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Bayesian Empirical Likelihood Methods for Quantile Comparisons

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Abstract

Bayes factors, practical tools of applied statistics, have been dealt with extensively in the literature in the context of hypothesis testing. The Bayes factor based on parametric likelihoods can be considered both as a pure Bayesian approach as well as a standard technique to compute p-values for hypothesis testing. We employ empirical likelihood methodology to modify Bayes factor type procedures for the nonparametric setting. The paper establishes asymptotic approximations to the proposed procedures. These approximations are shown to be similar to those of the classical parametric Bayes factor approach. The proposed approach is applied towards developing testing methods involving quantiles, which are commonly used to characterize distributions. We present and evaluate one and two sample distribution free Bayes factor type methods for testing quantiles based on indicators and smooth kernel functions. An extensive Monte Carlo study and real data examples show that the developed procedures have excellent operating characteristics for one-sample and two-sample data analysis.

Keywords

Bayes factor; empirical likelihood; Bayesian empirical likelihood; quantile hypothesis testing; nonparametric tests

1 Introduction

The use of Bayes Factors (BF) is commonly cited as an efficient approach to incorporating external information into the inference process about a given hypothesis of interest when the likelihood functions have parametric forms (e.g., Kass, 1993; Kass & Raftery, 1995). Although this method can be employed as a pure Bayesian alternative to frequentist techniques based on p-values, BF type statistics can also serve statistical decision making in the terms of traditional test statistics with controlled distributions under the null hypothesis, via, e.g., the use of theoretical asymptotic propositions. In this article, we propose to replace

the parametric likelihoods in BF with empirical likelihoods (Owen, 2001), constructing nonparametric BF type procedures.

In particular, we focus here on the development of BF based procedures for non-parametric testing about the quantiles in both the one- and two-sample settings. The following practical issues motivate the work regarding the development of novel tests for quantiles. In biomedical research, estimates of quantiles, e.g., median survival times, are frequently used to characterize outcome variables. For heavily censored data, summary measures based on sample quantiles are generally preferable to the estimated mean survival since they have smaller bias. Moreover, the difference in treatment effects might be estimated using the difference between quantiles under a location-shift assumption (e.g., Huges, 2000). Note that one of the main goals in applied research is to determine whether two independent groups (e.g., treatment and control groups) differ, and, if they do differ, to quantify the magnitude of this effect. Wilcox (1995) provided many two-sample test examples showing that evaluations and comparison of quantiles led the investigators to more satisfactory conclusions in terms of relative power compared to tests based on sample moments. Sometimes, the populations of the two groups differ not only by location but also by the shape and scale of the distribution. In such cases, 1) comparing quantiles, e.g., the medians may provide different information than comparing sample moments of the observations, e.g., the means, and 2) permutation based tests for differences in quantiles in this setting would violate the exchangeability assumption under the null hypothesis (e.g., Hutson 2007 or examples involving medians). An additional benefit of using sample quantiles in the two-group setting is that they provide a measure of robustness relative to their respective influence functions as compared to moment based tests (e.g., Yu et al., 2011; Yu et al., 2014). Similarly, the quantile approach is feasible and useful when tests based on moments are not. For example, when estimating the parameter of a Cauchy distribution the sample mean is not a consistent estimator of the location parameter while the sample median is (e.g., Zhou & Jing, 2003). Hughes (2000) emphasizes that in the context of censored measurement studies, the non-parametric approach, e.g., testing the median, is more robust to the censored data than the parametric approach based on moments. Yu et al. (2011) argue the relevance of median evaluation of biomarkers in the context of cancer research, such as enzyme-linked immunosorbent assay (ELISA) and Western blot analysis.

In certain studies, we may have prior information regarding the values of quantiles before evaluating a given dataset. In this article, we consider quantile evaluation conditional on prior information and observations, adopting well-developed Bayesian methods, specifically hypothesis testing based on BF mechanics. This approach provides practical and flexible tools for combining information and partial pooling of inferences, according to parametric likelihood principles (e.g., Carlin & Louis, 2008). BF type procedures are often discussed in the context of criticism of the traditional p-value approach, since the Bayesian approach to hypothesis testing can be shown to be much simpler and more sensible in principle in a variety of situations (e.g., Kass & Raftery, 1995; Carlin & Louis, 2008, p.39). Furthermore, asymptotic propositions related to analyzing null distributions of BF type statistics provide the possibility of considering traditional testing rules based on these statistics (e.g., Kass & Wasserman, 1995). The asymptotic results also simplify computations of BF based statistics that often consist of complex integrals, marginal distributions, calculated by numerical

methods. Within this framework we will consider the stated problems non-parametrically, whereas the classical Bayesian methodology involves parametric likelihoods.

Nonparametric tests offer a simple and reliable statistical technique in applications. They possess the desirable property of having the same sampling distribution for all continuous distributions. For the two-sample location shift setting commonly used non-parametric procedures include the Wilcoxon normal scores and median (Mood, 2003) tests. Results regarding the median test are varied. A number of authors (e.g., Freidlin & Gastwirth, 2000) demonstrate the poor performance of the median test in very small samples, as well as the loss of power of the median test relative to the Wilcoxon test in the case of highly unbalanced samples. Yu et al. (2011), on the other hand, note that the median test is valid under weaker conditions than other rank tests. Results of Yu et al. (2011) show that testing for two-group medians based on the empirical likelihood (EL) approach is workable even with the violation of the exchangeability assumption under the null hypothesis (Hutson, 2007). This article develops different tests for quantiles via the EL methodology.

The EL approach (e.g., Owen, 2001; Vexler, et al., 2009; Yu, et al., 2009, Lazar & Mykland, 1999) is a nonparametric method of statistical inference, which allows researchers to use traditional likelihood methodology without having to assume known forms of data distributions. The EL technique is very effective because it inherits many of the characteristics of parametric likelihood methods. This method is widely used to find efficient estimators and to construct tests with good power properties. The EL method is also very flexible. For this article's purposes, we refer to the research shown in Lazar (2003), which demonstrates that the EL is a valid function for Bayesian inference. To present this conclusion, Lazar (2003) uses the Monahan & Boos heuristic (e.g., Monahan & Boos, 1992) and examines the frequentist properties of Bayesian intervals. Thus, Lazar (2003) explores a simple way of proposing nonparametric Bayesian inference via an application of the standard theory with a prior on the functional of interest and EL taking the role of a model-based likelihood.

In this article, the methods mentioned above are applied to develop distribution free quantile testing procedures. The proposed technique is used for one-sample and two-sample analysis. We evaluate different mechanisms based on indicators and smooth kernel functions. The results we obtain can also be considered as certain extensions of the Bayesian nonparametric estimations of medians presented in Doss (1985a, b) and the EL estimation proposed in Chen & Hall (1993). Note that proof schemes used in this article can be of independent interest to investigators dealing with applications of adapted Laplace methods (e.g., Bleinstein & Handelsman, 1975, Davison, 1986, Kass & Raftery, 1995) to handle step functions and EL functions. These proof schemes allow us to obtain asymptotic results similar to those of parametric BFs.

The paper is organized as follows. In Section 2, we outline the relevant ideas of Bayes Factors and empirical likelihood. These methods are combined to develop the one and two sample EL BF inference for quantiles. In Section 3, we carry out an extensive Monte Carlo study to evaluate the new procedures. In Section 4 we present data examples that

demonstrate the excellent applicability of the proposed method in practice. Finally, in Section 5 we conclude by summarizing the most important points related to this research.

2 Method

In this section, we first outline basic techniques related to: 1) BF type tests; 2) ELs; 3) Bayesian ELs. The core of our inference procedure for quantiles is to combine these three.

2.1 One Sample Quantile-Based Testing

We assume that data D can be studied under one of the two hypotheses H_0 and H_a . Given prior densities p_{H_0} and p_{H_a} , the classic BF has the form

$$BF = Pr(D|H_a) / Pr(D|H_0),$$

(e.g., Kass & Raftery, 1995). In some cases, when the hypotheses correspond to a parameter space and are related to specifications of data distributions, the BF can be formulated as

$$BF = \frac{\int L(D|\beta) p_{H_a}(\beta) d\beta}{\int L(D|\theta) p_{H_0}(\theta) d\theta},$$

where β and θ are vectors of parameters of the probability densities, $L(D|\beta)$ and $L(D|\theta)$ are the likelihood functions of the data D . The density functions p_{H_0} and p_{H_a} can reflect initial information regarding values of the parameters θ and β . Note that classic BF procedures require parametric assumptions regarding the distribution of the data D . To relax this constraint, one can try to substitute nonparametric likelihoods for $L(D|\beta)$ and $L(D|\theta)$. In the context of one sample testing problems we will focus on BF type procedures for point null hypotheses discussed, e.g., in Berger (1985), pp.148–150.

The modern statistical literature has addressed the EL approach (e.g., Owen, 2001) in the context of developing powerful approximations of parametric likelihoods. For example, we assume data D consists of independent identically distributed observations $\{X_1, \dots, X_n\}$ and, say, we would like to test the hypotheses

$$H_0: E[g(x, \theta)] = 0 \text{ vs. } H_a: E[g(x, \theta)] \neq 0,$$

where E indicates expectation with respect to X_i , g is a given function and θ is a parameter. To test the hypotheses above in a nonparametric fashion, we define the EL function in the form of $\mathcal{L}_e(\theta) = L(X_1, \dots, X_n|\theta) = \prod_{i=1}^n p_i$, where $p_i = Pr\{x_i = X_i\} > 0$ and $\sum_{i=1}^n p_i = 1$. The values of the p_i s are unknown and should be evaluated under H_0 and H_a . Under the null hypothesis, the maximum likelihood approach requires one to find the values of the p_i s that maximize the EL given the empirical constraints $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n p_i g(X_i, \theta) = 0$. In this case, using Lagrange multipliers, one can show that

$$\mathcal{L}_e(\theta) = \sup_{0 < p_1, p_2, \dots, p_n < 1, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i g(X_i, \theta) = 0} \prod_{i=1}^n p_i = \prod_{i=1}^n \frac{1}{n + \lambda g(X_i, \theta)}, \quad (2.1)$$

where λ is a root of $\sum g(X_i, \theta)(n + \lambda g(X_i, \theta))^{-1} = 0$. Since under H_a only the constraint $\sum p_i = 1$ should be considered, we have

$$\mathcal{L}_e = \sup_{0 < p_1, p_2, \dots, p_n < 1, \sum_{i=1}^n p_i = 1} \prod_{i=1}^n p_i = \prod_{i=1}^n \frac{1}{n} = \left(\frac{1}{n}\right)^n. \quad (2.2)$$

Combining (2.1) and (2.2), we obtain the EL ratio (ELR) test statistic, $\mathcal{R}(\theta) = \mathcal{L}_e / \mathcal{L}_e(\theta)$, for the hypothesis H_0 vs. H_a . Owen (1988) shows that the non-parametric test statistic $2 \log \mathcal{R}(\theta)$ has asymptotically a chi-square distribution, under the null hypothesis. This result is an analog of Wilks Theorem (e.g., Wilks, 1938) that is known in the context of null distributions of parametric maximum likelihood ratio tests. Since ELs can be considered as candidates to substitute parametric likelihoods, when data distributions are completely unknown, one can propose to define a nonparametric analog to BFs via the EL methodology. However, this approach should be justified formally. Lazar (2003) shows that the EL technique can provide proper likelihoods that can serve as the basis for Bayesian inference, providing robustness to choices of priors when sample sizes are relatively large.

In this article, we focus on nonparametric quantile evaluations that employ prior information. To this end, we will consider the methods mentioned above in the following statistical problem. The formulation of the problem begins with data $D = \{X_1, \dots, X_n\}$ that consists of independent identically distributed observations with quantile q_a at the level of α , i.e. $\alpha = \Pr\{X_1 \leq q_a\}$. In this case, we can express the hypotheses H_0 and H_a as $H_0: q_a = q_0$ and $H_a: q_a \neq q_0$, respectively. To test for H_0 vs. H_a we define the ELs in the form of $\prod_{i=1}^n p_i$, where $0 < p_1, \dots, p_n < 1$ are constrained by $\sum_{i=1}^n p_i = 1$ only, and by $\sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i I\{X_i \leq q_0\} = \alpha$, under H_a and H_0 , respectively. (In the case of $q_0 < \min(X_i)$ or $q_0 > \max(X_i)$, the null hypothesis is assumed to be rejected.) One can show that under H_0 , the EL takes the form

$$\mathcal{L}_e(q_0) = \left(\frac{\alpha}{n F_n(q_0)}\right)^{n F_n(q_0)} \left(\frac{1 - \alpha}{n(1 - F_n(q_0))}\right)^{n(1 - F_n(q_0))}, \quad (2.3)$$

where $F_n(q) = (n^{-1}) \sum_{i=1}^n I\{X_i \leq q\}$ is the empirical distribution function. It is clear that, under H_a , the EL has the value n^{-n} . (For details, see the Appendix)

Following the BF methodology and Lazars results (2003), we propose the ELBF test statistic

$$ELBF = (\mathcal{L}_e(q_0))^{-1} \int_{X_{(1)}}^{X_{(n)}} e^{\log \mathcal{L}_e(q)} p(q) dq, \quad (2.4)$$

where $p(q)$ is the quantile's prior density under H_a , and $X_{(1)} \dots X_{(n)}$ denote the order statistics of $X_1, \dots, X_n$. In this case, we have

$$2 \log ELBF = 2 \log \int e^{\log \mathcal{L}_e(q) - \log(n^{-n})} p(q) dq + 2 \log(n^{-n} / \mathcal{L}_e(q_0)),$$

where by Wilks theorem, $2 \log(n^{-n} / \mathcal{L}_e(q_0))$ has a χ_1^2 distribution as $n \rightarrow \infty$. We reject the null hypothesis for large values of the test statistic (2.4). To control the Type I Error of the test based on the statistic (2.4), we assume that $p(q)$ is differentiable and show the next asymptotic result.

Proposition 2.1—*Suppose the true quantile $q_\alpha = q_0$, and observations are from a distribution function with density f that is differentiable. Then*

$$2 \log ELBF = \Omega + 2 \log(n^{-n} / \mathcal{L}_e(q_0)),$$

where $2 \log(n^{-n} / \mathcal{L}_e(q_0))$, which is simply $2 \log \mathcal{R}$, is asymptotically distributed as χ_1^2 and

$$\begin{aligned} \Omega &= 2 \log \int e^{\log \mathcal{L}_e(q) - \log(n^{-n})} p(q) dq = 2 \log \left(\frac{p(X_{(\lfloor \alpha n \rfloor)})}{f(X_{(\lfloor \alpha n \rfloor)})} \left(\frac{2\pi\alpha(1-\alpha)}{n} \right)^{0.5} \right) + o_p(1) \\ &= 2 \log \left(\frac{p(q_0)}{f(q_0)} \left(\frac{2\pi\alpha(1-\alpha)}{n} \right)^{0.5} \right) + o_p(1), \quad o_p(1) \xrightarrow[n \rightarrow \infty]{p} 0. \end{aligned} \quad (2.5)$$

In Proposition 2.1, we note that $X_{(\lfloor \alpha n \rfloor)}$ is the order statistic, where the subscript represent the integer part of αn . The proof is in the Appendix. In calculation of Ω , $f(q_0)$ is estimated based on the sample.

The standard method to approximate integrals such as Ω is based on the Laplace technique (e.g., Bleinstein & Handelsman, 1975, Davison, 1986). This method requires $\log \mathcal{L}_e(q) - \log(n^{-n})$ to be continuous and twice differentiable. However the function $\log \mathcal{L}_e(q) - \log(n^{-n})$ contains the step function $F_n(q)$. To prove Proposition 2.1, we therefore utilize the Bahadur theorem (e.g., Serfling, 1981) to adapt the Laplace method. The main idea of the Laplace method for $\int \exp(\varphi(u))g(u)du$ -type integrals is to apply the Taylor approximation $\varphi(u) \cong \varphi(u_0) + (u - u_0)\varphi'(u_0) + (u - u_0)^2\varphi''(u_0)/2$ to obtain approximate Gaussian forms of the integrals, where the minimum of φ occurs at the point u_0 . In the context of evaluations related to Bayesian posterior distributions and point estimators, Tierney & Kadane (1986) propose approximating $\int \exp(\varphi(u))g(u)du$ -type integrals using a Taylor approximation to $\varphi(u) + \log(g(u))$. Following the approach of Tierney & Kadane, one can easily modify the proof of Proposition 2.1 to show:

Corollary 2.1—The term Ω defined in Proposition 2.1 can be given asymptotically as

$$\Omega = \left(\frac{p'(q_0)}{p(q_0)} \right)^2 \left(\frac{n(f(q_0))^2}{\alpha(1-\alpha)} - (\log(p(q_0)))'' \right)^{-1} + 2 \log \left(p(q_0) \left(2\pi / \left(\frac{n(f(q_0))^2}{\alpha(1-\alpha)} - (\log(p(q_0)))'' \right) \right)^{0.5} \right) + o_p(1), \quad o_p(1) \xrightarrow{n \rightarrow \infty} 0.$$

Corollary 2.1 yields a result that is asymptotically equivalent to that of Proposition 2.1. It turns out that the use of Corollary 2.1's approximation emphasizes the role of prior information. This improves power characteristics of the proposed test based on samples with relatively small sizes. Our extensive Monte Carlo study confirms this result.

By the virtue of Proposition 2.1 and Corollary 2.1, we can obtain $(1 - \rho) \times 100\%$ confidence intervals for q_a in the form of

$$\begin{aligned} CI\{q\} &= \{q: 2 \log ELBF - \Omega \leq C_\rho\} \\ &\cong \left\{ q: 2 \log ELBF - 2 \log \left(\frac{p(q)}{f(q)} \left(\frac{2\pi\alpha(1-\alpha)}{n} \right)^{0.5} \right) \leq C_\rho \right\} \text{ or} \\ &\cong \left\{ q: 2 \log ELBF - \left(\frac{p'(q)}{p(q)} \right)^2 \left(\frac{n(f(q))^2}{\alpha(1-\alpha)} - (\log(p(q)))'' \right)^{-1} - 2 \log \left(p(q) \left(2\pi / \left(\frac{n(f(q))^2}{\alpha(1-\alpha)} - (\log(p(q)))'' \right) \right)^{0.5} \right) \leq C_\rho \right\}, \end{aligned}$$

where C_ρ is the corresponding ρ level critical value of the χ_1^2 distribution.

Remark 2.1—Choosing the prior $p(q)$. In the context of change point detection, Krieger et al. (2003), as well as Vexler & Wu (2009), propose several forms of priors. The method of Krieger et al. (2003) can be adapted for the problem stated in this paper. Two examples of possible priors are

$$p(q) = \frac{\partial}{\partial q} \Phi \left(\frac{\mu}{\sigma} \right)^{-1} \left(\Phi \left(\frac{q-\mu}{\sigma} \right) - \Phi \left(-\frac{\mu}{\sigma} \right) \right)^+, \quad \Phi(u) = \frac{1}{(2\pi)^{0.5}} \int_{-\infty}^u e^{-t^2/2} dt,$$

if we suspect that, under H_a , the observations have a distribution with q_a that differs greatly from q_0 ; a somewhat broader prior is

$$p(q) = 0.5 \frac{\partial}{\partial q} \left(\Phi \left(\frac{q-\mu}{\sigma} \right) + \Phi \left(\frac{q+\mu}{\sigma} \right) \right).$$

The hyper parameters μ and $\sigma > 0$ can be chosen arbitrarily, e.g. Krieger et al. (2003) recommend $\mu = 0$ and $\sigma = 1$ to simplify the possible forms of $p(q)$. Marden (2000) gives a general review of the Bayesian approach to hypothesis testing. The prior distribution $p(q)$ can be defined in accordance with rules mentioned in the literature cited in Section 3 of Mardens paper. The function $p(q)$ can be also chosen with respect to special areas of the

parameter q_α , where the maximum of the tests power is desired (for details see, e.g., Vexler et al., 2010).

Yu et al. (2011) propose and examine different EL tests for medians, demonstrating that, in the considered cases, an application of the kernel method (e.g., Nadaraya, 1964; Azzalini, 1981) can provide efficient results (see also Chen & Hall, 1993). In this article, we analyze situations where the constraint $\sum_{i=1}^n p_i I\{X_i \leq q\} = \alpha$, used in the construction of $\mathcal{L}_e(q)$, is substituted by $\sum_{i=1}^n p_i K_h\{X_i - q\} = \alpha$, where K_h is a kernel function with the bandwidth $h > 0$, $K_h(X - q) = \int_{-\infty}^{q-X} k_h(u) du$, and k_h is a non-negative, differentiable function satisfying $\int_{-\infty}^{\infty} k_h(u) du = 1$, $\int_{-\infty}^{\infty} u k_h(u) du = 0$, $\int_{-\infty}^{\infty} u^2 k_h(u) du < \infty$, and $\int_{-\infty}^{\infty} |k_h'(u)| du < \infty$. Defining the EL as

$$\mathcal{L}_e(q) = \max \left\{ \prod_{i=1}^n p_i : 0 < p_i < 1, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i K_h\{X_i - q\} = \alpha \right\},$$

one can show that

$$\mathcal{L}_e(q) = \prod_{i=1}^n (n + \lambda(K_h(X_i - q) - \alpha))^{-1}, \quad (2.6)$$

where λ is a root of the equation

$$\sum_{i=1}^n K_h(X_i - q)(n + \lambda(K_h(X_i - q) - \alpha))^{-1} = \alpha. \quad (2.7)$$

Thus, the ELBF statistic,

$$ELBF = (\mathcal{L}_e(q_0))^{-1} \int_{X_{(1)}}^{X_{(n)}} e^{\log \mathcal{L}_e(q)} p(q) dq, \quad \mathcal{L}_e(q) \text{ by (2.6)},$$

can be applied to test $H_0 : q_\alpha = q_0$ vs. $H_a : q_\alpha \neq q_0$, using the following proposition as an instrument in controlling the Type I error of the test.

Proposition 2.2—Assume observations $\{X_1, \dots, X_n\}$ are from a distribution function with density f that is differentiable. Then, under the null hypothesis H_0 , we have

$$2 \log ELBF = \Omega + 2 \log(n^{-n} / \mathcal{L}_e(q_0)),$$

where $2\log(n^{-n}\mathcal{L}_{\mathcal{A}}(q_0))$, $2\log\mathcal{R}$, is asymptotically distributed as χ_1^2 and

$$\begin{aligned}\Omega &= 2\log \int e^{\log \mathcal{L}_e(q) - \log(n^{-n})} p(q) dq \\ &= 2\log \left(p(q_M) \left(2\pi \frac{\sum_{i=1}^n (K_h(X_i - q_M) - \alpha)^2}{(\sum_{i=1}^n k_h(q_M - X_i))^2} \right)^{0.5} \right) + o_p(1),\end{aligned}$$

with $o_p(1) \xrightarrow[n \rightarrow \infty]{p} 0$ and $q_M : n^{-1} \sum_{i=1}^n K_h(X_i - q_M) = \alpha$ as well as

$$\begin{aligned}\Omega &= 2\log \left(p(q_0) \left(2\pi \frac{\sum_{i=1}^n (K_h(X_i - q_0) - \alpha)^2}{(\sum_{i=1}^n k_h(q_0 - X_i))^2} \right)^{0.5} \right) + o_p(1) \\ &= 2\log \left(\frac{p(q_0)}{f(q_0)} \left(\frac{2\pi\alpha(1-\alpha)}{n} \right)^{0.5} \right) + o_p(1),\end{aligned}$$

when $h \rightarrow 0$, $nh^2 \rightarrow \infty$, as $n \rightarrow \infty$.

Proposition 2.2 can be obtained by applying the classical tools based on the Laplace method (e.g., Bleistein & Handelsman, 1975, pp. 180–193) that are commonly used to approximate parametric BF's (e.g., Davison, 1986; Kass & Raftery, 1995; Gelfand & Dey, 1994, pp. 506–507). This approach uses considerations of parametric likelihoods around their maximum values. To apply the Laplace method to evaluate the ELBF statistic, we show that the function $\mathcal{L}_{\mathcal{A}}(q)$ behaves similarly to a parametric likelihood, in the context of the maximization problem, i.e. finding $\max \mathcal{L}_{\mathcal{A}}(q)$. This result is formulated in the following lemma that we propose in a general form, since the lemma below can have an independent interest, e.g., being considered as a nonasymptotic analog to several lemmas mentioned in Qin & Lawless (1994). To present the next result, we define the function

$$G(\theta) = \max \left\{ \prod_{i=1}^n p_i : 0 < p_i < 1, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i W(X_i, \theta) = 0 \right\},$$

where it is assumed, for simplicity, that $W(u, \theta) / \theta > 0$ or $W(u, \theta) / \theta < 0$, for all u .

Lemma 2.1—Let θ_M denote a root of the equation $n^{-1} \sum_{i=1}^n W(X_i, \theta_M) = 0$. Then θ_M gives a global maximum of the function $G(\theta)$ that increases and decreases monotonically for $\theta < \theta_M$ and $\theta > \theta_M$, respectively.

The proof is in the Appendix.

For example, defining $W(u, \theta) = u - \theta$, we obtain that $\theta_M = n^{-1} \sum_{i=1}^n X_i$ and the function $G(\theta)$ is the EL for the mean $EX_1 = \theta$. The result of Lemma 2.1, when $W(u, \theta) = K_h(u - \theta) - \alpha$, for fixed α , can be directly applied to prove Proposition 2.2. The proof of Proposition 2.2 is outlined in the Appendix.

Remark 2.2—Computing the test statistic. Note that, to obtain values of the test statistic $ELBF$ defined via the kernel functions, it is possible to apply packages such as “emplik” in R (www.r-project.org) without specifying the closed form of the empirical probabilities. We can also suggest, following Yu et al. (2011)’s conclusions and our extensive Monte Carlo experiments, using the kernel function $K_h(X-q) = (2\pi h^2)^{-0.5} \int_{-\infty}^{q-X} e^{-u^2/(2h^2)} du$ with $h = 0.2n^{-1/6}$. The related computer codes are available from the authors upon request.

In order to test for the null hypothesis, by virtue of Proposition 2.2, the following rule can be applied: Reject H_0 if and only if

$$2\log ELBF - 2\log(p(q_0)(2\pi/S_n)^{0.5}) \geq C_\rho,$$

where

$$S_n = \frac{(\sum_{i=1}^n k_h(q_0 - X_i))^2}{\sum_{i=1}^n (K_h(X_i - q_0) - \alpha)^2}.$$

In a similar manner to Corollary 2.1, we can also propose the test based on

$$2\log ELBF - \left(\frac{p'(q_0)}{p(q_0)} \right)^2 (S_n - (\log(p(q_0)))'')^{-1} - 2\log(p(q_0)(2\pi/(S_n - (\log(p(q_0)))''))^{0.5}) \geq C_\rho.$$

We evaluate performances of these tests in a Monte Carlo study (see Section 3).

Remark 2.3—Type I error control. The classical empirical likelihood ratio (ELR) methodology suggests applying H_a -maximum ELs obtained via a global maximization of their components, e.g. the H_a -maximum

H_a -maximum $EL = \max\{\prod_{i=1}^n p_i : 0 < p_i < 1, \sum_{i=1}^n p_i = 1\} = n^{-n}$. In this article, we propose instead test statistics that utilize a local maximization of the EL components, under H_a , with respect to a prior. Intuitively, this can give the proposed procedures a possibility to have better Type I errors control than that of the classical EL tests. Our extensive Monte Carlo study confirms this conclusion. Thus, in several situations, the proposed tests, even based on completely non-informative priors, can be recommended, e.g., when data are skewed, to obtain good type I error control. We’d point out in this context as well that, in some circumstances, the ELR tests have serious issues related to Type I error control in practice (e.g., Vexler et al., 2009; Yu et al., 2009); hence a result such as this one is beneficial.

Remark 2.4—In classical Bayesian analysis, the integrals related to BF type statistics may be evaluated analytically in some elementary situations. More often, these calculations are intractable and thus complicated numerical methods are employed. Alternatively, one can use the asymptotic techniques related to the Laplace method (e.g., Kass & Raftery, 1995). In

this work, as already noted, we adapt the asymptotic propositions developed for parametric BFs to the distribution-free EL analog. Also, note that the nonparametric posterior distributions are based on integrated EL functions. Often, the EL functions do not have closed analytical forms and thus require us to use numerical methods in order to obtain values of the EL functions depending on their arguments. That is, computation of the proposed BFs is not a simple task. In a similar manner to the classic Bayesian inference, the asymptotic results shown in this article provide a practical solution towards solving the complex computations related to the proposed techniques.

2.2 Two Sample Tests for Quantiles

Consider a comparison of two independent groups in terms of specific quantiles. Suppose that, for each group, $i = 1, 2$, there are n_i independent observations X_{ij} , where j denotes the j th unit from group i , $j = 1, \dots, n_i$. Let $q_{\alpha,i}$ indicate the true quantile of group i , $Pr\{X_{ij} \leq q_{\alpha,i}\} = \alpha$, $i = 1, 2$. We are interested in testing

$$H_0: q_{\alpha,1} = q_{\alpha,2} \text{ vs. } H_a: q_{\alpha,1} \neq q_{\alpha,2} \quad (2.8)$$

Following the EL concept, one can consider several schemes to construct tests that are based on different (yet equivalent) constraints under the hypotheses in (2.8). Yu et al. (2011) provide an extensive analysis of the relevant ELR tests for two sample medians. In this article, utilizing results in Yu et al. (2011), we focus on the ELs

$$\mathcal{L}_{e,I,i}(q) = \max \left\{ \prod_{j=1}^{n_i} p_j: 0 < p_j < 1, \sum_{j=1}^{n_i} p_j = 1, \sum_{j=1}^{n_i} p_j I\{X_{ij} \leq q\} = \alpha \right\}, i=1, 2$$

and

$$\mathcal{L}_{e,K,i}(q) = \max \left\{ \prod_{j=1}^{n_i} p_j: 0 < p_j < 1, \sum_{j=1}^{n_i} p_j = 1, \sum_{j=1}^{n_i} p_j K\{X_{ij} - q\} = \alpha \right\}, i=1, 2$$

to define the BF type test statistics

$$ELBF_S = \left(\int_{\max(X_{1(1)}, X_{2(1)})}^{\max(X_{1(n_1)}, X_{2(n_2)})} \exp(\log(\mathcal{L}_{e,S,1}(q)) + \log(\mathcal{L}_{e,S,2}(q))) p_0(q) dq \right)^{-1} \times \prod_{i=1}^2 \int_{X_{i(1)}}^{X_{i(n_i)}} e^{\log(\mathcal{L}_{e,S,i}(q))} p_i(q) dq, \quad S=I, K, \quad (2.9)$$

where $X_{i(j)}$ is the j th order statistic in the i th group and $p_r(q)$, $r = 0, 1, 2$, are corresponding priors. We reject the null hypothesis in (2.8) for large values of the test statistics $ELBF_S$, $S =$

I, K . In order to show asymptotic behaviors of $ELBF_S$, $S = I, K$, we adapt the conditions of Propositions 2.1 and 2.2, assuming that, under H_0 , $q_{a,1} = q_{a,2} = q_0$ and $n_1/(n_1 + n_2) \rightarrow \eta > 0$, as $n_i \rightarrow \infty$, $i = 1, 2$, where η is a constant.

Proposition 2.3—Let observations $\{X_{11}, \dots, X_{1n_1}\}$ and $\{X_{21}, \dots, X_{2n_2}\}$ be from distribution functions with differentiable densities f_1 and f_2 , respectively. Then under H_0 , we have

$$2\log ELBF_S = \Omega + 2\log \left(\prod_{i=1}^2 \max_q (\mathcal{L}_{e_{S,i}}(q)) / \max_q \left(\prod_{i=1}^2 \mathcal{L}_{e_{S,i}}(q) \right) \right),$$

where

$$2\log \left(\prod_{i=1}^2 \max_q (\mathcal{L}_{e_{S,i}}(q)) / \max_q \left(\prod_{i=1}^2 \mathcal{L}_{e_{S,i}}(q) \right) \right) = 2\log \left(n_1^{-n_1} n_2^{-n_2} / \max_q \left(\prod_{i=1}^2 (\mathcal{L}_{e_{S,i}}(q)) \right) \right),$$

the $2\log \mathcal{R}$, is asymptotically distributed as χ_1^2 and

$$\begin{aligned} \Omega &= 2\log \left((2\pi)^{0.5} \prod_{i=1}^2 p_i(q_0) / (D_{in_i})^{0.5} \right) - 2\log \left(p_0(q_0) / (D_0)^{0.5} \right) + o_p(1), \\ &= 2\log \left(\left(\frac{2\pi\alpha(1-\alpha)}{n_2} \left(\frac{1}{f_2(q_0)^2} + \frac{1-\eta}{\eta f_1(q_0)^2} \right) \right)^{0.5} \frac{p_1(q_0)p_2(q_0)}{p_0(q_0)} \right) + o_p(1) \end{aligned}$$

with

$$D_{in_i} = \frac{\left(\sum_{j=1}^{n_i} k_h(q_0 - X_{ij}) \right)^2}{\sum_{j=1}^{n_i} (K_h(X_{ij} - q_0) - \alpha)^2} \text{ and } D_0 = \sum_{i=1}^2 \frac{\left(\sum_{j=1}^{n_i} k_h(q_0 - X_{ij}) \right)^2}{\sum_{j=1}^{n_i} (K_h(X_{ij} - q_0) - \alpha)^2},$$

as $n_i \rightarrow \infty$, $i = 1, 2$.

The proof of Proposition 2.3 is outlined in the Appendix.

Following the proof scheme of this proposition, in a similar manner to Proposition 2.2 and Corollary 2.1, one can show the next result.

Corollary 2.2—The statistic

$$\begin{aligned}
& 2\log ELBF_K - \sum_{i=1}^2 \left(\frac{p'_i(q_0)}{p_i(q_0)} \right)^2 (D_{in_i} - (\log(p_i(q_0)))'')^{-1} \\
& - 2\log \left((2\pi)^{0.5} \prod_{i=1}^2 p_i(q_0) / (D_{in_i} - (\log(p_i(q_0)))'')^{0.5} \right) \\
& + \left(\frac{p'_i(q_0)}{p_i(q_0)} \right)^2 (D_0 - (\log(p_0(q_0)))'')^{-1} + 2\log (p_0(q_0) / (D_0 - (\log(p_0(q_0)))'')^{0.5})
\end{aligned}$$

is asymptotically distributed as χ^2_1 , under the null hypothesis.

3 Simulation Study

To examine the proposed procedures, we carried out an extensive Monte Carlo study under various data distributions and different prior functions.

3.1 One Sample Tests

Let the notation d_1 reflect situations in which generated data follow the $N(0, 1)$ distribution under H_0 and the $N(\delta, 1)$ distribution under H_a , where δ depicts the difference between q_0 and q_1 , and $q_a = q_0$ or q_1 under H_0 or H_a respectively; the notation d_2 reflects situations in which generated data follow the $LogNorm(0, 1)$ distribution under H_0 and the $LogNorm(q_0 + \delta, 1)$ distribution under H_a respectively.

Under d_1 , let $p_1, \dots, p_5$ depict the use of priors in the forms of $p(q) = (2\pi\sigma^2)^{-1/2} \exp(-(q - \mu)^2/2\sigma^2)$ with $(\mu = 0, \sigma = 1)$, $(\mu = \mu_p, \sigma = 1)$, $(\mu = \mu_p, \sigma = 1/2)$, $(\mu = \mu_p, \sigma = 1/1.5)$ and $(\mu = 2\mu_p, \sigma = 1)$, respectively, where $\mu_p = 1$. Under d_2 , $p_1, \dots, p_5$ are the same prior functions as under d_1 , except p_3 for which $(\mu = \mu_p, \sigma = 1/1.2)$.

Under both distributions, p_6 corresponds to the use of $p(q) = 0.5 (\Phi((q - \mu)/\sigma) + \Phi((q + \mu)/\sigma)) / q$ with $(\mu = \mu_p, \sigma = 1/1.2)$.

To give some sense of the differences among these priors, the prior p_3 can be used when we are reasonably sure that the possible values of q_a are tightly concentrated around μ_p ; whereas the prior p_5 considers values with a relatively higher variability around μ_p . The tests based on the proposed methodology are compared to the classical EL test (2.3) as well as the smoothed EL test proposed by Chen & Hall (1993). We present the definitions of the test statistics T1-T4, Tc, and Ts in Table 1.

We consider the quantile values $\alpha = 0.1, 0.5$, sample sizes $n = 15, 25, 30, 50, 75, 100, 150$ and $\delta = 0.6$ under d_1 and $\delta = 0.3$ under d_2 . For each scenario we have 10,000 repetitions. Table 2 presents the Monte Carlo Type I errors of the classical tests Tc and Ts, when the theoretical expected Type I error is fixed to be 0.05. The Monte Carlo Type I errors for tests T1-T4 under the distributions d_1 and d_2 are given in Table 3 and 4, respectively. Overall, the proposed tests provide better Type I error control than the classical tests when sample sizes are small. When using the prior $N(0, 1)$, which does not reflect correct information regarding the parameter q_a , or informative prior with relatively large variances (p_2), T1-T4 have better control of the Type I error than do the tests Tc and Ts. For example, in the scenario (d_2, p_2) with $\alpha = 0.1$, the classical tests do not control the Type I error unless $n \geq 75$, while the tests

T1-T4 have the Type I error under control when $n = 30$ with priors p_1 and p_2 . When the prior variance is relatively small, e.g., the priors p_3 and p_4 , under d_1 , the tests T1 and T3 do not control the Type I error for the quantile $\alpha = 0.1$, whereas T2 and T4 have better control of the Type I error than do the classical tests. For instance, in the case (d_1, p_4) with $\alpha = 0.1$, the tests T1 and T3 do not control the Type I error for any sample size, while the classical tests control the Type I error when $n = 50$. Under d_2 , the tests T1-T4 control the Type I error well for the quantile $\alpha = 0.1$, when $n = 30$. For the median, T1-T4 control the Type I error at 5% level under both d_1 and d_2 . The prior p_5 corresponds to a situation in which we mistakenly choose a prior that is focused on a point different from the true quantile under H_a ; our proposed methods are robust for testing medians in general as well as for testing the $\alpha = 0.1$ quantile with moderately large sample sizes. The prior p_6 depicts the case where we have some information about the true quantile under the alternative, but are not sure if it is positive or negative. The results show that T1-T4 control the Type I error well for testing medians in general and also for testing the quantile $\alpha = 0.1$ with sample size $n = 25$. The Monte Carlo powers of the classical tests and the tests T1-T4 with respect to distributions d_1 and d_2 are reported in Tables 5, 6 and 7 respectively. In the experiments with the non-informative prior p_1 and informative prior with relatively large variance p_2 , the tests T1-T4 do not provide significantly larger powers than the classical tests. In the cases of the informative priors with relatively small variances, e.g., p_3 and p_4 , the tests T2 and T4 provide about 15% more power than the classical tests. Even in the case where the hyperparameter μ_p is very different from the true quantile q_α under $H_a(p_5)$, the proposed methods T2 and T4 provide about 20% more power than the procedures Tc and Ts. When a more general informative prior p_6 is used, the tests T1-T4 provide about 10% more power than the classical tests.

3.2 Two Sample Tests

To evaluate the two-sample tests, we consider the following three data distributions: $d_3 : X_1 \sim \text{Norm}, X_2 \sim \text{Norm}$, $d_4 : X_1 \sim \text{LogNorm}, X_2 \sim \text{LogNorm}$, $d_5 : X_1 \sim \text{Norm}, X_2 \sim \text{LogNorm}$ and all the possible combinations of four prior functions (p_j) used in the kernel based statistics $ELBF_K, N(0, 1), N(\mu_i, 1), N(\mu_i, 0.5^2), N(2\mu_i, 0.7^2)$, where μ_i is the true quantile value of sample X_j under H_a , $i = 1, 2$.

For the details of μ_i refer to Yu et al. (2011). The prior under the null hypothesis, p_0 , is set to be $N(0, 1)$ in (2.9). The proposed tests can be compared with a test based on the well-known asymptotic properties of order statistics, i.e., since

$$\sqrt{n_1}(X_{1([\alpha n_1])} - q_0) \xrightarrow{d} N(0, \alpha(1-\alpha)/f_1^2(q_0)) \text{ and}$$

$$\sqrt{n_2}(X_{2([\alpha n_2])} - q_0) \xrightarrow{d} N(0, \alpha(1-\alpha)/f_2^2(q_0)), \text{ the alternative test can be constructed using}$$

the fact that $\text{Mood's } s = \frac{X_{1([\alpha n_1])} - X_{2([\alpha n_2])}}{\sqrt{\alpha(1-\alpha)/n_1 f_1^2(q_0) + \alpha(1-\alpha)/n_2 f_2^2(q_0)}} \xrightarrow{d} N(0, 1)$. We compare the methods based on Proposition 2.3 and Corollary 2.2 to median tests proposed in Yu et al. (2011) and the above asymptotic test. In this section, we use the definitions of the test statistics that are presented in Table 8 (Mood's, M2, K, T5 and T6).

We study the tests with the level of quantile $\alpha = 0.1, 0.5$, sample sizes $(n_1, n_2) = (15, 15), (15, 25), (100, 100), (100, 200)$. The Monte Carlo Type I errors are presented in Tables 9–10.

When using non-informative priors, e.g., $p_1 = N(0, 1)$, $p_2 = N(0, 1)$, the test statistics T5 and T6 have Type I error control comparable to M2,K for the median. For the quantile $\alpha = 0.1$, the tests T5 and T6 control the Type I error better than Mood's test does. For example, when $\alpha = 0.1$, under the distribution d_4 , the test statistic Mood's is too conservative whereas the tests T5 and T6 control the Type I error at 0.05 level when the two sample sizes are (15, 25) or bigger. When using informative priors with relatively large variance, e.g., $p_1 = N(\mu_i, 1)$, $p_2 = N(\mu_i, 1)$, the tests T5 and T6 still have the Type I error under control. When using informative priors with relatively small variance, e.g., $p_1 = N(\mu_i, 0.5^2)$, $p_2 = N(\mu_i, 0.5^2)$, the tests T5 and T6 control the Type I error under the distribution d_3 , whereas under d_4 and d_5 , both T5 and T6 control the Type I error for the quantile $\alpha = 0.1$ for large sample sizes; and when $\alpha = 0.5$ the test T6 control the Type I error for all the sample sizes. In the cases with wrong priors, e.g., $p_1 = N(2\mu_i, 0.7^2)$, $p_2 = N(2\mu_i, 0.7^2)$, the tests T5 and T6 control the Type I error under the distribution d_3 , but not under d_4 and d_5 . Table 11 and 12 present the Monte Carlo powers of the tests entitled Mood's, M2,K, T5 and T6, respectively. In the scenarios with non-informative priors, the tests T5 and T6 do not provide significantly bigger power than M2,K for the median tests. For testing $\alpha = 0.1$, the test T6 provides about 40% more power than Mood's test. When using informative priors with relatively large variances, the tests T5 and T6 provide comparable powers to M2,K for testing medians. For testing $\alpha = 0.1$, the test T6 provides about 50% more power than Mood's test. In the cases with informative priors with relatively small variances, the test T6 provides power that is comparable to M2,K under d_3 , and about 20% more power than M2,K under d_4 and d_5 for testing the median. When the "wrong" priors are used, the tests T5 and T6 provide about 15% more power than M2,K under d_3 for testing the median, and the test T6 provides about 15% more power than Mood's test.

The main conclusions from this investigation are that the methods T2 and T4 provide reliable one-sample quantile inference in terms of Type I error control and good power properties. For the two-sample quantile comparison, the method T6 performs the best among all the tests we considered.

4 Application to Chlorhexidine Gluconate Treatment on Oral Bacterial Pathogens Study

Ventilator-associated pneumonia (VAP) is a disease caused by proliferation of bacteria from the mouth region into the lung, and subsequent failure of host defenses to clear the bacteria. It is hypothesized that topical oral application of antiseptics, such as chlorhexidine gluconate (CHX), prevents oral colonization of the potential pathogens and subsequently reduces VAP. CHX is a cationic chlorophenyl bis-biguanide antiseptic that has been approved for use as an inhibitor of dental plaque formation and gingivitis. To test this hypothesis, intensive care unit, ICU, patients were randomly assigned to one of three arms: 1) a control arm 2) once daily oral topical treatment and 3) twice daily oral topical treatment (Scannapieco et al., 2009). The samples were assessed for the Clinical Pulmonary Infection Score (CPIS). CPIS is a commonly used tool for clinical estimation of VAP. It is made up of five components: temperature, blood leukocytes, tracheal secretions, oxygenation index and chest

roentgenogram. According to Zilberberg & Shorr (2010), $CPIS > 6$ is associated with VAP. In this analysis we look for decreased CPIS in the treatment group: ones or twice daily.

A total of 175 subjects are available, of which 116 were assigned to the treatment group, and 59 to the control group. 164 of the subjects stayed in the study until day 6, 109 from the treatment group and 55 from the control group. We focus on comparing the CPIS between the treatment group and control group on day 6. We also perform a paired test on CPIS between day 0 and day 6 within the treatment group and the control group.

The paired median test results are shown in Table 13. The classical EL tests T_c , T_s and the Wilcoxon test indicate that the paired medians are significantly different from 0 for the treatment group and the control group. The proposed tests T_1 - T_4 show consistent results with non-informative and informative priors. All the results suggest that the median CPIS values of the treatment and control groups are different from 0. Table 14 presents the results of a two sample test between the treatment and control groups on day 6. Because the sample medians of the control group and the treatment group are the same, Mood's test fails to detect any difference between the medians. With non-informative priors ($N(0, 1)$ and $N(0, 1)$), the tests T_5 and T_6 indicate that the medians of the control group and the treatment group are not significantly different. With informative priors ($N(6, 1)$ and $N(6, 1)$), by contrast, the test statistics T_5 and T_6 suggest rejecting the null hypothesis of no difference.

5 Discussion

In this article, we employed the empirical likelihood methodology to modify Bayes factor type procedures to the nonparametric setting. We proposed and examined a set of *ELBF* type methods for one-sample and two-sample quantile testing.

Bayes Factor type decision making procedures are known to be powerful methods based on likelihood function in comparisons between different models in Bayesian manners. In this context, asymptotic forms of Bayes Factor procedures commonly assist to compute values of the BF decision making mechanisms as well as providing a possibility to define asymptotic distributions of the BF type statistics. Then, a known way to use the BF statistics is similar to that related to the frequent testing strategies especially when tested hypotheses are nested (Kass & Raftery, 1995; Berger, 1985; Vexler et al., 2016), in which a test threshold is assumed to be fixed corresponding to a distribution of the test statistic under the null hypothesis. In the frequent perspective, Vexler and Wu (2009) and Vexler et al. (2010) showed that the BF type procedures provide integrated most powerful test statistics. We would like to mention that, when the BFs are used for decision making, their values are compared with different thresholds to make a decision related to the testing statement (Kass & Raftery, 1995). For the frequentist points of view, this strategy is also applicable, for example, when a classification problem needs to be evaluated when no null hypothesis can be defined. It is also interesting to note that Bayesian tools are different from those of frequentists, but their efficiency commonly is evaluated in a frequent manner (Carlin & Louis, 2008).

It may be of interest to compare the parametric BF and ELPF in the existing Bayesian testing procedure. With known and correct distributions, the parametric BF should perform very well. For the case of the wrong specification of distributions, we considered a simple scenario with $q_0 \sim Unif(0, 1)$ and $q_1 \sim Unif(1, 1.5)$ for 0.1-th quantile q . The parametric BF was built based on the exponential distribution with the parameter corresponding to q_0 while the observations were generated from the gamma distribution with a fixed scale parameter (0.1) and the shape parameter corresponding to q_0 . The generated BF values (25,000 simulations) were compared with the criteria indicating the strength of evidence of q_1 (Kass & Raftery, 1995). With the sample size of 25, the parametric BF supports q_0 only 74% times while the ELBF supports q_0 90% times (i.e., $BF < 1$). Some simple scenario and result described above can demonstrate the robustness of ELBF compared with the parametric BF. The complicated nature of the BF approach may require more extensive mathematical and numerical evaluations to compare the different methods in a broad perspective.

In this paper, we showed via the theoretical propositions that *ELBF* type procedures can be analyzed asymptotically in a similar fashion to the classical parametric BF schemes. Through theoretical investigations and an extensive Monte Carlo study under various scenarios, we illustrated that our proposed tests have good power characteristics while maintaining appropriate Type I error control. The simulation study further indicated that overall method T2 is the best for one-sample quantile testing and T6 is the best for two-sample quantile testing. It turns out that the technique applied by Tierney & Kadane to evaluate parametric posterior moments can be adapted to improve power characteristics of the nonparametric BF tests. According to the simulation studies, the proposed two-sample tests demonstrate clearly efficiency of the proposed methodology. Finally, we note that the problem of sensitivity to choice of priors familiar from the parametric BF (e.g., Kass, 1993; Kass & Vaidyanathan, 1992: Section 4) is also relevant to the proposed nonparametric procedures.

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Appendix A

The *EL* in an analytical form based on the order statistics

To find values of p_i that maximize $\mathcal{L}_e = \prod_{i=1}^n p_i$, given the constraints $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n p_i I\{X_i \leq q_0\} = \alpha$, we can use the Lagrange function

$$\Lambda = \sum_{i=1}^n \log p_i + \lambda_1 (1 - \sum_{i=1}^n p_i) + \lambda_2 (\alpha - \sum_{i=1}^n p_i I\{X_i \leq q_0\}),$$

where λ_1 and λ_2 are the Lagrange multipliers. Simple considerations of $\Lambda / p_i = 0$, $i = 1, \dots, n$, show that

$$p_i = (n + \lambda_2 (I\{X_i \leq q_0\} - \alpha))^{-1} \quad (\text{A.1})$$

where λ_2 is the root of the equation

$\sum_{i=1}^n p_i I\{X_i \leq q_0\} = \sum_{i=1}^n (n + \lambda_2 (I\{X_i \leq q_0\} - \alpha))^{-1} I\{X_i \leq q_0\} = \alpha$ that can be simplified as

$$\sum_{i=1}^n (n - \lambda_2 \alpha + \lambda_2)^{-1} I\{X_i \leq q_0\} = \alpha. \quad (\text{A.2})$$

Define the empirical distribution function $F_n(u) = n^{-1} \sum_{i=1}^n I\{X_i \leq u\}$. Thus by (A.2), we have $\lambda_2 = (\alpha(1 - \alpha))^{-1} n(F_n(q_0) - \alpha)$. Now, by virtue of (A.1),

$$p_i = \left[n - \frac{nF_n(q_0) - n\alpha}{(1-\alpha)} + \frac{nF_n(q_0) - n\alpha}{\alpha(1-\alpha)} I\{X_i \leq q_0\} \right]^{-1}.$$

Let $X_{(1)}, \dots, X_{(n)}$ be the order statistics based on $X_1, \dots, X_n$. Suppose k_0 is the index such that $X_{(k_0)} \leq q_0$ and $X_{(k_0+1)} > q_0$. Note that $k_0 = nF_n(q_0)$. Define

$$p_{[i]} = \left[n - \frac{nF_n(q_0) - n\alpha}{(1-\alpha)} + \frac{nF_n(q_0) - n\alpha}{\alpha(1-\alpha)} I\{X_{(i)} \leq q_0\} \right]^{-1},$$

then

$$p_{[i]} = \frac{\alpha}{nF_n(q_0)} I\{i \leq k_0\} + \frac{\alpha}{n(1-F_n(q_0))} I\{i > k_0\}.$$

Finally in this case, we can rewrite the EL as

$$\mathcal{L}_e(q_0) = \prod_{i=1}^n p_{[i]} = \prod_{i=1}^{k_0} \frac{\alpha}{nF_n(q_0)} \prod_{i=k_0+1}^n \frac{\alpha}{n(1-F_n(q_0))}$$

and

$$\begin{aligned} \log(\mathcal{L}_e(q_0)) &= nF_n(q_0)\log(\alpha) - nF_n(q_0)\log(F_n(q_0)) + n\log(1-\alpha) \\ &\quad - nF_n(q_0)\log(1-\alpha) - n\log(n) - n\log(1-F_n(q_0)) + nF_n(q_0)\log(1-F_n(q_0)). \end{aligned} \quad (\text{A.3})$$

Proof of Proposition 1

Using definition (2.4) of the $ELBF$, we have

$$2\log ELBF = 2\log \int e^{\log \mathcal{L}_e(q) - \log(n^{-n})} p(q) dq + 2\log(n^{-n} / \mathcal{L}_e(q_0)).$$

In order to find the asymptotic distribution of the statistic $2\log ELBF$, consider the $\log \mathcal{R}$, utilizing the conclusions mentioned above in the Appendix.

$$\begin{aligned} \log \mathcal{R}(q) &= \log \mathcal{L}_e(q) - \log(n^{-n}) = n[F_n(q_0)\log(\alpha) - F_n(q_0)\log(F_n(q_0)) + \log(1-\alpha) \\ &\quad - F_n(q_0)\log(1-\alpha) - \log(1-F_n(q_0)) + F_n(q_0)\log(1-F_n(q_0))]. \end{aligned}$$

In this case, we can find

$$\begin{aligned}\partial \log \mathcal{R}(q) / \partial F_n(q) &= n [\log(\alpha) - \log(F_n(q)) - \log(1-\alpha) + \log(1-F_n(q))], \\ \frac{\partial^2 \log \mathcal{R}(q)}{\partial F_n(q)^2} &= -\frac{n}{F_n(q)(1-F_n(q))}.\end{aligned}\quad (\text{A.4})$$

By setting the function $\log \mathcal{R}(q) / F_n(q)$ at (A.4) to be equal to 0, we have

$$\max_q \{\log \mathcal{R}(q)\} = \log \mathcal{R}(X_{(\lfloor \alpha n \rfloor)}) = 0, \text{ when } F_n(q) = \alpha, q = X_{(\lfloor \alpha n \rfloor)}.$$

Defining $a = \alpha n - (2\alpha(1-\alpha)n \log(n))^{0.5}$ and $b = \alpha n + (2\alpha(1-\alpha)n \log(n))^{0.5}$, we consider the numerator of the ratio $ELBF$,

$$\begin{aligned}\int_{X_{(1)}}^{X_{(n)}} e^{\log \mathcal{R}(q)} p(q) dq &= \int_{X_{(1)}}^{X_{(a)}} e^{\log \mathcal{R}(q)} p(q) dq + \int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq \\ &\quad + \int_{X_{(b)}}^{X_{(n)}} e^{\log \mathcal{R}(q)} p(q) dq.\end{aligned}$$

We will show that the integrals $\int_{X_{(1)}}^{X_{(a)}} e^{\log \mathcal{R}(q)} p(q) dq$ and $\int_{X_{(b)}}^{X_{(n)}} e^{\log \mathcal{R}(q)} p(q) dq$ are remainder terms, whereas the integral $\int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq$ is the dominant part of the numerator of $ELBF$.

To analyze $\int_{-\infty}^{X_{(a)}} e^{\log \mathcal{R}(q)} p(q) dq$, we will show that the function $\log \mathcal{R}(q)$ is increasing when $q < X_{(\lfloor \alpha n \rfloor)}$ and decreasing when $q > X_{(\lfloor \alpha n \rfloor)}$. This follows from (A.4), since the first derivative is greater than 0 when $q < X_{(\lfloor \alpha n \rfloor)}$, and less than 0 when $q > X_{(\lfloor \alpha n \rfloor)}$. Thus

$$\begin{aligned}\int_{X_{(1)}}^{X_{(a)}} e^{\log \mathcal{R}(q)} p(q) dq &\leq \int_{X_{(1)}}^{X_{(a)}} e^{\log \mathcal{R}(X_{(a)})} p(q) dq \\ &\leq e^{\log \mathcal{R}(X_{(a)})} \int_{X_{(1)}}^{X_{(a)}} p(q) dq = e^{\log \mathcal{R}(X_{(a)})}.\end{aligned}$$

Now, taking into account (A.3), one can show

$$\int_{-\infty}^{X_{(a)}} e^{\log \mathcal{R}(q)} p(q) dq \leq e^{\log \mathcal{R}(X_{(a)})} = o_p(1/n^{1-\varepsilon}) \rightarrow 0, \quad 0 < \varepsilon < 0.1, \text{ as } n \rightarrow \infty.$$

Similarly, $\int_{X_{(b)}}^{\infty} e^{\log \mathcal{R}(q)} p(q) dq \rightarrow 0$ with an order of $o_p(1/n^{1-\varepsilon})$.

Consider $\int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq$ that is a main part of the numerator of $ELBF$.

Define the function

$$\begin{aligned}L(F_n) &= n [F_n \log(\alpha) - F_n \log(F_n) + \log(1-\alpha) - F_n \log(1-\alpha) \\ &\quad - \log(1-F_n) + F_n \log(1-F_n)] = \log \mathcal{R}(q).\end{aligned}$$

Since $L(F_n)/F_n = n[\log(\alpha) - \log(F_n) - \log(1 - \alpha) + \log(1 - F_n)]$ and $\partial^2 L(F_n)/\partial F_n^2 = -n[F_n(1 - F_n)]^{-1}$, the Taylor expansion of $L(F_n)$ around α gives

$$L(F_n) = -(\alpha - F_n)^2 \frac{n}{2\alpha(1-\alpha)} + o_p(1/n^{0.5-\varepsilon}),$$

where arguments of $F_n(q)$ belong to $[X_{(a)}, X_{(b)}]$. Thus

$$\int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq = \int_{X_{(a)}}^{X_{(b)}} e^{-(\alpha - F_n(q))^2 \frac{n}{2\alpha(1-\alpha)}} p(q) dq (1 + o_p(1/n^{0.5-\varepsilon})). \quad (\text{A.5})$$

The standard method to approximate such integrals is based on the Laplace technique (e.g., Bleistein & Handelsman, 1975, pp. 180–185). This method requires $(\alpha - F_n(q))^2$ to be continuous and twice differentiable. However $F_n(q)$ is a step function. To prove Proposition 2.1, we will utilize the Bahadur theorem (see, e.g., Lemma E in Serfling, 1981, p. 97) to approximate (A.5). To this end, we write, for all q^* ,

$$F_n(q) = F(q) - \Delta(q) - F(q^*) + F_n(q^*), \quad (\text{A.6})$$

where $\Delta(q) = [(F_n(q^*) - F_n(q)) - (F(q^*) - F(q))]$. Defining

$$\Delta = \sup_{|q^* - q| \leq c_n} |(F_n(q^*) - F_n(q)) - (F(q^*) - F(q))|, \quad c_n \sim (2\alpha(1-\alpha) \log(n)/n)^{0.5},$$

by the Bahadur theorem, we have $\Delta = O_p(n^{-3/4} \log n)$, as $n \rightarrow \infty$. Thus, applying (A.6) to (A.5),

$$\begin{aligned} & \int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq \\ &= \int_{X_{(a)}}^{X_{(b)}} e^{-(\alpha - F(q) + \Delta(q) + F(q^*) - F_n(q^*))^2 \frac{n}{2\alpha(1-\alpha)}} p(q) dq (1 + o_p(1/n^{0.5-\varepsilon})). \end{aligned}$$

Using $q^* = X_{([\alpha n])}$, we obtain

$$\begin{aligned} & \int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq \\ &= \int_{X_{(a)}}^{X_{(b)}} e^{-[(F(X_{([\alpha n])}) - F(q))^2 + 2\Delta(q)(F(X_{([\alpha n])}) - F(q)) + \Delta(q)^2] \frac{n}{2\alpha(1-\alpha)}} p(q) dq (1 + o_p(1/n^{0.5-\varepsilon})). \end{aligned}$$

Since $n^{-2} = O_p(n^{-0.5} \log^2 n)$ and $n [F(X_{([an])}) - F(q)] = o_p(n^{-1/4+\varepsilon})$ as $n \rightarrow \infty$ (where $q \in [X_{(a)}, X_{(b)}]$, $F(X_{([an])}) - F(q) \sim (X_{([an])} - q)f(q_0)$, $X_{(u)} = q_0 + (u/n - F_n(q_0))/f(q_0) + O_p(n^{-3/4} \log n)$, $u \in [a, b]$, see Serfling, 1981, pp. 91–101),

$$\begin{aligned} & \int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq \\ &= \int_{X_{(a)}}^{X_{(b)}} e^{-(F(X_{([an])}) - F(q))^2 \frac{n}{2\alpha(1-\alpha)}} p(q) dq (1 + o_p(1/n^{0.25-\varepsilon})) \text{ as } n \rightarrow \infty. \end{aligned}$$

Using the Laplace method, we obtain

$$\begin{aligned} \int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq &= \frac{p(X_{([an])})}{f(X_{([an])})} \sqrt{\frac{2\pi\alpha(1-\alpha)}{n}} + o_p(1/n^{0.75-\varepsilon}) \\ &= \frac{p(q_0)}{f(q_0)} \sqrt{\frac{2\pi\alpha(1-\alpha)}{n}} + o_p(1/n^{0.75-\varepsilon}). \end{aligned}$$

This and the inequalities provide

$$\begin{aligned} 2\log\left(\int_{X_{(1)}}^{X_{(n)}} e^{\log \mathcal{R}(q)} p(q) dq\right) &\geq 2\log\left(\int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq\right), \\ 2\log\left(\int_{X_{(1)}}^{X_{(n)}} e^{\log \mathcal{R}(q)} p(q) dq\right) &\leq 2\log\left(\int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq\right) \\ &\quad + 2 \frac{\int_{X_{(1)}}^{X_{(a)}} e^{\log \mathcal{R}(q)} p(q) dq + \int_{X_{(b)}}^{X_{(n)}} e^{\log \mathcal{R}(q)} p(q) dq}{\int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq} \\ &= 2\log\left(\int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq\right) + \frac{o_p(1/n^{1-\varepsilon_1})}{o_p(1/n^{1/2-\varepsilon_2})} \\ &= 2\log\left(\int_{X_{(a)}}^{X_{(b)}} e^{\log \mathcal{R}(q)} p(q) dq\right) + o_p(1/n^{1/2-\varepsilon}), \varepsilon_1 > 0, \varepsilon_2 > 0, \end{aligned}$$

where $\varepsilon = \varepsilon_1 - \varepsilon_2$ satisfying $0 < \varepsilon < 1/2$. This completes the proof of Proposition 1.

Proof of Lemma 2.1

It is clear that the argument $\theta_M, n^{-1} \sum_{i=1}^n W(X_i, \theta_M) = 0$, maximizes the function $G(\theta)$, since, in this case, $G(\theta_M) = n^{-n}$ with $p_i = n^{-1}$, $i = 1, \dots, n$, that maximize $\prod_{i=1}^n p_i$ just under one constraint $\sum_{i=1}^n p_i = 1$, $0 \leq p_i \leq 1$, $i = 1, \dots, n$. The Lagrange method provides the form of $G(\theta)$ as

$$G(\theta) = \prod_{i=1}^n p_i, \quad 0 < p_i = \frac{1}{n + \lambda W(X_i, \theta)} < 1, \quad i = 1, \dots, n,$$

where the Lagrange multiplier λ is the root of $\sum W(X_i, \theta) (n + \lambda W(X_i, \theta))^{-1} = 0$. Thus

$$\frac{\log(G(\theta))}{d\theta} = -\lambda \sum_{i=1}^n \frac{\partial W(X_i, \theta)/\partial \theta}{n + \lambda W(X_i, \theta)} - \sum_{i=1}^n \frac{W(X_i, \theta)}{n + \lambda W(X_i, \theta)} \frac{\partial \lambda}{\partial \theta} = -\lambda \sum_{i=1}^n \frac{\partial W(X_i, \theta)/\partial \theta}{n + \lambda W(X_i, \theta)}, \quad (\text{A.7})$$

where we assume $W(X_i, \theta)/\partial \theta > 0$, $i = 1, \dots, n$ for simplicity.

Define the function $L(\lambda) = \sum_{i=1}^n W(X_i, \theta)(n + \lambda W(X_i, \theta))^{-1}$. Since $dL(\lambda)/d\lambda < 0$, the function $L(\lambda)$ decreases with respect to λ and has just one root of $L(\lambda) = 0$.

Consider situations when $\theta > \theta_M$. In this case, we set $\lambda_0 = 0$ and conclude that

$$L(\lambda_0) = \sum_{i=1}^n W(X_i, \theta)(n)^{-1} > \sum_{i=1}^n W(X_i, \theta_M)(n)^{-1} = 0.$$

Thus, since the function $L(\lambda)$ decreases, a root of $L(\lambda) = 0$ should be located on the right side from $\lambda_0 = 0$ (for details, see Figure 1) and hence this root is positive. This and (A.7) imply that the function $G(\theta)$ decreases when $\theta > \theta_M$. In a similar manner to the considerations above, one can show that the function $G(\theta)$ increases when $\theta < \theta_M$. The proof of Lemma 2.1 is complete.

Proof of Proposition 2

Note that Chen & Hall (1993) prove that $2\log(n^{-n}\mathcal{L}_e(q_0))$ is asymptotically distributed as χ^2_1 . Since Lemma 2.1 is in effect, we directly apply the Laplace method (Davison, 1986; Bleistein & Handelsman, 1975, pp. 180–185; Gelfand & Dey, 1994, pp. 506–507; see also Erkanli, 1994) to show Proposition 2.2, in a similar manner to evaluations of parametric BF's (e.g., Kass & Raftery, 1995). To this end, we derive

$$\left. \frac{\partial^k \log(\mathcal{L}_e(q)/n^{-n})}{\partial q^k} \right|_{q=q_M} = - \left(\left. \frac{\partial^{k-1} \lambda}{\partial q^{k-1}} \right|_{q=q_M} \right) \sum_{i=1}^n k_h(X_i - q_M)/n, \quad k=2, \dots,$$

where (2.6) and the fact $\lambda = 0$, if $q = q_M$ are utilized. Now, for example, by virtue of (2.7), we have

$$\frac{\partial}{\partial q} \sum_{i=1}^n \frac{K_h(X_i - q)}{(n + \lambda(K_h(X_i - q) - \alpha))} = 0 \Rightarrow \left. \frac{\partial \lambda}{\partial q} \right|_{q=q_M} = n \frac{\sum_{i=1}^n k_h(X_i - q_M)}{\sum_{i=1}^n (K_h(X_i - q_M) - \alpha)^2}$$

(here the definition of q_M , $\sum_{i=1}^n (K_h(X_i - q_M) - \alpha) = 0$ is used). This implies

$$\left. \frac{\partial^2 \log(\mathcal{L}_e(q)/n^{-n})}{\partial q^2} \right|_{q=q_M} = - \frac{(\sum_{i=1}^n k_h(X_i - q_M))^2}{\sum_{i=1}^n (K_h(X_i - q_M) - \alpha)^2}$$

that can be plugged in Equation (5.1.9) of Bleistein & Handelsman 1975 to verify Proposition 2.2.

Proof of Proposition 3

Proposition 2.3 can be proven by directly applying the proof schemes of Propositions 2.1 and (2.2). In this case, Propositions 1 and 2 of Yu et al. (2011) provide the needed asymptotic conclusions regarding the corresponding ELRs.

Figure 1.

The schematic behavior of $L(\lambda)$ plotted against λ (the axis of abscissae), when (a): $\theta > \theta_M$ and (b): $\theta < \theta_M$, respectively.

Table 1
The notations and definitions of the one sample test statistics.

Notation	Definition of the test statistics	Description
T1	$2\log ELBF - 2\log \left(\frac{p(q_0)}{f(q_0)} \left(\frac{2\pi\alpha(1-\alpha)}{n} \right)^{0.5} \right)$	Test based on Proposition 2.1
T2	$2\log ELBF - \left(\frac{p'(q_0)}{p(q_0)} \right)^2 \left(\frac{n(f(q_0))^2}{\alpha(1-\alpha)} - (\log(p(q_0)))'' \right)^{-1} - 2\log \left(p(q_0) \left(2\pi / \left(\frac{n(f(q_0))^2}{\alpha(1-\alpha)} - (\log(p(q_0)))'' \right) \right)^{0.5} \right)$	Test based on Corollary 2.1
T3	$2\log ELBF - 2\log \left(p(q_0) \left(2\pi \frac{\sum_{i=1}^n (K_h(X_i - q_0) - \alpha)^2}{\left(\sum_{i=1}^n K_h(q_0 - X_i) \right)^2} \right)^{0.5} \right)$	Test based on Proposition 2.2
T4	$2\log ELBF - \left(\frac{p'(q_0)}{p(q_0)} \right)^2 (S_n - (\log(p(q_0)))'')^{-1} - 2\log(p(q_0)(2\pi / (S_n - \log(p(q_0)))''))^{0.5}$	Test using kernel version of Corollary 2.1
Tc	$2\log(n^{-n}\mathfrak{L}_c(q_0))$	EL test (2.3)
Ts	$2\log \left(n^{-n} / \prod_{i=1}^n (n + \lambda(K_h(X_i - q_0) - \alpha))^{-1} \right)$	Smoothed EL test of Chen & Hall (1993)

Table 2

The Monte Carlo Type I errors of Tc and Ts. The expected Type I error is 5%.

n	α	$d_1: X \sim N$				$d_2: X \sim \log N$			
		Test Tc	Test Ts	Test Tc	Test Ts	Test Tc	Test Ts	Test Tc	Test Ts
15	0.1	0.236	0.214	0.219	0.143				
25		0.103	0.098	0.099	0.078				
30		0.067	0.076	0.072	0.081				
50		0.056	0.055	0.054	0.056				
75		0.031	0.051	0.031	0.051				
100		0.045	0.053	0.042	0.055				
150		0.057	0.051	0.054	0.053				
15	0.5	0.040	0.060	0.039	0.060				
25		0.045	0.057	0.040	0.050				
30		0.057	0.054	0.040	0.051				
50		0.063	0.052	0.070	0.051				
75		0.058	0.048	0.062	0.045				
100		0.056	0.050	0.059	0.052				
150		0.056	0.046	0.059	0.048				

Table 3

The Monte Carlo Type I errors of the proposed methods under d_1 . The expected Type I error is 5%.

n	α	$p_1 = N(0,1)$				$p_2 = N(0.6, (0.6)^2)$			
		T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.218	0.219	0.208	0.208	0.291	0.208	0.306	0.260
25		0.079	0.079	0.096	0.096	0.150	0.073	0.166	0.123
50		0.044	0.043	0.048	0.048	0.109	0.035	0.112	0.077
75		0.035	0.034	0.045	0.045	0.091	0.048	0.089	0.061
100		0.039	0.039	0.045	0.045	0.084	0.052	0.084	0.063
150		0.037	0.037	0.042	0.042	0.073	0.053	0.076	0.060
15	0.5	0.032	0.033	0.033	0.037	0.030	0.031	0.041	0.041
25		0.027	0.028	0.028	0.029	0.028	0.029	0.036	0.036
50		0.029	0.029	0.032	0.032	0.042	0.042	0.046	0.046
75		0.037	0.038	0.043	0.044	0.037	0.037	0.039	0.039
100		0.042	0.042	0.046	0.047	0.041	0.041	0.045	0.045
150		0.042	0.042	0.049	0.049	0.040	0.040	0.049	0.049
n	α	$p_3 = N(0.6, (0.6/1.5)^2)$				$p_4 = N(0.6, (0.6/2)^2)$			
		T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.958	0.214	0.792	0.345	1.000	0.229	0.983	0.398
25		0.878	0.082	0.691	0.214	0.998	0.091	0.983	0.284
50		0.667	0.041	0.513	0.141	0.975	0.030	0.934	0.203
75		0.500	0.058	0.389	0.118	0.934	0.044	0.865	0.169
100		0.404	0.074	0.320	0.119	0.872	0.067	0.771	0.162
150		0.276	0.074	0.225	0.093	0.754	0.101	0.635	0.145
15	0.5	0.054	0.043	0.060	0.053	0.099	0.043	0.115	0.067
25		0.045	0.040	0.055	0.048	0.114	0.055	0.119	0.074
50		0.053	0.047	0.056	0.052	0.095	0.062	0.100	0.068
75		0.057	0.052	0.064	0.060	0.086	0.059	0.089	0.068
100		0.046	0.043	0.049	0.047	0.079	0.061	0.083	0.065

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$\zeta_1 = N(0,1)$									
$p_2 = N(0,6)$									
α	T1	T2	T3	T4	T1	T2	T3	T4	T4
150	0.044	0.043	0.046	0.045	0.073	0.061	0.072	0.060	0.060
Case p_6									
$p_3 = N(1.2,(0.6)^2)$									
α	T1	T2	T3	T4	T1	T2	T3	T4	T4
15	0.1	0.643	0.214	0.462	0.290	0.200	0.201	0.208	0.209
25	0.444	0.083	0.317	0.166	0.080	0.081	0.103	0.103	0.103
50	0.220	0.039	0.185	0.088	0.043	0.043	0.056	0.057	0.057
75	0.168	0.052	0.153	0.084	0.043	0.046	0.045	0.047	0.047
100	0.135	0.059	0.125	0.076	0.036	0.038	0.048	0.050	0.050
150	0.098	0.058	0.096	0.063	0.043	0.043	0.043	0.043	0.043
$p_5 = N(0,1)$									
α	T1	T2	T3	T4	T1	T2	T3	T4	T4
15	0.5	0.082	0.055	0.092	0.068	0.034	0.049	0.038	0.056
25	0.064	0.048	0.077	0.061	0.035	0.042	0.042	0.054	0.054
50	0.054	0.046	0.061	0.057	0.041	0.050	0.048	0.053	0.053
75	0.055	0.046	0.058	0.052	0.038	0.042	0.045	0.049	0.049
100	0.048	0.043	0.052	0.050	0.037	0.044	0.048	0.050	0.050
150	0.054	0.050	0.052	0.051	0.042	0.045	0.047	0.044	0.044

Table 4

The Monte Carlo Type I errors of the proposed tests under d_2 . The expected Type I error is 5%.

n	α	$p(q)=N(0,1)$				$p(q)=N(q_1^*,(0.3)^2)$			
		T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.220	0.220	0.113	0.113	0.211	0.211	0.125	0.126
25		0.079	0.079	0.071	0.071	0.079	0.079	0.087	0.087
30		0.047	0.047	0.065	0.065	0.044	0.044	0.067	0.067
50		0.017	0.018	0.047	0.047	0.028	0.029	0.046	0.046
75		0.023	0.023	0.047	0.048	0.023	0.023	0.050	0.050
100		0.020	0.020	0.048	0.048	0.024	0.024	0.046	0.046
150		0.022	0.022	0.047	0.047	0.024	0.024	0.043	0.043
15	0.5	0.022	0.022	0.023	0.023	0.017	0.019	0.033	0.036
25		0.028	0.028	0.028	0.028	0.024	0.026	0.046	0.048
30		0.032	0.032	0.032	0.032	0.027	0.030	0.038	0.039
50		0.039	0.039	0.038	0.038	0.029	0.030	0.041	0.043
75		0.035	0.035	0.035	0.035	0.049	0.051	0.053	0.054
100		0.032	0.032	0.034	0.034	0.047	0.047	0.056	0.057
150		0.046	0.046	0.045	0.045	0.048	0.048	0.054	0.054
n	α	$p(q)=N(q_1^*,(0.3/1.2)^2)$				$p(q)=N(q_1^*,(0.3/1.5)^2)$			
		T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.210	0.210	0.132	0.131	0.209	0.209	0.149	0.140
25		0.079	0.078	0.090	0.089	0.078	0.078	0.101	0.096
30		0.044	0.044	0.071	0.071	0.043	0.043	0.083	0.078
50		0.029	0.029	0.048	0.048	0.030	0.029	0.056	0.053
75		0.024	0.023	0.052	0.052	0.029	0.025	0.058	0.056
100		0.024	0.023	0.047	0.047	0.027	0.025	0.050	0.048
150		0.025	0.025	0.043	0.043	0.026	0.026	0.046	0.045
15	0.5	0.018	0.018	0.035	0.036	0.025	0.019	0.046	0.039
25		0.028	0.028	0.051	0.050	0.047	0.033	0.069	0.056

n	α	$p(q)=N(0,1)$				$\chi^2_2(q)=N(0,1)$			
		T1	T2	T3	T4	T1	T2	T3	T4
30		0.032	0.032	0.044	0.042	0.044	0.034	0.060	0.051
50		0.036	0.033	0.046	0.046	0.051	0.041	0.067	0.051
75		0.054	0.052	0.062	0.061	0.073	0.058	0.085	0.070
100		0.054	0.052	0.062	0.060	0.071	0.062	0.086	0.072
150		0.052	0.051	0.062	0.060	0.075	0.063	0.079	0.068

n	α	$p(q)=N(q_1+0.1,(0.3)^2)$				Case P6			
		T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.212	0.212	0.118	0.119	0.210	0.211	0.132	0.134
25		0.081	0.081	0.081	0.081	0.079	0.079	0.091	0.093
30		0.047	0.047	0.065	0.066	0.044	0.044	0.072	0.074
50		0.021	0.022	0.043	0.044	0.030	0.031	0.048	0.049
75		0.023	0.023	0.047	0.047	0.024	0.025	0.052	0.053
100		0.022	0.022	0.044	0.044	0.024	0.025	0.047	0.049
150		0.022	0.022	0.044	0.044	0.025	0.025	0.044	0.044

15	0.5	0.023	0.024	0.036	0.039	0.018	0.047	0.035	0.051
25		0.030	0.032	0.045	0.048	0.028	0.062	0.051	0.072
30		0.030	0.031	0.037	0.040	0.032	0.047	0.044	0.060
50		0.033	0.034	0.036	0.037	0.036	0.050	0.046	0.056
75		0.042	0.043	0.049	0.051	0.054	0.061	0.062	0.069
100		0.041	0.041	0.042	0.044	0.054	0.061	0.062	0.069
150		0.048	0.049	0.048	0.048	0.052	0.058	0.062	0.066

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Table 5

The Monte Carlo powers of Tc and Ts.

n	α	$d_2: X \sim N$		$d_2: \log X \sim N$	
		Test Tc	Test Ts	Test Tc	Test Ts
15	0.1	0.627	0.608	0.717	0.619
25		0.476	0.495	0.579	0.603
30		0.402	0.453	0.500	0.608
50		0.558	0.579	0.706	0.809
75		0.621	0.749	0.770	0.932
100		0.809	0.865	0.934	0.982
150		0.962	0.966	0.994	0.999
15	0.5	0.380	0.481	0.094	0.139
25		0.623	0.694	0.168	0.207
30		0.698	0.759	0.187	0.228
50		0.935	0.938	0.342	0.334
75		0.987	0.988	0.483	0.487
100		0.998	0.998	0.550	0.579
150		1.000	1.000	0.735	0.753

Table 6

The Monte Carlo powers of the proposed methods under d_1 .

n	α	$p_1 = \text{prior } N(0,1)$				$p_2 = N(0.6, (0.6)^2)$			
		T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.625	0.639	0.658	0.658	0.852	0.640	0.800	0.730
25		0.454	0.456	0.526	0.525	0.866	0.490	0.749	0.634
50		0.457	0.445	0.520	0.515	0.931	0.496	0.817	0.620
75		0.634	0.623	0.680	0.699	0.966	0.775	0.910	0.743
100		0.814	0.810	0.830	0.843	0.989	0.921	0.968	0.870
150		0.960	0.958	0.964	0.966	0.997	0.989	0.996	0.969
<hr/>									
15	0.5	0.320	0.341	0.348	0.378	0.412	0.432	0.456	0.499
25		0.554	0.566	0.587	0.606	0.669	0.675	0.698	0.713
50		0.896	0.900	0.923	0.925	0.947	0.947	0.962	0.964
75		0.986	0.986	0.991	0.992	0.990	0.990	0.994	0.994
100		0.997	0.997	0.999	0.999	0.999	0.999	0.999	0.999
150		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
<hr/>									
n	α	$p_3 = N(0.6, (0.6/1.5)^2)$				$p_4 = N(0.6, (0.6/2)^2)$			
		T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.986	0.625	0.926	0.733	0.986	0.643	0.983	0.748
25		0.998	0.476	0.938	0.657	0.999	0.505	0.986	0.684
50		1.000	0.435	0.967	0.622	1.000	0.342	0.996	0.609
75		1.000	0.717	0.991	0.697	1.000	0.502	0.999	0.665
100		1.000	0.894	0.996	0.809	1.000	0.747	1.000	0.727
150		1.000	0.995	1.000	0.942	1.000	0.968	1.000	0.886
<hr/>									
15	0.5	0.558	0.508	0.579	0.570	0.740	0.530	0.713	0.595
25		0.780	0.750	0.800	0.784	0.911	0.788	0.900	0.823
50		0.974	0.970	0.978	0.972	0.991	0.981	0.989	0.977
75		0.996	0.996	0.997	0.997	0.998	0.997	0.999	0.997
100		0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.000

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n	α	$p_1 = \text{prior } N(0,1)$				$p_5 = N(1.2, (0.6)^2)$				Case p_6			
		T1	T2	T3	T4	T1	T2	T3	T4	T1	T2	T3	T4
150		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
<hr/>													
15	0.1	0.973	0.640	0.876	0.744	0.624	0.643	0.666	0.666	0.624	0.643	0.666	0.666
25		0.984	0.485	0.848	0.657	0.460	0.461	0.543	0.547	0.460	0.461	0.543	0.547
50		0.987	0.445	0.904	0.632	0.517	0.587	0.584	0.701	0.517	0.587	0.584	0.701
75		0.993	0.745	0.958	0.730	0.719	0.771	0.705	0.808	0.719	0.771	0.705	0.808
100		0.997	0.912	0.985	0.818	0.867	0.887	0.862	0.912	0.867	0.887	0.862	0.912
150		0.999	0.994	0.998	0.953	0.971	0.975	0.975	0.982	0.971	0.975	0.975	0.982
<hr/>													
15	0.5	0.715	0.591	0.703	0.622	0.345	0.449	0.383	0.517	0.345	0.449	0.383	0.517
25		0.851	0.803	0.856	0.804	0.609	0.663	0.636	0.725	0.609	0.663	0.636	0.725
50		0.979	0.973	0.982	0.972	0.912	0.928	0.928	0.944	0.912	0.928	0.928	0.944
75		0.997	0.996	0.998	0.998	0.983	0.985	0.989	0.993	0.983	0.985	0.989	0.993
100		0.999	0.999	0.999	0.999	0.996	0.996	0.997	0.998	0.996	0.996	0.997	0.998
150		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 7

The Monte Carlo powers of the proposed methods under d_2 .

		$p(q)=N(0,1)$				$p(q)=N(0.3,(0.3)^2)$			
n	α	T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.702	0.702	0.619	0.620	0.725	0.725	0.680	0.682
25		0.558	0.558	0.618	0.619	0.565	0.565	0.671	0.673
30		0.516	0.516	0.632	0.633	0.487	0.487	0.693	0.696
50		0.356	0.365	0.786	0.789	0.599	0.636	0.863	0.868
75		0.734	0.737	0.932	0.933	0.761	0.768	0.970	0.970
100		0.848	0.851	0.987	0.987	0.929	0.930	0.997	0.997
150		0.978	0.978	1.000	1.000	0.992	0.992	1.000	1.000
15	0.5	0.041	0.041	0.069	0.069	0.058	0.072	0.103	0.119
25		0.090	0.090	0.132	0.133	0.141	0.160	0.219	0.234
30		0.113	0.114	0.161	0.161	0.181	0.199	0.250	0.265
50		0.234	0.234	0.289	0.289	0.353	0.360	0.430	0.442
75		0.378	0.378	0.438	0.438	0.531	0.537	0.586	0.593
100		0.490	0.490	0.558	0.558	0.657	0.659	0.716	0.717
150		0.702	0.702	0.760	0.760	0.810	0.810	0.849	0.849
		$p(q)=N(0.3,(0.3/1.2)^2)$							
n	α	T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.710	0.710	0.680	0.677	0.721	0.721	0.725	0.705
25		0.576	0.576	0.702	0.698	0.575	0.575	0.755	0.725
30		0.501	0.501	0.716	0.711	0.524	0.524	0.791	0.760
50		0.664	0.670	0.904	0.905	0.691	0.679	0.936	0.924
75		0.809	0.802	0.978	0.979	0.888	0.839	0.986	0.984
100		0.943	0.942	0.995	0.995	0.961	0.944	0.997	0.997
150		0.992	0.991	1.000	1.000	0.995	0.993	1.000	1.000

		p(q)=N(0,1)				p(q)=N(0.3,(0.3) ²)			
n	α	T1	T2	T3	T4	T1	T2	T3	T4
15	0.5	0.063	0.077	0.117	0.125	0.076	0.069	0.140	0.120
25		0.155	0.162	0.218	0.226	0.190	0.161	0.266	0.235
30		0.217	0.221	0.291	0.299	0.245	0.210	0.335	0.297
50		0.397	0.391	0.479	0.475	0.463	0.409	0.543	0.497
75		0.563	0.554	0.628	0.622	0.631	0.578	0.695	0.656
100		0.677	0.673	0.729	0.724	0.757	0.717	0.803	0.774
150		0.841	0.836	0.873	0.871	0.884	0.872	0.902	0.892
p(q)=N(0.3,(0.6) ²)									
n	α	T1	T2	T3	T4	T1	T2	T3	T4
15	0.1	0.709	0.709	0.656	0.657	0.707	0.707	0.676	0.687
25		0.592	0.592	0.667	0.667	0.567	0.567	0.698	0.721
30		0.521	0.521	0.677	0.681	0.501	0.501	0.717	0.745
50		0.473	0.532	0.821	0.826	0.664	0.704	0.896	0.916
75		0.776	0.778	0.954	0.959	0.810	0.874	0.976	0.979
100		0.900	0.905	0.989	0.989	0.937	0.952	0.997	0.997
150		0.980	0.981	1.000	1.000	0.992	0.995	1.000	1.000
15	0.5	0.075	0.086	0.128	0.145	0.067	0.162	0.113	0.189
25		0.154	0.164	0.209	0.225	0.156	0.256	0.236	0.315
30		0.182	0.192	0.244	0.261	0.198	0.288	0.277	0.356
50		0.358	0.367	0.418	0.427	0.368	0.436	0.455	0.511
75		0.488	0.492	0.549	0.553	0.578	0.614	0.646	0.682
100		0.595	0.602	0.660	0.663	0.682	0.702	0.733	0.753
150		0.774	0.776	0.813	0.816	0.839	0.847	0.869	0.878

Table 8

The notations and definitions of the two sample test statistics

Notation	Definition of the test statistics	Description
T5	$2\log ELBF_S = \Omega + 2\log \left(\prod_{i=1}^2 \max_q (\mathcal{L}_{e_{S,i}}(q)) / \max_q (\prod_{i=1}^2 \mathcal{L}_{e_{S,i}}(q)) \right)$	Test statistic based on Proposition 2.3
T6	$2\log ELBF_K - \sum_{i=1}^2 \left(\frac{p'(q_0)}{p(q_0)} \right)^2 (D_{in_i} - (\log(p_i(q_0)))'')^{-1} \\ - 2\log ((2\pi)^{0.5} \prod_{i=1}^2 p_i(q_0) / (D_{in_i} - (\log(p_i(q_0)))'')^{0.5}) \\ + \left(\frac{p'(q_0)}{p(q_0)} \right)^2 (D_0 - (\log(p_0(q_0)))'')^{-1} \\ + 2\log(p_0(q_0) / (D_0 - (\log(p_0(q_0)))'')^{0.5})$	Test statistic based on Corollary 2.2
M2,K	$-2\log n_1^{-n_1} n_2^{-n_2} \prod_{i=1}^2 \prod_{j=1}^{n_j} p_{ij}$	Median test proposed by Yu et al. (2011)
Mood's	$\frac{X_1([\alpha n_1]) - X_2([\alpha n_2])}{\sqrt{\alpha(1-\alpha)/n_1 f_1^2(q_0) + \alpha(1-\alpha)/n_2 f_2^2(q_0)}}$	Test statistic based on the asymptotic properties of the sample quantiles

Table 9

The Monte Carlo Type I errors of Mood's and M2,K.

α	n	N(0,1) vs N(0,1)		logN(0,1) vs logN(0.986,1.769)		N(1.559,1) vs logN(0,1)	
		Mood's	M2,K	Mood's	M2,K	Mood's	M2,K
0.1	15,15	0.006		0.000		0.005	
	15,25	0.030		0.000		0.013	
	100,100	0.035		0.000		0.034	
	100,200	0.038		0.000		0.045	
α	n	N(0,1) vs N(0,1)		logN(0,1) vs logN(0.1,769)		N(1,1) vs logN(0,1)	
		Mood's	M2,K	Mood's	M2,K	Mood's	M2,K
0.5	15,15	0.013	0.034	0.029	0.033	0.021	0.034
	15,25	0.019	0.041	0.029	0.037	0.017	0.038
	100,100	0.034	0.038	0.054	0.041	0.039	0.039
	100,200	0.035	0.042	0.064	0.047	0.042	0.042

Table 10

The Monte Carlo Type I errors of T5 and T6.

n_1, n_2	α	T5	T6	T5	T6	T5	T6	T5	T6
		$p1:N(0,1), p2:N(0,1)$		$p1:N(-1.28,1), p2:N(-0.58,1)$		$p1:N(-1.28,0.5), p2:N(-0.58,0.5)$		$p1:N(-2.56,0.7), p2:N(-1.16,0.7)$	
15,15	0.1	0.031	0.055	0.033	0.061	0.029	0.033	0.044	0.047
		$x1 \sim N(0,1)$ $x2 \sim N(0,1)$							
15,25		0.035	0.048	0.034	0.061	0.018	0.028	0.048	0.038
100,100		0.026	0.026	0.028	0.029	0.024	0.021	0.039	0.035
100,200		0.026	0.025	0.032	0.032	0.031	0.029	0.049	0.043
		$p1:N(0,1), p2:N(0,1)$		$p1:N(0,1), p2:N(0.7,1)$		$p1:N(0,0.5), p2:N(0.7,0.5)$		$p1:N(0,0.7), p2:N(1.4,0.7)$	
15,15	0.5	0.011	0.013	0.012	0.014	0.017	0.011	0.039	0.025
		$x1 \sim N(0,1)$ $x2 \sim N(0,1)$							
15,25		0.014	0.016	0.019	0.021	0.017	0.010	0.029	0.021
100,100		0.035	0.036	0.036	0.036	0.036	0.033	0.044	0.039
100,200		0.040	0.040	0.038	0.038	0.035	0.034	0.036	0.035
		$p1:N(0,1), p2:N(0,1)$		$p1:N(0.28,1), p2:N(1.29,1)$		$p1:N(0.28,0.5), p2:N(1.29,0.5)$		$p1:N(0.28,0.7), p2:N(1.29,0.7)$	
15,15	0.1	0.069	0.093	0.120	0.134	0.258	0.225	0.576	0.453
		$x1 \sim \log N(0,1)$ $x2 \sim \log N(0.99,1.77)$							
15,25		0.049	0.051	0.074	0.077	0.170	0.142	0.364	0.285
100,100		0.049	0.049	0.055	0.055	0.073	0.065	0.090	0.085
100,200		0.047	0.047	0.049	0.049	0.070	0.068	0.075	0.071
		$p1:N(0,1), p2:N(0,1)$		$p1:N(1,1), p2:N(2.01,1)$		$p1:N(1,0.5), p2:N(2.01,0.5)$		$p1:N(1,0.7), p2:N(2.01,0.7)$	
15,15	0.5	0.003	0.003	0.038	0.048	0.155	0.049	0.965	0.604
		$x1 \sim \log N(0,1)$ $x2 \sim \log N(0.1,1.77)$							
15,25		0.008	0.008	0.035	0.044	0.152	0.060	0.959	0.627
100,100		0.025	0.025	0.056	0.057	0.133	0.089	0.558	0.343
100,200		0.033	0.033	0.047	0.047	0.092	0.072	0.294	0.186
		$p1:N(0,1), p2:N(0,1)$		$p1:N(1.29,1), p2:N(0.28,1)$		$p1:N(1.29,0.5), p2:N(0.28,0.5)$		$p1:N(1.29,0.7), p2:N(0.27,0.7)$	
15,15	0.1	0.078	0.114	0.087	0.119	0.156	0.151	0.241	0.176
		$x1 \sim N(1.559,1)$ $x2 \sim \log N(0,1)$							

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n_1, n_2	α	T_5	T_6	T_5	T_6	T_5	T_6	T_5	T_6	T_5	T_6
15,25	0.5	0.127	0.155	0.122	0.141	0.190	0.184	0.260	0.197	$p1:N(1,1), p2:N(0,1)$	$p1:N(1,1), p2:N(0,1)$
100,100		0.045	0.045	0.045	0.045	0.061	0.052	0.072	0.050	$p1:N(1,1), p2:N(0,1)$	$p1:N(1,1), p2:N(0,1)$
100,200		0.034	0.034	0.043	0.043	0.063	0.053	0.072	0.050	$p1:N(1,1), p2:N(0,1)$	$p1:N(1,1), p2:N(0,1)$
15,15	0.5	0.015	0.015	0.042	0.047	0.150	0.075	0.893	0.607	$p1:N(1,1), p2:N(0,1)$	$p1:N(1,1), p2:N(0,1)$
15,25		0.017	0.019	0.030	0.036	0.117	0.072	0.726	0.448	$p1:N(1,1), p2:N(0,1)$	$p1:N(1,1), p2:N(0,1)$
100,100		0.035	0.035	0.056	0.057	0.071	0.062	0.163	0.110	$p1:N(1,1), p2:N(0,1)$	$p1:N(1,1), p2:N(0,1)$
100,200		0.038	0.038	0.042	0.042	0.058	0.053	0.088	0.069	$p1:N(1,1), p2:N(0,1)$	$p1:N(1,1), p2:N(0,1)$

Table 11

The Monte Carlo powers of Mood's and M2.K.

n_1, n_2	α	N(0,1) vs N(0,1)		logN(0,1) vs logN(2.523,1.769)		N(2.573,1) vs logN(0,1)	
		Mood's	M2.K	Mood's	M2.K	Mood's	M2.K
15,15	0.1	0.013		0.000		0.001	
15,25		0.040		0.000		0.002	
100,100		0.554		0.000		0.099	
100,200		0.930		0.001		0.041	
n_1, n_2	α	N(0,1) vs N(0,1)		logN(0,1) vs logN(0.7,1.769)		N(1,1) vs logN(0.7,1)	
		Mood's	M2.K	Mood's	M2.K	Mood's	M2.K
15,15	0.5	0.173	0.299	0.065	0.191	0.228	0.336
15,25		0.139	0.382	0.039	0.280	0.181	0.474
100,100		0.967	0.978	0.712	0.916	0.986	0.991
100,200		0.990	0.997	0.943	0.984	1.000	1.000

Table 12

The Monte Carlo powers of T5 and T6.

n1,n2	α	T5	T6	T5	T6	T5	T6	T5	T6	
15,15	0.1	$x_1 \sim N(0,1)$ $x_2 \sim N(0,1)$	$p_1:N(0,1), p_2:N(0,1)$	0.049	0.255	0.027	0.262	$p_1:N(-1.28,1), p_2:N(-0.58,1)$	$p_1:N(-1.28,0.5), p_2:N(-0.58,0.5)$	$p_1:N(-2.56,0.7), p_2:N(-1.16,0.7)$
15,25				0.063	0.172	0.054	0.179		0.146	
100,100				0.759	0.784	0.674	0.735		0.762	
100,200				0.952	0.952	0.902	0.912		0.939	
			$p_1:N(0,1), p_2:N(0,1)$			$p_1:N(0,1), p_2:N(0,1)$		$p_1:N(0,0.5), p_2:N(0.7,0.5)$		$p_1:N(0,0.7), p_2:N(1.4,0.7)$
15,15	0.5	$x_1 \sim N(0,1)$ $x_2 \sim N(0,1)$		0.139	0.179	0.229	0.270	0.307	0.233	0.544
15,25				0.236	0.268	0.311	0.346	0.460	0.386	0.651
100,100				0.984	0.984	0.989	0.989	0.994	0.994	0.998
100,200				0.992	0.992	0.997	0.997	0.999	0.999	0.999
			$p_1:N(0,1), p_2:N(0,1)$			$p_1:N(0,28,1), p_2:N(1.29,1)$		$p_1:N(0,28,0.5), p_2:N(1.29,0.5)$		$p_1:N(0,28,0.7), p_2:N(1.29,0.7)$
15,15	0.1	$x_1 \sim \log N(0,1)$ $x_2 \sim \log N(0.99,1.77)$		0.193	0.601	0.360	0.771	0.502	0.810	0.682
15,25				0.270	0.534	0.480	0.742	0.674	0.815	0.820
100,100				0.994	0.995	0.999	0.999	1.000	1.000	1.000
100,200				1.000	1.000	1.000	1.000	1.000	1.000	1.000
			$p_1:N(0,1), p_2:N(0,1)$			$p_1:N(1,1), p_2:N(2.01,1)$		$p_1:N(1,0.5), p_2:N(2.01,0.5)$		$p_1:N(1,0.7), p_2:N(2.01,0.7)$
15,15	0.5	$x_1 \sim \log N(0,1)$ $x_2 \sim \log N(0.1,7.7)$		0.010	0.012	0.211	0.269	0.436	0.203	0.981
15,25				0.028	0.030	0.352	0.414	0.621	0.345	0.996
100,100				0.673	0.674	0.920	0.922	0.987	0.960	1.000
100,200				0.909	0.909	0.989	0.989	0.999	0.998	1.000
			$p_1:N(0,1), p_2:N(0,1)$			$p_1:N(1.29,1), p_2:N(0.28,1)$		$p_1:N(1.29,0.5), p_2:N(0.28,0.5)$		$p_1:N(1.29,0.7), p_2:N(0.28,0.7)$
15,15	0.1	$x_1 \sim N(1.56,1)$ $x_2 \sim \log N(0,1)$		0.145	0.720	0.207	0.778	0.325	0.791	0.448
										0.790

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n_1, n_2	σ	$p_1: N(0, 1), p_2: N(0, 1)$	$p_1: N(1, 1), p_2: N(2, 0.1, 1)$	$p_1: N(1, 0.5), p_2: N(2, 0.1, 0.5)$	$p_1: N(1, 0.7), p_2: N(2, 0.1, 0.7)$	T5	T6	T5	T6	T5	T6	T5	T6
15, 25	0.5	$x_1 \sim N(1, 1)$ $x_2 \sim \log N(0, 1)$	0.003	0.063	0.574	0.654	0.801	0.614	0.998	0.587	0.845	0.952	0.864
100, 100			0.020	0.134	0.736	0.788	0.901	0.806	0.998	0.967	0.984	0.952	0.944
100, 100			0.770	0.919	0.999	0.999	1.000	1.000	1.000	0.963	0.987	0.952	0.944
100, 200			0.965	0.991	1.000	1.000	1.000	1.000	1.000	0.963	0.987	0.952	0.944

Table 13

The paired median test of the treatment group and the control group.

CPIS of Treatment group					CPIS of control group				
prior N(0,1)					prior N(0,1)				
$\alpha=0.5$	T1	T2	T3	T4	$\alpha=0.5$	T1	T2	T3	T4
Test statistic	12.793	12.830	100.827	100.855	Test statistic	12.793	12.830	100.827	100.855
p-value	0.000348	0.000341	<0.0001	<0.0001	p-value	0.000348	0.000341	<0.0001	<0.0001
prior N(6,1)					prior N(6,1)				
	T1	T2	T3	T4		T1	T2	T3	T4
Test statistic	32.047	30.747	118.428	117.464	Test statistic	11.276	9.946	48.047	47.187
p-value	<0.0001	<0.0001	<0.0001	<0.0001	p-value	0.000785	0.00161	<0.0001	<0.0001
Tc					Tc				
Ts					Ts				
Wilcox					Wilcox				
Test statistic	139.7326	102.47	519.5		Test statistic	64.90056	37.35683	519.5	
p-value	<0.0001	<0.0001	<0.0001		p-value	<0.0001	<0.0001	<0.0001	

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Table 14

The two sample median test.

$\alpha=0.5$	N(0,1) vs N(0,1)		N(6,1) vs N(6,1)		Mood's
	T5	T6	T5	T6	
Test statistic	1.156	1.101	9.058	9.061	0.000
p-value	0.282	0.294	0.003	0.003	0.500