accordingly, Zermelo's Method also enjoys these properties if no ballots are cast with abstentions. The purpose of this section is to point out that Zermelo's Method does *not* necessarily satisfy these properties when some ballots are cast with abstentions. That is, Zermelo's method allows abstentions, but at the cost of not satisfying these four properties. On the other hand, if there are no abstentions, then we have shown that Zermelo's method just reproduces Borda's results.

These four properties have been called (see, e.g., Young 1974, 1975): positive involvement, negative involvement, strong participation, and multi-district consistency.

Positive Involvement

A ranked voting method has *positive involvement* if the following property holds. Given a set of ranked ballots, suppose that the voting method in question selects alternative a as the winner. Suppose that the selection process were to be re-held with additional voters, all of whom rank a as their unique first choice. Then a must still be selected if none of the original voters change their ballots, no matter how the additional voters rank the alternatives below a on their ballots.

Negative Involvement

A ranked voting method satisfies *negative involvement* if the following property holds. Given a set of ranked ballots, suppose that the voting method in question does not select alternative b as a winner. Suppose that the selection process were to be re-held with additional voters, all of whom rank all the alternatives, and all of whom rank b as their

unique last choice. Then b must still not be selected if none of the original voters change their ballots, no matter how the additional voters rank the alternatives above b on their ballots.

Strong Participation

A ranked voting method has *strong participation* if the following property holds. Given a set of ranked ballots, suppose that the voting method in question selects alternative a as the winner. Suppose that the selection process were to be re-held with additional voters, all of whom rank a ahead of alternative b . Then b must not be selected if none of the original voters change their ballots, no matter how the additional voters rank the other alternatives on their ballots.

A voting method that has strong participation necessarily has both positive involvement and negative involvement, but not vice versa.

Multi-district consistency

A voting method satisfies the property of multi-district consistency if the following holds. Suppose the voters can be divided into two groups, or panels, in such a manner that, considering the ballots of each panel separately, the method would declare some alternative, say a , to be winner for each panel. Then, if the ballots are not changed, the method must declare a to be the winner if the two groups are combined into a single panel.

Zermelo's Method satisfies neither positive involvement nor strong participation, as the following example illustrates.

Consider a voting situation involving 5 alternatives— a , b , c , d , and e —and 116 voters who cast the following ballots: 89 voters cast $a > c$. 8 cast $b > d$. 10 cast $c > a$. 1 casts $c > e > a$. 6 cast $d > b$. 1 casts $d > b > c$. And 1 casts $d > e > b$. Note that each voter here casts a ballot with abstentions. Given these ballots, Zermelo's Method selects the ranking $d > a > b > e > c$, and so it declares d to be the winner. Now suppose a new voter is added who casts the ballot: $d > a > c > b > e$. Then Zermelo's Method selects the ranking $a > d > b > c > e$, and so it now says that a is the winner.

Assume that all of the voting methods being considered satisfy unanimity (that is, if all of the voters rank an alternative as their unique first choice, that alternative must be the unique winner). Then positive involvement is a special case of multi-district consistency in that any example showing that a voting method fails positive involvement also shows that it fails multi-district consistency. The example just above does this here.

Finally, Zermelo's Method can violate *negative involvement*.

Given the ballots of the 116 voters in the example above, d is the winner, and so a is a loser. If a voter who casts the ballot $c > e > b > d > a$ is added, Zermelo's Method will produce the ranking $a > e > b = d > c$, thus changing that new voter's last place alternative, a , from being a loser into being the unique winner.

At the same time, it is not clear that these four properties are as important as they first might seem. Few who follow team sports would seriously argue that adding a set of extra games to a season should necessarily leave joint winners the same. If this is not expected in sports, where intransitivities are obvious, why should it be required in voting situations, where intransitive relationships also occur?

Due to these properties, we do not recommend the use of Zermelo's Method in partisan political contests, where each voter is primarily interested in selecting alternatives that are in the best interest of *that* voter. Rather, if it is at all appropriate, it might be so for situations wherein the voters agree on the attributes desired of the alternatives, and are trying to judge which alternatives have the best mix of those attributes. The ranking of research proposals by independent, unbiased evaluators is a likely example of such a case.

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