Form finding in elastic gridshells

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Elastic gridshells comprise an initially planar network of elastic rods that are actuated into a shell-like structure by loading their extremities. The resulting actuated form derives from the elastic buckling of the rods subjected to inextensibility. We study elastic gridshells with a focus on the rational design of the final shapes. Our precision desktop experiments exhibit complex geometries, even from seemingly simple initial configurations and actuation processes. The numerical simulations capture this nonintuitive behavior with excellent quantitative agreement, allowing for an exploration of parameter space that reveals multistable states. We then turn to the theory of smooth Chebyshev nets to address the inverse design of hemispherical elastic gridshells. The results suggest that rod inextensibility, not elastic response, dictates the zeroth-order shape of an actuated elastic gridshell. As it turns out, this is the shape of a common household strainer. Therefore, the geometry of Chebyshev nets can be further used to understand elastic gridshells. In particular, we introduce a way to quantify the intrinsic shape of the empty, but enclosed regions, which we then use to rationalize the nonlocal deformation of elastic gridshells to point loading. This justifies the observed difficulty in form finding. Nevertheless, we close with an exploration of concatenating multiple elastic gridshell building blocks.

elastic structures | gridshells | buckling | Chebyshev nets | mechanical instabilities

Camping tents create curved shell-like structures in three dimensions (3D) through elastic buckling of a network of rods. In the 1970s, the architectural community transferred this idea into the realm of large-scale construction (1) by introducing the concept of an elastic gridshell, a regular grid of elastic rods that is actuated into a shell-like structure by loading its extremities. Research has since focused mostly on case studies (1–3) and computational form finding of the static actuated geometry (4–6). Elastic gridshells fit with the recent trend in extreme mechanics to design through buckling, by harnessing the underlying geometry (7, 8). Examples of these geometry-driven structures range from microscopic buckling of a series of ribbons attached to a prestretched substrate (9) to macroscopic origami-inspired engineering (10, 11).

The buckled forms of elastic gridshells (Fig. 1A) suggest a nontrivial functional relationship between the initially flat, regular grid and the ensuing actuated geometry, making both forward and inverse design challenging tasks. This relationship juxtaposes rod elasticity and inextensibility, which are well understood in 1D as classical Euler's elastica (12), with a network of 2D constraints that oversee the emergence of a shell-like structure. This network can be modeled, on the one hand, from the perspective of interacting constrained elastica (13–15), an approach that is common in the study of random 3D polymer systems (16), but not as well explored for structured 2D networks. On the other hand, an elastic network can be modeled as a continuum of inextensible rods subjected to bending or shearing (17–20), which results in Euler–Lagrange equations but whose tractable solutions are often restricted to a plane or particular simple geometries. Instead of the variational approach, one can encode an elastic grid as a purely geometric object using the differential geometry of surfaces (21). This approach was taken by P. L.

Chebyshev in the 1880s (22) while investigating the deformations of woven fabric. The resulting theory of smooth Chebyshev nets is an ongoing topic of research which is one of the central themes of our study.

Here, we combine both perspectives of an elastic gridshell as an ensemble of constrained elastica and a continuum of inextensible rods to provide a quantitative description of their behavior. We focus on rationalizing the design challenges that underlie the actuation process and the resulting shape. Our physical model implements a 2D network of inextensible elastic rods in a desktop-scale setting that allows for high-precision validation against numerical simulations and theory. For simulations, we extend the discrete elastic rod (DER) method (23, 24) to elastic gridshells. We find excellent agreement with experiments (Fig. 1), allowing us to explore the parameter space of a seemingly simple elastic gridshell. This exploration reveals the complexity of actuation and forward design, as well as the presence of multistability. We then interpret elastic gridshells as both a network of smooth rods and a continuum by finitely sampling a Chebyshev net. This perspective enables an inverse design process, whose results suggest that rod inextensibility—not elasticity—is primary in dictating the zeroth-order shape of an actuated elastic gridshell. Moreover, our combined perspective allows us to harness theorems about smooth Chebyshev nets to rationalize elastic gridshell shape. This leads to quantitative investigations of their nonlocal response to loading and provides justification of the difficulties in form finding. We, therefore, expose the geometry-driven nature of elastic gridshells.

Significance

Elastic gridshells arise from the buckling of an initially planar grid of rods. The interaction of elasticity and geometric constraints makes their actuated shapes difficult to predict using classical methods. However, recent progress in extreme mechanics reveals the benefits of structures that buckle by design, when exploiting underlying geometry. Here, we demonstrate the geometry-driven nature of elastic gridshells. We use a geometric model, originally for woven fabric, to rationalize their actuated shapes and describe their nonlocal response to loading. Validation is provided with precision experiments and rod-based simulations. The prominence of geometry in elastic gridshells that we identify should allow for our results to transfer across length scales from architectural structures to micro/nano-1-df mechanical actuators and self-assembly systems.

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Fig. 1. (A and B) Actuation of an elastic gridshell from (A) experiments and (B) DER simulation. The edge points of a planar and unloaded footprint (A1 and B1) are gradually moved toward a prescribed actuated boundary (A2 and B2) to yield an actuated shape (A3 and B3). (Scale bar, 20 mm.)

Anatomy and Actuation of an Elastic Gridshell

Our model system comprises a planar, completely unloaded footprint of elastic rods that is actuated by loading its extremities (Fig. 1 A1). A footprint is constructed from straight elastic rods that are assumed to be inextensible (*Materials and Methods*). First, two sets of parallel rods with spacing d are laid perpendicularly. Then, the points of contact between cross-laid rods are constrained by joints. A square grid results, where each empty unit cell is enclosed by four adjacent rod segments of length d . Finally, the square grid is cut along an original boundary that encloses the footprint. A footprint thus contains a region of square grid and a boundary layer of “legs” extending to edge points along the original boundary. Pinning these edge points onto an actuated boundary of smaller perimeter induces out-of-plane buckling. The ensuing 3D actuated form is dictated by a combination of the elastic deformation of the rods and the geometric constraints of the grid. Without joints, each rod would act as an independent Euler’s elastica (12). Instead, the system behaves as a shell-like structure, whose shape, in contrast to a continuum shell, must be rationalized by resolving its actuated unit cells. We explored the form finding of elastic gridshells, using precision desktop-scale experiments (*Materials and Methods*). Elastomeric joints enforce positional constraints between cross-laid Nitinol rods, with negligible resistance to both shear of the unit cells and transport of twist along each rod. The super-elastic nature of Nitinol ensures that the gridshell remains elastic throughout actuation. Pinned boundary conditions at the edge points are set by 3D-printed ball joints, which, for actuation, were manually fitted onto a series of laser-cut holes on the designated actuated boundary (Fig. 1A).

In parallel to the experiments, we performed numerical simulations using the DER method, which was originally developed in the computer graphics community for the realistic visualization of filamentous structures such as hair and fur (23, 24). More recently, DER has been shown to be remarkably predictive in engineering contexts to quantify large deformations of elastic rods (25). Our current implementation of DER simulates elastic gridshells using rods under Kirchhoff’s assumptions, with the same physical parameters as in experiments. The positional constraints imposed by joints are modeled by stiff effective springs (*Materials and Methods*).

In Fig. 1 and Movie S1, we show representative examples, from both physical experiments (Fig. 1 A1–A3) and numerical simulations (Fig. 1 B1–B3), of a footprint with a circular original boundary, with radius $R_o = 126.0$ mm, that was gradually compressed to circular actuated boundaries with radii R_a [mm] = {126.0, 115.9, 94.5}. Excellent qualitative agreement is found between the two. As the actuation progresses, the height of the midsection increases and a dome-shaped gridshell emerges. We also consider noncircular boundaries.

Elastic Gridshells with Circular Boundaries

Even though the original and actuated boundaries in the example of Fig. 1 were both circular, the resulting actuated geometries were surprisingly complex (we expected to find entirely concave domes). This behavior is a signature of higher-order modes, which are known to arise in the theory of constrained elastica (13–15), and suggests the presence of multistability. Moreover, these complex geometries point to a nontrivial functional relationship between footprints and their actuated forms, which we address next.

In Fig. 2 A1–A3, we show three multistable states of the same elastic gridshell, in both experiment and simulation. The experimental state is represented by the positions of its joints (red solid circles) acquired via 3D laser scanning (*Materials and Methods*) and was obtained by manually point loading the actuated form and allowing it to snap into a new state. The corresponding multistable numerical solutions (solid lines) were computed by a jittering algorithm (*Materials and Methods*) and are in excellent quantitative agreement with experiments; the maximum joint positional mismatch is within two joint radii.

Given the accurate predictions from our simulations, we now rely on DER to systematically explore the parameter space of elastic gridshells with circular boundaries. The radius of the actuated boundary is set to $R_a = 0.75R_o$. To investigate boundary layer effects, we consider the longest “leg” in a footprint with n rods per direction, normalized by its unit cell spacing d ; i.e., $\bar{L}_d = \frac{L_d}{d} = \frac{R_o}{d} - \frac{n-1}{2}$. We further enforce $\bar{L}_d > 0$ (otherwise, the number of rods would reduce to $n - 2$). In Fig. 2B, we show the

Fig. 2. Elastic gridshells with circular boundaries. (A) Multistable states of a circular elastic gridshell with $(\bar{L}_d, n) = (0.8, 12)$ observed from experiments (red solid circles) and simulations (black lines). (B) Phase diagram of elastic gridshells with circular boundaries. In addition to the three states in A, a convex dome-shaped state (B, Inset) is obtained by extending the legs. Each data point is obtained from DER simulations under a jittering procedure. (C) Four representative modes of the circular elastic gridshells with different symmetry properties.

is represented by $\mathbf{G}: \mathcal{F}_b^d \rightarrow \mathbb{R}^3$. A specific Chebyshev net $\mathbf{C}$ is regarded as a viable ansatz if, for each position $b \in \mathbf{b}$ of the original boundary that maps to an actuated position $a \in \mathbf{a}$, we have $\mathbf{C}(b) = \mathbf{g}(b) = a$. Determining a viable ansatz for an elastic gridshell involves three steps. First, we specify an actuated boundary $\mathbf{a}$ in the Chebyshev net surface. Second, we compute the original boundary in the domain by solving for the contour $\mathbf{C}(\mathbf{b}) = \mathbf{a}$. Third, we fix $d > 0$ to determine the footprint $\mathcal{F}_b^d$ enclosed by $\mathbf{b}$.

Our ansatz procedure does not consider the pinned boundary conditions on the actuated boundary, which lead to higher-order constraints from variational considerations of the bending energy (19). Therefore, we quantify deviations, as follows, to attest the accuracy of viable Chebyshev net ansatzes. We first restrict $\mathbf{C}$ to points of the footprint $\mathcal{F}_b^d$ (d sampling of the Chebyshev net). Then, for each footprint point, $x \in \mathcal{F}_b^d$, we define $\Delta(x) = \|\mathbf{C}(x) - \mathbf{G}(x)\|$, the pointwise distance between an actuated elastic gridshell and a d -sampled viable ansatz.

Elastic Gridshells That Resemble Spherical Caps. With the goal of designing hemispherical and spherical-cap gridshells, we consider Chebyshev's hemisphere $\mathbf{C}_h$ (22, 26, 38, 39) (see *SI Appendix, S1. Hemispherical Ansatz* for details on its construction). We then use $\mathbf{C}_h$ as a viable ansatz by solving for the original boundary (shown as the outermost contour of Fig. 3B) that corresponds to an actuated boundary of the equator and fixing a unit cell spacing $d = 20$ mm. This yields a footprint $\mathcal{F}_b^d$, which is then actuated into an elastic gridshell $\mathbf{G}_h$. In Fig. 3C1 and C2, we quantify the deviation between the computed d sampling of $\mathbf{C}_h$ (blue curves) and the elastic gridshells corresponding to $\mathbf{G}_h$ obtained from both DER simulations (black curves) and experiments (red solid circles). A visual comparison is shown in Fig. 3C1, while Fig. 3C2 plots the data using spherical coordinates without longitudinal information (radius r , latitude ϕ); there is excellent agreement between theory, simulations, and experiments. Hereafter, we compare a viable ansatz only with a DER actuated elastic gridshell. Quantitatively, we find that there is $\lesssim 2\%$ deviation between the d -sampled $\mathbf{C}_h$ and the DER elastic gridshell $\mathbf{G}_h$ (using Δ as introduced above).

To further affirm the strength of our ansatz protocol, we use the same $\mathbf{C}_h$ to compute original boundaries of footprints for other spherical caps. Circles of latitude are used to slice the hemisphere by planes at varying heights ($0 \leq z^*/\rho < 1$) and understood as actuated boundaries $\mathbf{a}_{z^*}$. Note that ρ is the fixed radius of the sphere and agrees with the radius of $\mathbf{a}_{z^*}$ only when $z^*/\rho = 0$ (entire hemisphere). After fixing $d = 20$ mm, the corresponding original boundaries $\mathbf{b}_{z^*}$ (Fig. 3B) yield a series of actuated elastic gridshells $\mathbf{G}_{z^*}$. In Fig. 3D, we quantify the resulting deviations. As z^*/ρ increases, the deviation decreases from 1.87% ($z^*/\rho = 0.3$) to 0.17% ($z^*/\rho = 0.9$). This agreement is surprising since the smaller footprints contain even fewer rods; a continuum Chebyshev net ansatz may continue to be valid even for very coarse footprints within the limits of our inextensibility assumptions, whose applicability is investigated in *SI Appendix, S5. Inextensibility Limits of Elastic Gridshells*. To emphasize the geometric nature of these results, we considered a common household strainer (Fig. 3E), which also forms a spherical cap from a network of plastic inextensible rods. Interestingly, its flattened shape (after being cut along its “actuated boundary” rim) shows excellent agreement with a corresponding “original boundary” from $\mathbf{C}_h$ (Fig. 3F), thereby pointing to the generality of our framework.

Deformation of the Unit Cells and Nonlocal Response

Success in deriving elastic gridshells from smooth Chebyshev net ansatzes suggests broader applicability. We now use smooth Chebyshev net theory to rationalize the shape of elastic gridshells. In particular, we introduce a notion of integrated Gauss

curvature that relates directly to the shearing of the actuated unit cells. Subsequently, we use this notion to explore the non-local and highly anisotropic response of elastic gridshell systems under point-load indentation.

Integrated Gauss Curvature per Unit Cell. Even though the surface of a unit cell is ill-defined, it does have well-defined crossing angles at its joints. Using Hazzidakis' result, Eq. 2, for each unit cell of an actuated gridshell, we define its integrated Gauss curvature as $K^\square = 2\pi - \sum_{i=1}^4 \alpha_i$. If the enclosing rods of an actuated unit cell were to lie along geodesics, then the Gauss–Bonnet theorem (21) would state that the integrated Gauss curvature of the enclosed region would be $\sum_{i=1}^4 \alpha_i - 2\pi$, exactly the negative of our quantity $K^\square$. We will see that, generally, $K^\square$ carries the sign matching our intuition, implying that we do not interpret elastic gridshell rods as lying along geodesics of an underlying surface.

Next, we evaluate the validity of the notion of integrated Gauss curvature, $K^\square$, to design and describe elastic gridshells. We study viable ansatzes from surfaces of constant (i) positive, (ii) zero, and (iii) negative Gauss curvature, for which we use, respectively, (i) Chebyshev's hemisphere $\mathbf{C}_h$, (ii) half a circular cylinder parameterized by helices $\mathbf{C}_c$, and (iii) half an analytic parameterization for a pseudosphere of revolution $\mathbf{C}_p$. The latter two are special cases of Chebyshev nets on surfaces of revolution (40) (*SI Appendix, S2. Cylindrical and Pseudospherical Ansatzes*).

In Fig. 4A, *Left to Right*, we show that the obtained elastic gridshells closely resemble their ansatz d samplings, with only small deviations. In Fig. 4B, we plot $K^\square$ vs. the centroid of its four vertices in space along the shown axis. We find that $K^\square$ matches our expectation, with actuated unit cells that visually resemble regions of positive, zero, and negative curvature having an integrated Gauss curvature that is correctly signed. Our chosen ansatzes do not incorporate bending, which impacts the shape of the actuated forms. Most prominently, Fig. 4A, *Right* shows $\mathbf{G}_p$ protruding above its negatively curved Chebyshev ansatz $\mathbf{C}_p$ immediately after leaving its actuated boundary, before dipping in the middle. This behavior is consistent with a change from

Fig. 4. Notion of integrated Gauss curvature. (A) Elastic gridshells (DER simulation; black solid lines) and their d -sampled Chebyshev ansatzes (blue solid lines), which are from *Left to Right*, respectively, a hemisphere, a cylinder, and a saddle (each with half-span L). (B) Distribution of $K^\square$, the integrated Gauss curvature of a unit cell obtained from Eq. 2, for three elastic gridshells. Each unit cell is positioned by its normalized centroid $\bar{x} = x/L$. (C) Three indentation cases of an actuated hemispherical gridshell: (*Left*) inward and (*Center*) outward normal displacement at the north pole and (*Right*) inward displacement at $\pi/4$ latitude and longitude. The normalized displacement field is $|\mathbf{u}|/\delta_o$. (D) Binned $K^\square$ for C (*Left to Right*). Averages (bins) and standard deviations (error bars) show the spatial heterogeneity (i.e., nonlocality) of the shearing response.

Materials and Methods

Fabrication of Elastic Gridshells and Their 3D Imaging. A square grid ($d = 20$ mm) of grooves (depth $t_g = 500$ μm) was etched onto an acrylic plate, and holes (diameter $D_r = 3$ mm) were laser cut at the intersections. Nitinol rods (diameter $D_r = 254$ μm and Young's modulus $E_r = 83$ GPa) were placed onto the positioning grooves of the acrylic plate and the holes served to cast vinylpolysiloxane (VPS) cylindrical joints (Young's modulus $E_j = 0.23 \pm 0.01$ MPa, diameter $D_j = 3$ mm, height $H_j = 5$ mm). After the joints cured, the grid was cut along the original boundary and a speck of liquid glue (Loc-tite) was applied at each joint to prevent sliding. The 3D-printed ball joints (diameter $D_b = 4$ mm) were attached at the edge-point extremities of each rod and manually fixed onto the laser-cut holes along the actuated boundary, resulting in the 3D actuated form (Movie S4). Actuated forms were digitized using a 3D laser scanner (NextEngine), with a resolution of 100 μm . Partly because the diameter of rods was too small to be scanned and partly because the Nitinol rods reflect the laser beam away, only the joints could be successfully scanned, each yielding a point cloud. The scanned points were clustered by proximity and then averaged to obtain each joint position.

DER Simulations and Jittering Algorithm. A detailed account of DERs is found in refs. 23 and 24, with further details provided in *SI Appendix, S4. Numerical Simulations*. The DER method derives equations of motion using the discretized geometry and is well suited for problems involving elastic rod-like structures under geometrically nonlinear deformations. Our implementation generates discretized geometry with joint constraints (modeled as stiff effective springs) according to an input footprint. To explore multistability for elastic gridshell domes, we applied a jittering algorithm to each configuration. Partially actuated forms were subjected to an "artificial gravitational field" at different levels of intensity. The force field was removed and actuation progressed until the final actuated boundary was reached. The resulting actuated forms were classified by symmetry and rod-buckling modes, as discussed in the main text.

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