Published in final edited form as:

J Behav Decis Mak. 2017 April ; 30(2): 527–532. doi:10.1002/bdm.1969.

Alternative Lens Model Equations for Dichotomous Judgments about Dichotomous Criteria

Robert M. Hamm, Ph.D.¹ and Huiqin Yang, Ph.D.²

¹Department of Family and Preventive Medicine, University of Oklahoma, USA

²Centre for Reviews and Dissemination, University of York, UK

Abstract

Objective—The Brunswik lens model typically represents a judge's accuracy using parameters derived from linear regression. This is not optimal if the judgment or the ecological criterion is dichotomous. Alternative approaches, modeling dichotomies using logistic regression, or linearizing judgments with confidence ratings, have not been compared with the same data.

Method—Four techniques for deriving lens model equation parameters were compared: 1) linear and 2) logistic regression applied to dichotomous patient outcomes and judgments; 3) linear regression with confidence-adjusted judgments but dichotomous patient outcomes; and 4) a hybrid with a linear model of the confidence-adjusted judgments and a logistic model of the patient outcomes.

Results—Judgment accuracy (r_a) was slightly higher with confidence adjustment of the categorical judgments. The logistic lens model accounted for a higher proportion of r_a than the linear lens model; the confident-linear and hybrid lens models were intermediate. For up to a quarter of participants, different methods identified different cues as most important. Display condition differences in achievement r_a and in lens model components are similar with all lens model methods.

Conclusion—Each of the three alternative lens model equation methods improves on the linear lens model equation's decomposition of the accuracy of dichotomous judgments. Confidence adjustment improves achievement although it requires additional work from the subjects. The logistic lens model equation explains the highest proportion of achievement, but with a small stimulus set it is more vulnerable to cue intercorrelations than either the linear or the confident linear lens model equation.

INTRODUCTION

The Brunswik Lens Model is a well-established general theory for analysis of the accuracy and variability of judgments made in environments where perceivable cues are probabilistically related to the criterion state of the world (Brunswik, 1952). The theory, embodied in the lens model equation (Hammond, Hursch, & Todd, 1964; Stewart, 1988; Tucker, 1964), accounts for accuracy in terms of the relative weight that a judge accords to different cues (compared to the relative weight those same features have in predicting the ecological criterion) and the consistency of the judge's use of the cues.

Most research with the lens model equation has used linear regression to model the relations between perceivable cues and people's judgments, and between the cues and the ecology's criterion (Cooksey, 1996; see Table 1 and Supplement B1; Hammond & Stewart, 2001; Stewart, 1988). It is appropriate when the criterion and the judgment are measured on continuous scales. The linear lens model has also been applied to dichotomous ecological criteria and judgments; as noted by Cooksey (1996) this is inelegant and potentially inaccurate.

Those studying dichotomous judgments have the choice to modify the task so that it can appropriately be modeled with linear regression, or to use a more general form of the lens model equation that is capable of handling prediction models produced by methods other than the general linear model (Cooksey, 1996; Stewart, 2004). The judgment task could be modified by asking the judges, in addition to their dichotomous response, to state how probable the diagnostic category they chose is, how likely they are to take the action, or how confident they are that their choice is correct. Similarly, in addition to each case's actual category membership, an objective model could be used to measure how prototypical it is. With such modifications in procedure, the standard linear regression lens model could be applied (see Table 1 and Supplement B). In our study, we modified the categorical judgments using the judge's stated confidence in each judgment.

$$Y_{dcs} = \frac{(1 - Y_s) * (100 - C_s) + Y_s (100 + C_s)}{200} \quad \text{Equation 1}$$

This distributes the binary judgments over a linear scale; the confidence-adjusted judgments may be modelled using linear regression. Though this linearizes the judgment part of the lens model, the ecological criterion remains dichotomous, still awkwardly handled by the linear lens model.

The more general lens model equation accommodates the use of logistic regression, which produces models that estimate the probability of a category, and in theory would be superior for analyzing dichotomous judgments or dichotomous ecological criteria. Cooksey (1996, p. 299) recognized that three C terms would be needed for such a logistic lens model, reflecting the correlation between the residuals of the ecology and judgment models, as in the linear lens model, and also the cross correlations between the prediction of one model and the residual of the other: $C_1 = r(\tilde{Z}_e, \tilde{Z}_s)$, $C_2 = r(\tilde{Y}_e, \tilde{Z}_s)$, $C_3 = r(\tilde{Z}_e, \tilde{Y}_s)$ (see Table 1). With logistic regression, unlike with linear, the best fit predictions are produced in the log(odds) scale, $\log[p/(1-p)]$, but then they are rescaled to probabilities p using the logistic function $1/(1+e^{-\log(\text{odds})})$ before the lens model correlations are calculated. Due to this rescaling, the predictions of a variable -- the probabilities the patients are members of a category -- are not necessarily completely uncorrelated with their residuals, nor with the residuals from the prediction of a different variable by the same set of predictors, as they would be with linear regression. Stewart (1976, 2004) showed further that it is necessary to

replace the R_e measuring ecological predictability with $\frac{\sigma_{\tilde{Y}_e}}{\sigma_{Y_e}}$ and the $\sqrt{1-R_e^2}$ with $\frac{\sigma_{\tilde{Z}_e}}{\sigma_{Y_e}}$, and

similarly for the R_s measuring cognitive consistency. Therefore the logistic lens model equation is this covariance decomposition:

$$r_a = G \frac{\sigma_{\hat{Y}_e} \sigma_{\hat{Y}_s}}{\sigma_{Y_e} \sigma_{Y_s}} + C_1 \frac{\sigma_{\hat{Z}_e} \sigma_{\hat{Z}_s}}{\sigma_{Y_e} \sigma_{Y_s}} + C_2 \frac{\sigma_{\hat{Y}_e} \sigma_{\hat{Z}_s}}{\sigma_{Y_e} \sigma_{Y_s}} + C_3 \frac{\sigma_{\hat{Z}_e} \sigma_{\hat{Y}_s}}{\sigma_{Y_e} \sigma_{Y_s}}. \quad \text{Equation 2}$$

The general lens model equation allows construction of a hybrid lens model, using logistic regression to predict the dichotomous ecological criterion and linear regression to predict the confidence adjusted judgments. This sacrifices the desirable use of the same form of model to describe both the ecology and the judgment. Such a hybrid approach was suggested by Stewart (1976) and Cooksey (1996), and Beckstead (2015) has applied it to two data sets.

There are few published examples of logistic lens model equations with dichotomous judgments and criteria, and no examples using confidence ratings to linearize dichotomous judgments so they may be incorporated in a lens model with linear regression. Therefore it is not known whether either has any practical advantages over linear regression for the lens model equation analysis of dichotomous categorical judgments. The contribution of this paper is to compare these four different methods of deriving the lens model equation parameters.

We constructed lens models based on linearizing dichotomous judgments using confidence ratings, and on logistic regression, and compared them with the linear lens model. We used data from Yang's dissertation (2009), nurses' judgments predicting whether or not hospital patients, described with a few clinical signs, will soon have a crisis; the judgments' accuracy is determined with reference to data on the hospital history of those patients. In addition to comparing the accuracy of alternative lens model approaches, we looked at whether the lens model method had an impact on the study results: do descriptions of characteristics of individual nurses' judgments and comparisons between those characteristics when the nurses judged the patients from the paper and simulation stimuli, or between nurses and students, vary as a function of the four alternative methods for producing the lens model equations? This question is relevant for those who might consider which method to use for their own data analyses: does one method better describe the data, or more often reveal differences between conditions?

METHODS

Each nurse participant assessed the risk of clinical deterioration in a set of patient scenarios, as described previously (Yang & Thompson, 2011; Yang, Thompson, Hamm, Bland, & Foster, 2013), once when patients were described with cues listed on paper and again when embodied in a manikin controlled by a patient simulator computer. They stated their confidence in each assessment. Relevant details are summarized in Supplement A.

DATA ANALYSIS

Regression models were fit to the actual outcomes of the 20 cases and to each nurse's judgments, as a function of four case cues. This is a different method for analyzing the

accuracy of these judgments than reported by Thompson, Yang, and Crouch (2012). Logistic and linear models were fit to the dichotomous outcomes and judgments, and linear models were fit to the confidence adjusted judgments (see Table 1). Estimates of the cues' relative weights were computed using the regression parameters. Lens model parameters were produced using correlations among the actual data, the models' predictions, and the residuals; or ratios among these variables' standard deviations, as detailed in Supplement B. Because the behavior of the parameters of the various lens model equations have not previously been compared, they are provided in Supplement C1. To assess the impact of the different modeling techniques, the effects of nursing experience (nurse versus student) and of task perceptual representativeness (paper cases versus manikin simulation), as described by each lens model method, were compared (Supplement C2).

Comparisons of Lens Model Equation Parameters

The empirical focus of judgment studies using the lens model is the size of the lens model parameters and whether the parameters and judge-ecology relations differ between conditions or groups, such as between the paper case and manikin simulation tasks (Yang, Thompson, & Bland, 2012b; Yang et al., 2013), or between nurses and students (Yang & Thompson, 2010, 2011; Yang, Thompson, & Bland, 2012a). This paper's methodological question is whether those empirical comparisons differ with different approaches to producing the lens model equations. To address this, we compared the elemental lens model

equation parameters (r_a , G , C_p , R_s , $\frac{\sigma_{\tilde{Y}_s}}{\sigma_{Y_s}}$, $\frac{\sigma_{\tilde{Z}_s}}{\sigma_{Y_s}}$, and the cues' relative weights) and the

combinations, the G -Product (GR_eR_s or $G\frac{\sigma_{\tilde{Y}_e}}{\sigma_{Y_e}}\frac{\sigma_{\tilde{Y}_s}}{\sigma_{Y_s}}$) and the C -Product sum (composed of all the C -Products in the equations), of nurses' risk assessments among four lens model approaches. We also inspect the different C_p s and C_f -Products (Supplement C3).

RESULTS

The tasks were completed by 97 (34 experienced and 63 student) UK nurses (Yang et al., 2013). Participants correctly categorized on average 15.3 of the 20 paper cases as high versus low risk, and 14.8 of the same cases when presented as simulations.

Judgment Accuracy and Lens Model Equation Parameters

Method comparisons—Figure 1 summarizes the participants' lens model equation parameters for judgments of paper cases and manikin simulations, for the linear, logistic, and confident linear lens model equations and for the hybrid lens model equation comparing the logistic ecology model and linear confidence-adjusted judgment model (see also tables C1–C4 in Supplement C). The judgment accuracy (r_a) was slightly, but statistically significantly, higher when the categorical “at risk/not at risk” judgment was adjusted considering the nurse's confidence. Adjusting the risk judgments for the paper cases increased the r_a from 0.552 to 0.570 ($t(96) = 4.28$, $p < .001$). The mean physical simulation judgment r_a increased from 0.503 to 0.535 ($t(96) = 8.73$, $p < .001$). Judgment accuracy with the manikin simulations was significantly less than with paper cases for both the dichotomous judgments (0.503 versus 0.552; $t(96) = 2.78$, $p = 0.006$) and the confidence-

adjusted judgments (0.535 versus 0.570; $t(96) = 2.14$, $p = 0.036$), as reported previously (Yang et al., 2013).

Do all four variants of the lens model equation produce the same decomposition-based explanation of these differences? The additive decomposition into the G-Product (the agreement between the model of the participant's knowledge and the model of the ecology) and the C-Product or C-Product-Sum (the portion that has not been captured by both models) is shown in the shaded bars in Figure 1. Lens model equations relating the dichotomous assessment to the dichotomous ecological criterion using logistic regression accounted for a higher proportion of the nurses' achievement r_a than using linear regression (paper cases: 0.906 versus 0.832, $t(96) = 10.1$, $p < .001$; manikin simulations: 0.917 versus 0.860, $t(96) = 5.9$, $p < .001$).

As noted above, adding information to the dichotomous judgment by adjusting for the nurse's confidence increased the achievement, r_a . The proportion of the r_a explained by the confidence-adjusted linear lens model is greater than the proportion the linear lens model equation explains for judgments of paper cases (0.858 versus 0.832, $t(96) = 3.1$, $p = 0.002$) but not for judgments of the manikin cases (0.870 versus 0.860, $t(96) = 1.0$, $p = .33$). However, the proportion of achievement the lens models of confidence-adjusted judgments explain is less than the proportion explained by the logistic lens models (paper: 0.858 versus 0.906, $t(96) = 4.3$, $p < 0.001$; physical: 0.870 versus 0.917, $t(96) = 3.47$, $p = 0.001$).

Task condition and respondent group comparisons—The study was done (Yang, 2009) to compare achievement between the paper case judgments and the manikin simulation judgments. The various components of r_a for the two task conditions are compared for each method in Tables C1–C4 of Supplement C1. The task differences in achievement r_a and knowledge G differ comparably with all 4 lens model methods. The study also compared the decomposition of achievement between experienced nurses and students (Supplement C2). No differences were found, with any of the methods.

Relative weights—Comparisons between the relative weights produced by logistic and linear regression analyses of the ecological criterion and the dichotomous and confidence-adjusted judgments are provided in Supplement C4. While respiration rate was the most important cue in both ecology models, in their judgment models only 43 to 62 of the 97 participants gave it the most weight (depending on method; see Table C7 in Supplement C4). Thus the choice of lens modeling method could affect conclusions about how many participants use qualitatively similar judgment policies in different settings. In comparing modeling methods for exactly the same judgment data, the number of individuals where two methods identified the same cue as most important ranged from 74 to 90 of 97 (bottom two rows of Table C8). Thus for up to a quarter of participants, the choice of lens model equation method could affect which cue is identified as most important.

DISCUSSION

The linear regression based lens model equation (Hammond et al., 1964; Tucker, 1964) has been the standard for 50 years, but is inelegant for predicting binary categories. We

demonstrated and compared three alternative approaches for dealing with dichotomous judgments or dichotomous ecological criteria, which employ logistic regression and modification of the categorical judgment according to rated confidence to produce a continuous measure.

To use logistic regression, the applicable lens model formula includes the 4 component decomposition described by Cooksey (1996, p. 299), modified by substituting ratios of standard deviations for R_e and R_s correlations, as described by Stewart (1976, p. 115). In addition to Cooksey's (1996) illustration, the full logistic lens model equation has been applied recently (Adderley & Thompson, 2015), including with the present data (Yang et al., 2013). The present paper is the first to compare the logistic and linear lens model equations for the same data.

Although confidence judgments have been elicited in policy capturing studies (e.g., Yaniv & Choshen-Hillel, 2012) and Brunswikian lens model studies (e.g., Yang et al., 2012a, 2012b), and Brunswik's theory has been used to explain judgments of confidence (Juslin & Olsson, 1997), this is the first lens model study to use Dalglish's formula to adjust dichotomous judgments according to their confidence in order to derive a continuous judgment variable to which linear regression can be applied.

The general form of the Brunswik lens model that accommodates logistic regression when both the criterion and the judgment are dichotomous also works when one is dichotomous and the other continuous. Beckstead (2015) has used the hybrid lens model in two studies, to compare each nurse's judgments of patient risk with the same nurse's decision whether to refer. The present study is the first to apply the hybrid model with a dichotomous ecological criterion and (confidence-adjusted) linear judgment.

Logistic compared to linear regression—Using logistic regression, theoretically appropriate for dichotomous dependent variables, explained more of the nurses' dichotomous judgments. For paper cases, for example, it increased the median G -product

$G \frac{\sigma_{\hat{Y}_e} \sigma_{\hat{Y}_s}}{\sigma_{Y_e} \sigma_{Y_s}}$ from 0.431 (79.1% of the r_a) to 0.481 (86.9%). Although the logistic lens model formula includes three C correlations with their corresponding variance-weighted C -products, the total proportion of all these terms was less than the C -product

$C \sqrt{1-R_e^2} \sqrt{1-R_s^2}$ in the linear lens model equation. Further, the unmodeled components no longer contributed to the explanation of the difference between judgment accuracy when judging cases represented in manikin simulations as compared to paper descriptions. However, fitting logistic models required more attention, manual dropping of predictors for many of the participants, due to the degree of cue intercorrelation and the small number of stimuli.

Confidence adjustment of dichotomous judgments—Making the judgment variable continuous by considering the nurse's stated confidence in each judgment raised the r_a s (e.g., from a median of 0.577 to 0.585 for the paper cases; and more for the manikin simulations; a statistically significant but very small difference). If there were a standard

which an individual's achievement must achieve, say in an evaluative context, then using confidence adjustment could change the proportion of people meeting the standard. Additionally, the choice of method can influence which cue is identified as being most important. In the confidence-adjusted lens model equation, the median G -Product increased from the linear model's 0.431 (79.1% of the r_a) to 0.466 (83.2%), about the same as using logistic regression. Thus, in this study, asking respondents for confidence ratings about each judgment utilizes some of their knowledge not expressed in their categorical judgments, increasing the overall accuracy (r_a), but it does not increase the amount that is modeled (proportion of r_a).

Hybrid lens model—It is theoretically most appropriate to apply logistic regression to the dichotomous ecological criterion, but linear regression to the continuous confidence adjusted judgments. However, the G -product parameters from such hybrid lens model equations were identical to those for the confidence-adjusted linear lens model equations. This may be understood from recognizing that the only difference between the two is in whether the ecological criterion is predicted with logistic or linear regression (Table 1), and these predictions correlated 0.922. Thus using the hybrid model made no practical difference with these data.

In sum, the impacts of the logistic regression and confidence-adjustment approaches for making lens models of dichotomous data, compared to applying linear regression, are relatively small, though statistically significant in this study with a relatively large number of participants (97). Rather than applying the standard linear lens model equation with dichotomous data, researchers could use logistic regression to explain a higher proportion of the r_a or elicit confidence judgments to increase the r_a . Researchers can pick the method according to their judgment of the theoretical and analytical aesthetics and their preference for easy analysis or limited participant effort.

Limitations—Because we compared these alternative lens model equations for judgments of dichotomous concepts using just one data set, with only 20 cases and with a different proportion of the two classes in the sample than in the ecology, as discussed above, we don't know how generalizable our findings are. Fitting logistic models using this small stimulus set with its correlated cues required some individualized adjustments; the impact of this on model parameters is not known. We suspect *ad hoc* adjustments would not be necessary if there were more stimuli or if the cue distributions were different. In his two studies, Beckstead (2015) used a 7-cue task with 100 cases, and an 8-cue task with 70 cases; the only difficulty finding unique logistic regression models was with nurses who only made one response, or who relied on just one cue.

Supplementary Material

Refer to Web version on PubMed Central for supplementary material.

Acknowledgments

We appreciate the contributions of Carl Thompson, Martin Bland, and Alison Foster on the original project. We thank Christian P. Subbe for consenting for us to use his dataset for our model of the ecology and as the source for

the simulated cases. Stacy Wigley, Michelle Ward, and Jenny Bauer entered data in support of the lens model comparisons. The collection of data was funded by the Department of Health Sciences of University of York, United Kingdom. We appreciate comments from Thomas R. Stewart, the late Lenard I. Dalgleish, Olga Kostopoulou, Robin M. Hogarth, George Wright, Dale Barr, and 5 anonymous reviewers.

References

- Adderley UJ, Thompson C. Community nurses' judgement for the management of venous leg ulceration: A judgement analysis. *Int J Nurs Stud.* 2015; 52(1):345–354. DOI: 10.1016/j.ijnurstu.2014.09.00425277880 [PubMed: 25277880]
- Beckstead, JW. The bifocal lens model and equation: Examining the linkage between clinical judgments and decisions. University of South Florida College of Nursing; Tampa, FL: 2015.
- Brunswik, E. The conceptual framework of psychology. Chicago: University of Chicago Press; 1952.
- Cooksey, RW. Judgment analysis: Theory, methods, and applications. San Diego, CA: Academic Press; 1996.
- Hammond KR, Hursch CJ, Todd FJ. Analyzing the components of clinical inference. *Psychol Rev.* 1964; 71:438–456. [PubMed: 14216893]
- Hammond, KR., Stewart, TR. Introduction. In: Hammond, KR., Stewart, TR., editors. *The essential Brunswik: Beginnings, explications, applications.* Oxford: Oxford University Press; 2001. p. 3-11.
- Juslin P, Olsson H. Thurstonian and Brunswikian origins of uncertainty in judgment: A sampling model of confidence in sensory discrimination. *Psychol Rev.* 1997; 104(2):344–366. DOI: 10.1037/0033-295X.104.2.3449162950 [PubMed: 9162950]
- Stewart TR. Components of correlation and extensions of the lens model equation. *Psychometrika.* 1976; 41:101–120.
- Stewart, TR. Judgment analysis: Procedures. In: Brehmer, B., Joyce, CRB., editors. *Human Judgment: The SJT View.* Oxford: North-Holland; 1988. p. 41-74.
- Stewart, TR. Notes on a form of the lens model equation for logistic regression analysis. Paper presented at the 20th Annual International Meeting of the Brunswik Society; Minneapolis, MN. 2004.
- Thompson C, Yang H, Crouch S. Clinical simulation fidelity and nurses' identification of critical event risk: a signal detection analysis. *J Adv Nurs.* 2012; 68(11):2477–2485. DOI: 10.1111/j.1365-2648.2012.05945.x [PubMed: 22309309]
- Tucker LR. A suggested alternative formulation in the developments by Hursch, Hammond, and Hursch, and by Hammond, Hursch, and Todd. *Psychol Rev.* 1964; 71:528–530. [PubMed: 14216901]
- Yang, H. PhD. University of York; 2009. *The Effects of Improved Representative Design on Nurses' Risk Assessment Judgements and Confidence Calibration.*
- Yang H, Thompson C. Nurses' risk assessment judgements: a confidence calibration study. *J Adv Nurs.* 2010; 66(12):2751–2760. DOI: 10.1111/j.1365-2648.2010.05437.x [PubMed: 20825516]
- Yang H, Thompson C. The effects of clinical experience on nurses' critical event risk assessment judgements in paper based and high fidelity simulated conditions: a comparative judgement analysis. *Int J Nurs Stud.* 2011; 48(4):429–437. DOI: 10.1016/j.ijnurstu.2010.09.010 [PubMed: 20943223]
- Yang H, Thompson C, Bland M. The effect of clinical experience, judgment task difficulty and time pressure on nurses' confidence calibration in a high fidelity clinical simulation. *BMC Med Inform Decis Mak.* 2012a; 12(1):113.doi: 10.1186/1472-6947-12-113 [PubMed: 23034048]
- Yang H, Thompson C, Bland M. Effect of improving the realism of simulated clinical judgement tasks on nurses' overconfidence and underconfidence: Evidence from a comparative confidence calibration analysis. *Int J Nurs Stud.* 2012b; 49(12):1505–1511. DOI: 10.1016/j.ijnurstu.2012.08.005 [PubMed: 22947410]
- Yang H, Thompson C, Hamm RM, Bland M, Foster A. The effect of improving task representativeness on capturing nurses' risk assessment judgements: a comparison of written case simulations and physical simulations. *BMC Med Inform Decis Mak.* 2013; 13:62.doi: 10.1186/1472-6947-13-62 [PubMed: 23718556]

Yaniv I, Choshen-Hillel S. Exploiting the wisdom of others to make better decisions: Suspending judgment reduces egocentrism and increases accuracy. *Journal of Behavioral Decision Making*. 2012; 25(5):427–434. DOI: 10.1002/bdm.740

Author Manuscript

Author Manuscript

Author Manuscript

Author Manuscript

Figure 1.

Lens model equation decompositions of judgment achievement r_a , paper case based and physical simulation (manikin) based judgments. Each participant did both tasks and each performance was modeled with 4 different lens model equation approaches.

Table 1

Judgment scale, judgment model type, and ecological model type for each of the 4 lens model equation approaches compared.

Judgment Type	Model of Judgment	Model of Ecology	
		Linear	Logistic
Dichotomous Judgment	Linear	Linear Lens Model Equation $r_a = GR_e R_s + C \sqrt{1 - R_e^2} \sqrt{1 - R_s^2}$	X
	Logistic	X	Logistic Lens Model Equation $r_a = G \frac{\sigma_{Y_e} \sigma_{Y_s}}{\sigma_{Y_e} \sigma_{Y_s}} + C_1 \frac{\sigma_{Y_e}}{\sigma_{Y_e}} \frac{\sigma_{Y_s}}{\sigma_{Y_s}} + C_2 \frac{\sigma_{Y_e}}{\sigma_{Y_e}} \frac{\sigma_{Y_s}}{\sigma_{Y_s}} + C_3 \frac{\sigma_{Y_e}}{\sigma_{Y_e}} \frac{\sigma_{Y_s}}{\sigma_{Y_s}}$
Confidence-Adjusted Judgment	Linear	Confidence-Adjusted Lens Model Equation $r_a = G \frac{\sigma_{Y_e} \sigma_{Y_{des}}}{\sigma_{Y_e} \sigma_{Y_{des}}} + C_1 \frac{\sigma_{Y_e}}{\sigma_{Y_e}} \frac{\sigma_{Y_{des}}}{\sigma_{Y_{des}}}$	Hybrid Lens Model Equation $r_a = G \frac{\sigma_{Y_e} \sigma_{Y_{des}}}{\sigma_{Y_e} \sigma_{Y_{des}}} + C_1 \frac{\sigma_{Y_e}}{\sigma_{Y_e}} \frac{\sigma_{Y_{des}}}{\sigma_{Y_{des}}} + C_2 \frac{\sigma_{Y_e}}{\sigma_{Y_e}} \frac{\sigma_{Y_{des}}}{\sigma_{Y_{des}}} + C_3 \frac{\sigma_{Y_e}}{\sigma_{Y_e}} \frac{\sigma_{Y_{des}}}{\sigma_{Y_{des}}}$

Note: The use of different superscripts, such as $\hat{Y}_e$ and $\hat{Z}_e$ for the linear regression prediction and residual, but $\hat{Y}_e$ and $\hat{Z}_e$ for logistic regression prediction and residual, is simply to distinguish the source.