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Identification of the joint effect of a dynamic treatment intervention and a stochastic monitoring intervention under the no direct effect assumption

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Abstract

The management of chronic conditions is characterized by frequent re-assessment of therapy decisions in response to the patient's changing condition over the course of the illness. Evidence most suitable to inform care thus often concerns the contrast of adaptive treatment strategies that repeatedly personalize treatment decisions over time using the latest accumulated data available from the patient's previous clinic visits such as laboratory exams (e.g., hemoglobin A1c measurements in diabetes care). The frequency at which such information is monitored implicitly defines the causal estimand that is typically evaluated in an observational or randomized study of such adaptive treatment strategies. Analytic control of monitoring with standard estimation approaches for time-varying interventions can therefore not only improve study generalizability but also inform the optimal timing of clinical surveillance. Valid inference with these estimators requires the upholding of a positivity assumption that can hinder their applicability. To potentially weaken this requirement for monitoring control, we introduce identifiability results that will facilitate the derivation of alternate estimators of effects defined by general joint treatment and monitoring interventions in the context of time-to-event outcomes. These results are developed based on the nonparametric structural equation modeling framework using a no direct effect assumption originally introduced in a prior paper that inspired this work. The relevance and scope of the results presented here are illustrated with examples in diabetes comparative effectiveness research.

Keywords

counterfactual identifiability; dynamic intervention; effect of monitoring; no direct effect assumption; stochastic intervention; time-varying treatment

1 Introduction

Effective management of chronic conditions such as diabetes involves frequent re-evaluation of treatment decisions over the course of the patient's illness. Patients and their clinicians

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must choose not only adequate therapies among many options, but they must also decide throughout the course of the illness when to initiate, switch or intensify treatments and how to titrate medication to balance health benefits and harms. These clinical decisions are informed by periodic monitoring of laboratory test results (e.g., hemoglobin A1c for patients with diabetes), changes in symptoms, and more general considerations of patients' preferences, concurrent conditions, adherence to previously prescribed treatments, and overall health. Evidence most suitable to guide the real-world management of chronic conditions must therefore involve the evaluation of complex treatment strategies that continuously adapt to new information about the patient's changing circumstances collected during routine clinic visits. For instance, in diabetes research, contrasting (e.g., by randomization) adaptive treatment regimens of the type:

“Initiate insulin the first time the patient's observed A1c drifts above X%” with $X=7\%, 7.5\%, 8\%$, or 8.5%

will generate evidence that is clinically more relevant than contrasting static treatment regimens of the type

“Initiate insulin X months from now” with $X=3, 6, 9$, or 12

because the former regimens emulate (albeit simplistically) how treatment decisions are made in real-world clinical practice by personalizing treatment decisions over time based on the latest A1c information available for the patient. With the help of healthcare providers, the above adaptive treatment regimens can be made more complex so they better emulate how treatment decisions are informed in real-world clinical settings and thus ensure that the evidence generated is most useful by contrasting decision making procedures that are (or could be realistically) used by clinicians in practice.

Causal inference methods to estimate the effects of treatment regimens in the presence of time-dependent confounding have been developed specifically to generate such evidence using data from observational studies (Murphy, van der Laan, and Robins, 2001, Hernan, Lanoy, Costagliola, and Robins, 2006, van der Laan and Petersen, 2007, Robins, Orellana, and Rotnitzky, 2008, Cain, Robins, Lanoy, Logan, Costagliola, and Hernan, 2010, Petersen, Schwab, Gruber, Blaser, Schomaker, and van der Laan, 2014) or randomized experiments (Thall, Millikan, and Sung, 2000, Lavori and Dawson, 2000, 2008, Murphy, 2005, Lavori and Dawson, 2004, Murphy, 2003). Until recently (Ewings, Ford, Walker, Carpenter, and Copas, 2014, Ford, Robins, Petersen, Gibb, Gilks, Mugenyi, Grosskurth, Hakim, Katabira, Babiker, and Walker on behalf of the DART Trial Team, 2015) however, applications have ignored the role of clinical monitoring frequency in the definition of the effects of adaptive treatment strategies. As a result, conclusions about health benefits and harms remain specific (Robins et al., 2008, Section 6) to the particular clinical monitoring conditions of the study which may or not reflect a particular real-world practice of interest. For example, two perfect randomized trials (no non-compliance and no loss to follow-up) that are conducted in the same patient population but using two distinct clinical monitoring schedules could provide different conclusions (both valid) about the effectiveness of an adaptive treatment strategy compared to the same, for example, static (i.e., non-adaptive) treatment strategy (e.g., never treat).

Evaluating the joint effect of an adaptive treatment strategy and a particular clinical monitoring intervention is thus desirable to, not only, improve the generalizability (Pearl and Bareinboim, 2011, 2014, Robins et al., 2008) of study results, but also, more importantly, to generate evidence on how clinical monitoring modify the health effects of adaptive treatment strategies. For instance, in diabetes research, contrasting joint treatment and monitoring regimens of the type:

“Monitor the patient’s A1c every X months and initiate insulin the first time her A1c drifts above 8%” with $X = 3, 6, 9, 12$

can generate evidence about the value of increased A1c monitoring (‘value of information’ (Robins et al., 2008)) that can be highly relevant for both patients living with chronic conditions and for the health systems that care for them: A patient’s quality of life is impacted by the burdens imposed by frequent laboratory and other clinical surveillance and this surveillance also contributes to the significant burdens that chronic conditions put on healthcare systems. Thus, not only is evidence on how to possibly minimize the intensity of clinical monitoring without significantly undermining health outcomes relevant to inform care, but also evidence that aim to optimize outcomes by adapting monitoring decisions over time based on a patient’s evolving condition. The latter type of evidence can, for example, be generated by contrasting joint treatment and monitoring regimens of the type:

“Monitor the patient’s A1c every X months as long as her latest A1c remain below 7% and increase monitoring to every 3 months otherwise. Initiate insulin the first time her A1c drifts above 8%” with $X = 3, 6, 9, 12$.

Standard estimation approaches for evaluating the effect of general time-varying exposures with observational data can be readily applied to the evaluation of the joint effect of an adaptive treatment strategy and a particular clinical monitoring intervention by simply defining the exposure variable of interest at each time point as a two-dimensional variable where the first and second element represents the treatment and monitoring decision, respectively. These methods include not only the most popular Inverse Probability Weighting (IPW) (Robins, 1999) and G-computation (Robins, 1986) estimation approaches but also the two doubly robust and locally efficient augmented-IPW (Robins, 2000, van der Laan and Robins, 2003, Bang and Robins, 2005, Tsiatis, 2006, Kang and Schafer, 2007) and Targeted Minimum Loss based Estimation (TMLE) (van der Laan and Rubin, 2006, van der Laan, Mark J. and Rose, 2011, van der Laan and Gruber, 2012) approaches. Effect identifiability with these various estimators relies on both a no unmeasured confounders assumption and a positivity assumption. The latter assumption requires that each patient in the study can possibly experience the time-varying interventions of interest at each time point whatever the levels of the time-dependent confounders. Upholding of this assumption when the time-varying interventions include rigid monitoring control (e.g., “patient should have her A1c monitored exactly every 3 months”) might often be unrealistic in observational studies conducted in real-world settings. Concerns over effect non-identifiability due to violation of the positivity assumption associated with monitoring decisions have motivated prior work (Robins et al., 2008, Section 6) and led to the development of new IPW estimators which remain consistent under a weaker positivity assumption for monitoring decisions as long as it can also be assumed that these monitoring decisions have no direct

effects on covariates (including the outcome) once treatment decisions are made. This assumption was named the No Direct Effect (NDE) assumption and the resulting NDE-based IPW estimators were developed in problems with an end-of-study outcome and for static or more general (adaptive or not) stochastic monitoring interventions.

In this article, we extend this prior work in the context of time-to-event outcomes with the presentation of two counterfactual identifiability results. These two counterfactual equalities are developed based on the nonparametric structural equation model (NPSEM) framework (Pearl, 1995, 2009a,b, 2010a) and applicable to the evaluation of the joint effect of an adaptive treatment strategy and either a static or general stochastic clinical monitoring intervention under the NDE assumption. The practical implication of the results presented here is the straight-forward derivation of new IPW, G-computation, augmented IPW and TMLE estimators that rely on a weaker positivity assumption for the monitoring process compared to their standard analogs (i.e., not based on the NDE assumption). We note, however, that while we illustrate the implication of the results in this paper for estimator construction using the IPW methodology with the simple example of static monitoring interventions, the detailed derivation and evaluation of alternate estimators and of the required assumptions for estimation consistency (e.g., sequential randomization assumption) are beyond the scope of this article. In separate work, we will further exploit the two identifiability results presented here to demonstrate that they lead, not only, to an other innovative NDE-based IPW estimator for general stochastic monitoring interventions derived in a prior paper (Robins et al., 2008) (without formally evoking the results of this article), but also to the straightforward derivation of NDE-based TMLE estimators for either static or general stochastic monitoring interventions.

In the next section, we present the notation adopted in this paper to represent the general longitudinal observational data available in practice to evaluate the joint treatment and monitoring effects of interest. In Section 3, we introduce the causal model defined by a general NPSEM that is assumed to have generated the observed data and that defines the various classes of counterfactual variables considered in this work. We detail the definition of these counterfactuals in Section 4 before formalizing the NDE assumption. In Sections 6 and 7, we present the two identifiability results for evaluating joint effects defined by static and general stochastic monitoring interventions, respectively. We illustrate the practical implication of these results for estimator construction in Section 8 and conclude with a discussion of the contribution of this paper to the existing literature on the same topic in Section 9. For clarity, we continue to use examples in diabetes comparative effectiveness research (CER) throughout the manuscript to demonstrate the applicability of the formalism introduced in the next sections.

2 Observed Data

We consider an observational study (e.g., retrospective study based on Electronic Health Records from a healthcare database) in which longitudinal data are collected on a random sample of n patients from a target population starting at the index date (i.e., cohort entry date denoted by $t = 0$) and until end of follow-up defined by the earliest occurrence of a failure event of interest (e.g., onset of diabetic peripheral neuropathy) or a right-censoring event

(e.g., health plan disenrollment). Right-censoring events include reaching the administrative end of the study which occurs for all patients at time $t = K + 1$ at the latest. At each time point t during follow-up (expressed in an arbitrary unit of time, e.g., 30-day interval), measurements on the following variables are assembled. The sequence in which we describe these variables reflects the actual temporal ordering of the underlying information these variables encode. The binary outcome variable $Y(t)$ represents whether the failure event occurred since time point $t - 1$ ($Y(0) = 0$ by convention). The vector of covariates $Z(t)$ represents various patient attributes (e.g., demographic, vital signs, comorbidities, etc.) at time t . We note that some of these attributes may not be monitored at each time point in which case indicators of monitoring are included as part of the definition of $Z(t)$ and the unmonitored attribute measurements are then equal to 0 by convention. The covariate $I(t)$ represents the patient's attribute at time t (e.g., A1c) that is involved in the definition of the adaptive treatment regimen of interest and for which a monitoring intervention is also of interest. By convention, $I(0)$ is always monitored and $I(t) \equiv 0$ if the attribute is not monitored at time t . The binary variable $A_2(t)$ represents whether the patient experiences a right-censoring event at time t . The variable $A_1(t)$ represents the treatment decision of interest at time t (e.g., insulin exposure). The binary variable $N(t)$ represents the monitoring decision of interest at time t and indicates whether a measurement for $I(t + 1)$ will be collected (e.g., whether an A1c test will be conducted at the next time point). We thus have $I(t + 1) = 0$ if $N(t) = 0$. In short, the observed data are realizations of n independent and identically distributed copies of the following temporally ordered sequence of random variables:

$$O = (L(0), A_2(0), A_1(0), N(0), \dots, L(K), A_2(K), A_1(K), N(K), L(K+1), A_2(K+1)),$$

where $L(t) \equiv (Y(t), Z(t), I(t), A_2(K+1) = 1 \text{ if } Y(K+1) = 0 \text{ (i.e., right-censoring from reaching the administrative end of the study is deterministic at } K+1), \text{ and where, by convention, all variables become degenerate (defined at their last observed value) after occurrence of a failure or right-censoring event. Time-independent covariates (e.g., sex) are included in the vector of baseline covariates } L(0) \text{ alongside the baseline measurements of time-varying covariates. We denote the distribution of these observed data } O \text{ by } P_0. \text{ To simplify notation below, we use the overbar symbol to denote the history of a variable. More specifically, } \bar{X}(t) \text{ and } \bar{X}(t, t') \text{ represent the vectors of measurements of the time-varying attribute } X \text{ from baseline to time point } t. \bar{X}(t) \equiv (X(0), \dots, X(t)) \text{ and from time point } t \text{ to } t' : \bar{X}(t, t') \equiv (X(t), \dots, X(t')), \text{ respectively. By convention, } \bar{X}(t) \text{ and } X(t) \text{ are nil for } t < 0 \text{ and } \bar{X}(t, t') \text{ is nil for } t' < t. \text{ We also use lower case letters to represent realizations of random variables. Finally, we denote the support of the treatment and monitoring processes } A_1(K) \text{ and } N(K) \text{ by } \mathcal{A} \text{ and } \mathcal{N}, \text{ respectively.}$

3 Causal Model

We assume the following nonparametric structural equation model (NPSEM) that links (Appendix A) the observed data distribution P_0 to the probability distribution (denoted by P_U) of a vector of random disturbances $U \equiv ((U_{Y(t)}, U_{Z(t)}, U_{I(t)}, U_{A_2(t)})_{t=0, \dots, K+1}, (U_{A_1(t)},$

$U_{N(t)}_{t=0,\dots,K}$) and a fixed vector of functions $f \equiv ((f_{Y(t)}, f_{Z(t)}, f_{I_0(t)}, f_{A_2(t)})_{t=0,\dots,K+1}, (f_{A_1(t)}, f_{N(t)})_{t=0,\dots,K})$:

- $Y(t) = f_{Y(t)}(P_{Y(t)}^0, U_{Y(t)})$ with parents
 $P_{Y(t)}^0 \equiv (\bar{Y}(t-1), \bar{Z}(t-1), \bar{I}_0(t-1), \bar{A}_2(t-1), \bar{A}_1(t-1), \bar{N}(t-1))$ for $t = 0, \dots, K+1$,
- $Z(t) = f_{Z(t)}(P_{Z(t)}^0, U_{Z(t)})$ with parents
 $P_{Z(t)}^0 \equiv (\bar{Y}(t), \bar{Z}(t-1), \bar{I}_0(t-1), \bar{A}_2(t-1), \bar{A}_1(t-1), \bar{N}(t-1))$ for $t = 0, \dots, K+1$,
- $I_0(t) = f_{I_0(t)}(P_{I_0(t)}^0, U_{I_0(t)})$ with parents
 $P_{I_0(t)}^0 \equiv (\bar{Y}(t), \bar{Z}(t), \bar{I}_0(t-1), \bar{A}_2(t-1), \bar{A}_1(t-1), \bar{N}(t-1))$ for $t = 0, \dots, K+1$,
- $I(t) = Y(t)I(t-1) + (1 - Y(t))N(t-1)I_0(t)$ for $t = 0, \dots, K+1$ where $N(-1) \equiv 1$ by convention,
- $A_2(t) = f_{A_2(t)}(P_{A_2(t)}^0, U_{A_2(t)})$ with parents
 $P_{A_2(t)}^0 \equiv (\bar{Y}(t), \bar{Z}(t), \bar{I}_0(t), \bar{A}_2(t-1), \bar{A}_1(t-1), \bar{N}(t-1))$ for $t = 0, \dots, K+1$,
- $A_1(t) = f_{A_1(t)}(P_{A_1(t)}^0, U_{A_1(t)})$ with parents
 $P_{A_1(t)}^0 \equiv (\bar{Y}(t), \bar{Z}(t), \bar{I}_0(t), \bar{A}_2(t), \bar{A}_1(t-1), \bar{N}(t-1))$ for $t = 0, \dots, K$,
- $N(t) = f_{N(t)}(P_{N(t)}^0, U_{N(t)})$ with parents
 $P_{N(t)}^0 \equiv (\bar{Y}(t), \bar{Z}(t), \bar{I}_0(t), \bar{A}_2(t), \bar{A}_1(t), \bar{N}(t-1))$ for $t = 0, \dots, K$,

where the functions f above are left unspecified except for the following constraints:

- $f_{Y(0)}(U_{Y(0)}) = 0$,
- $f_{Y(t)}(P_{Y(t)}^0, U_{Y(t)}) = Y(t-1)$ if $A_2(t-1) = 1$ or $Y(t-1) = 1$ for $t = 1, \dots, K+1$,
- $f_{Z(t)}(P_{Z(t)}^0, U_{Z(t)}) = Z(t-1)$ if $A_2(t-1) = 1$ or $Y(t) = 1$ for $t = 1, \dots, K+1$,
- $f_{I_0(t)}(P_{I_0(t)}^0, U_{I_0(t)}) = I_0(t-1)$ if $A_2(t-1) = 1$ or $Y(t) = 1$ for $t = 1, \dots, K+1$,
- $f_{A_2(t)}(P_{A_2(t)}^0, U_{A_2(t)}) = A_2(t-1)$ if $A_2(t-1) = 1$ or $Y(t) = 1$ for $t = 1, \dots, K+1$,
- $f_{A_1(t)}(P_{A_1(t)}^0, U_{A_1(t)}) = A_1(t-1)$ if $A_2(t) = 1$ or $Y(t) = 1$ for $t = 1, \dots, K$,
- $f_{N(t)}(P_{N(t)}^0, U_{N(t)}) = N(t-1)$ if $A_2(t) = 1$ or $Y(t) = 1$ for $t = 1, \dots, K$,
- $f_{A_2(K+1)}(P_{A_2(K+1)}^0, U_{A_2(K+1)}) = 1$ if $Y(K+1) = 0$.

In this NPSEM, the observed covariate $I(t)$ is explicitly linked to a latent variable $I_0(t)$ which represents the patient's underlying attribute measurement of interest at time t . More specifically, as long as the patient has not experienced failure yet, if a decision was made not to collect the measurement $I_0(t)$ at the previous time point (i.e., when $N(t-1) = 0$) then $I(t) = 0$ (which follows the convention described in the previous section to define $I(t)$) and otherwise $I(t) = I_0(t)$ (i.e., when $N(t-1) = 1$).

The combination of the latent variable $I_0(t)$ and observed variables $Y(t)$, $Z(t)$, $A_2(t)$, $A_1(t)$, and $N(t)$ are referred to as endogenous variables¹ and we note that the previous constraints on the functions f merely encode 1) the convention that the baseline outcome is 0, 2) the convention that the endogenous variables are defined as degenerate random variables after end of follow-up from occurrence of a failure or right-censoring event, and 3) deterministic right-censoring from administrative end of study at $t = K + 1$.

We note also that the definition of the observed measurement $I(t)$ above implies that, like all other endogenous variables, it becomes degenerate at its last observed value after the occurrence of, not only, the failure event, but also a right-censoring event, i.e., $I(t) = I(t-1)$ if $Y(t) = 1$ or $A_2(t-1) = 1$.

Finally, we introduce the notation $L_0(t) \equiv (Y(t), Z(t), I_0(t))$ for $t = 0, \dots, K + 1$ and bring attention to the distinction between $L_0(t)$ and the observed vector of covariates $L(t) = (Y(t), Z(t), I(t))$.

4 Counterfactual variables of interest

Whether conducted with a randomized experiment or an observational study, CER typically aims to evaluate causal estimands defined by the distribution of 'potential outcomes', i.e., outcomes if all patients in the target population experienced the 'same' (possibly adaptive or stochastic) exposure intervention. In this section, we differentiate five classes of general counterfactual variables considered in this work. Each is defined by a particular modification of the previous NPSEM as described below.

The NPSEM is modified by changing the structural equations associated with a set (denoted by $\mathcal{X}$) of endogenous variables referred to as the action variables. More specifically, for each

action variable $X(t) \in \mathcal{X}$, the original structural equation $X(t) = f_{X(t)}(P_{X(t)}^0, U_{X(t)})$ is

replaced with a new structural equation $X(t) = f_{X(t)}^*(P_{X(t)}^0, U_{X(t)}, U_{X(t)}^*)$ and a joint conditional distribution (denoted by $P_{U^*|U}$) is specified for the new random disturbances

associated with the action variables, $(U_{X(t)}^*)_{X(t) \in \mathcal{X}}$, given the original disturbances U . The endogenous variables resulting from such a modified NPSEM are referred to as counterfactual variables and their joint distribution is entirely defined by the joint

distribution of the disturbances $(U, (U_{X(t)}^*)_{X(t) \in \mathcal{X}})$ which we denote by P_{U, U^*} . To differentiate such counterfactual variables from the original endogenous variables in the

¹by opposition to exogenous variables not included explicitly in the analysis but represented abstractly by the vector of random disturbances U (Pearl, 1995).

unmodified NPSEM, we use the generic superscript notation $*$, i.e., the endogenous variables in any given modified NPSEM are denoted by $Y^*(t)$, $Z^*(t)$, $I_0^*(t)$, $F^*(t)$, $A_2^*(t)$, $A_1^*(t)$, and $N^*(t)$ and the modified NPSEM is then expressed as the following collection of equations:

- $X^*(t) = f_{X(t)}^*(P_{X^*(t)}^0, U_{X(t)}, U_{X(t)}^*)$ for each counterfactual variable $X^*(t)$ corresponding with an action variable $X(t) \in \mathcal{X}$, where $P_{X^*(t)}^0$ denotes the counterfactual analog of the parents $P_{X(t)}^0$ in the modified NPSEM (e.g., $P_{N^*(t)}^0 \equiv (\bar{Y}^*(t), \bar{Z}^*(t), \bar{I}_0^*(t), \bar{A}_2^*(t), \bar{A}_1^*(t), \bar{N}^*(t-1))$),
- $X^*(t) = f_{X(t)}(P_{X^*(t)}^0, U_{X(t)})$ for each counterfactual variables $X^*(t)$ corresponding with a non-action variable $X(t) \notin \mathcal{X}$.

4.1 Static intervention on the treatment and right-censoring variables

In this section, we formalize the definition of counterfactual outcomes involved in the evaluation of the following types of CER question in a cohort of diabetes patients whose glycemia has become out of control (e.g. A1c 7%) despite prior therapies with oral agents (e.g., metformin therapy): *What is the difference in the risk of onset diabetic peripheral neuropathy (DPN) if patients initiate insulin therapy at study entry versus delay initiation by 12 months?* We note that such risk evaluation would require that all patients remain followed-up for the entire duration of the study (i.e., no loss to follow-up). Because the treatment interventions of interest in the example above are not adapted to the patient's covariates, the treatment interventions are referred to as static.

We denote such static intervention regimens on the treatment and right-censoring variables through time K by $\bar{a}^* \equiv (\bar{a}_1^*(K), \bar{a}_2^*(K))$ where $\bar{a}_2^*(t) = 0$ for $t = 0, \dots, K$. The effect of such an intervention on all other endogenous variables in the causal model is defined by the modified NPSEM with the following structural equations for the action variables $\mathcal{X} = \{A_1(K), A_2(K)\}$:

- $A_2^*(t) = f_{A_2(t)}^*(P_{A_2^*(t)}^0, U_{A_2(t)}, U_{A_2(t)}^*) = Y^*(t) f_{A_2(t)}(P_{A_2^*(t)}^0, U_{A_2(t)}) + (1 - Y^*(t)) \bar{a}_2^*(t)$
- $A_1^*(t) = f_{A_1(t)}^*(P_{A_1^*(t)}^0, U_{A_1(t)}, U_{A_1(t)}^*) = Y^*(t) f_{A_1(t)}(P_{A_1^*(t)}^0, U_{A_1(t)}) + (1 - Y^*(t)) \bar{a}_1^*(t)$

for any given conditional distribution $P_{U^*|U}$. We note that, with this definition, the intervention is restricted to the treatment and censoring variables from time 0 to the minimum of time K or the time of failure and, therefore, $A_1^*(t)$ is not necessarily equal to $a_1^*(t)$ after failure. We will also denote the counterfactual variables $Y^*(t)$, $Z^*(t)$, $I_0^*(t)$, $F^*(t)$, $A_2^*(t)$, $A_1^*(t)$ and $N^*(t)$ defined by the modified NPSEM described above by $Y^*(t)$, $Z^*(t)$, $I_0^*(t)$, $F^*(t)$, $A_2^*(t)$, $A_1^*(t)$ and $N^*(t)$, respectively, to differentiate them from the other counterfactual variables defined in the following subsections. We denote the support of the counterfactual treatment and monitoring processes $A_1^*(K)$ and $N^*(K)$ by $\mathcal{A}_1^*$ and $\mathcal{N}^*$, respectively.

4.2 Static intervention on the treatment, right-censoring, and monitoring variables

The static treatment and right-censoring interventions defined in the previous sections are now supplemented by a static intervention on the monitoring process. We denote any given static intervention regimen of interest on monitoring through time K by $\bar{n}^* \equiv \bar{n}^*(K)$. The joint effect of this intervention and the static interventions on treatment and right-censoring, Y^* , on all other endogenous variables in the causal model is defined by the modified NPSEM with the following structural equations for the action variables $\mathcal{X} = \{A_1(K), A_2(K), \bar{N}(K)\}$:

- $A_2^*(t) = f_{A_2(t)}^*(P_{A_2^*(t)}^0, U_{A_2(t)}, U_{A_2^*(t)}^*) = Y^*(t) f_{A_2(t)}(P_{A_2^*(t)}^0, U_{A_2(t)}) + (1 - Y^*(t)) a_2^*(t)$
- $A_1^*(t) = f_{A_1(t)}^*(P_{A_1^*(t)}^0, U_{A_1(t)}, U_{A_1^*(t)}^*) = Y^*(t) f_{A_1(t)}(P_{A_1^*(t)}^0, U_{A_1(t)}) + (1 - Y^*(t)) a_1^*(t)$
- $N^*(t) = f_{N(t)}^*(P_{N^*(t)}^0, U_{N(t)}, U_{N^*(t)}^*) = Y^*(t) f_{N(t)}(P_{N^*(t)}^0, U_{N(t)}) + (1 - Y^*(t)) n^*(t)$

for any given conditional distribution $P_{U^*|U}$. We will also denote the counterfactual variables $Y^*(t)$, $Z^*(t)$, $I_0^*(t)$, $I^*(t)$, $A_2^*(t)$, $A_1^*(t)$ and $N^*(t)$ defined by the modified NPSEM just described by $Y_{*,\bar{n}^*}^*(t)$, $Z_{*,\bar{n}^*}^*(t)$, $I_{0*,\bar{n}^*}^*(t)$, $I_{*,\bar{n}^*}^*(t)$, $A_{2*,\bar{n}^*}^*(t)$, $A_{1*,\bar{n}^*}^*(t)$ and $N_{*,\bar{n}^*}^*(t)$, respectively. We note again here that, after failure, $A_{1*,\bar{n}^*}^*(t)$ and now also $N_{*,\bar{n}^*}^*(t)$ are not necessarily equal to $a_1^*(t)$ and $n^*(t)$, respectively. We denote the support of the counterfactual treatment and monitoring processes $A_{1*,\bar{n}^*}^*(K)$ and $\bar{N}_{*,\bar{n}^*}^*(K)$ by $\bar{\mathcal{A}}_{*,\bar{n}^*}$ and $\bar{\mathcal{N}}_{*,\bar{n}^*}$, respectively.

4.3 Static intervention on the right-censoring and monitoring variables combined with a dynamic intervention on the treatment variables

In this section, we formalize the definition of counterfactual outcomes involved in the evaluation of the following types of diabetes CER questions: *What is the difference in the risk of onset DPN if patients have their A1c monitored every 6 months and initiate insulin therapy the first time their A1c drifts above 7.5% versus 8.5%? What is the difference in the risk of onset DPN if patients have their A1c monitored every 3 months versus every 6 months and initiate insulin therapy the first time their A1c drifts above 8.5%?* We note that the intervention regimens of interest now involve adaptive treatment decisions combined with static (i.e., non-adaptive) monitoring decisions for the biomarker that is used to personalize treatment decisions.

The effect of such intervention regimens on all other endogenous variables in the causal model is defined by the modified NPSEM with the following structural equations for the action variables $\mathcal{X} = \{A_1(K), A_2(K), \bar{N}(K)\}$:

- $A_2^*(t) = f_{A_2(t)}^*(P_{A_2^*(t)}^0, U_{A_2(t)}, U_{A_2^*(t)}^*) = Y^*(t) f_{A_2(t)}(P_{A_2^*(t)}^0, U_{A_2(t)}) + (1 - Y^*(t)) a_2^*(t)$
- $A_1^*(t) = f_{A_1(t)}^*(P_{A_1^*(t)}^0, U_{A_1(t)}, U_{A_1^*(t)}^*) = Y^*(t) f_{A_1(t)}(P_{A_1^*(t)}^0, U_{A_1(t)}) + (1 - Y^*(t)) \times d_x(t)(P_{A_1^*(t)}^0)$

where $P_{A_1^*(t)}^0$ is defined as $P_{A_1^*(t)}^0$ with the difference that the variables $\bar{I}_0^*(t)$ are replaced with the variables $\bar{I}^*(t)$, i.e.,

$$P_{A_1^*(t)}^0 \equiv (\bar{Y}^*(t), \bar{Z}^*(t), \bar{I}^*(t), \bar{A}_2^*(t), \bar{A}_1^*(t-1), \bar{N}^*(t-1))$$

$$\bullet \quad N^*(t) = f_{N(t)}^*(P_{N^*(t)}^0, U_{N(t)}, U_{N(t)}^*) = Y^*(t) f_{N(t)}(P_{N^*(t)}^0, U_{N(t)}) + (1 - Y^*(t)) n^*(t)$$

for any given distribution $P_{U^*|U}$ and a particular choice of mapping $d_x(t): P_{A_1^*(t)} \mapsto a_1^*(t)$ for $t = 0, \dots, K$ from past observed variables into a treatment intervention at time t corresponding with the dynamic intervention of interest. We will also denote the counterfactual variables $Y^*(t)$, $Z^*(t)$, $I_0^*(t)$, $I^*(t)$, $A_2^*(t)$, $A_1^*(t)$ and $N^*(t)$ defined by the modified NPSEM just described by $Y_{d_X, \bar{N}^*}(t)$, $Z_{d_X, \bar{N}^*}(t)$, $I_{0, d_X, \bar{N}^*}(t)$, $I_{d_X, \bar{N}^*}(t)$, $A_{2, d_X, \bar{N}^*}(t)$, $A_{1, d_X, \bar{N}^*}(t)$ and $N_{d_X, \bar{N}^*}(t)$, respectively. Note that to simplify notation, we omit the use of the subscript $\bar{a}_2^*(K)$ and thus make the intervention on the right-censoring variables implicit in the notation of these counterfactual variables. We denote the support of the counterfactual treatment and monitoring processes $1_{d_X, \bar{N}^*}(K)$ and $\bar{N}_{d_X, \bar{N}^*}(K)$ by $\bar{\mathcal{A}}_{d_X, \bar{N}^*}$ and $\bar{\mathcal{N}}_{d_X, \bar{N}^*}$, respectively.

4.4 Static intervention on the right-censoring variables combined with a stochastic intervention on the monitoring variables and a dynamic intervention on the treatment variables

In this section, we consider a general class of monitoring interventions which can be used to address four types of CER questions that are each illustrated below before formalizing the definition of these interventions with a unifying notation.

So far, the static monitoring intervention considered required that patients adhere to a rigid A1c testing schedule with no tolerance for even slight deviations (i.e., advanced or delayed monitoring). Here, we consider more realistic monitoring interventions that can reflect real-world adherence to monitoring guidelines. More specifically, we formalize the definition of counterfactual outcomes involved in the evaluation of the following types of diabetes CER questions: *(Q1) What is the difference in the risk of onset DPN if patients' A1c are monitored on average every 3 versus 6 months and patients initiate insulin therapy the first time their A1c drifts above 8.5%?* The monitoring interventions of interest are now stochastic. In addition, we also consider here adaptive (deterministic or stochastic) monitoring interventions which allow the personalization of monitoring decisions over time based on the patient's latest observed condition. More specifically, we also formalize below the definition of counterfactual outcomes involved in the evaluation of the following types of diabetes CER questions: *(Q2) What is the difference in the risk of onset DPN if patients initiate insulin therapy the first time their A1c drifts above 8.5% and their A1c is monitored on average every 3 versus 6 months as long as their latest A1c remain below 7% and on average every 3 months otherwise?* The monitoring intervention of interest is now, not only, stochastic, but also adaptive. We note that, to formally evaluate the effects of interest in the previous two examples, the analyst must first define the monitoring interventions of interest more precisely by specifying monitoring distributions that would result in patients having *on average* their A1c collected every 3 or 6 months (possibly conditional on the last observed A1c value). Many such distributions can be chosen and each would lead to a different effect measure. While real-world adherence to the stochastic interventions above is more realistic than adherence to the static intervention described in the previous sections, they do not accommodate for the occurrence of unexpected events (e.g., severe hypoglycemia) which would require additional monitoring in practice. To avoid arbitrary specification of

monitoring distributions and to further increase the relevance of the evidence generated to real-world settings, we also consider here alternate monitoring interventions which require that gaps between two A1c test not exceed a given threshold. More specifically, we also formalize here the definition of counterfactual outcomes involved in the evaluation of the following types of diabetes CER questions: (Q3) *What is the difference in the risk of onset DPN if A1c monitoring precludes gaps between measurements greater than 6 versus 12 months and patients initiate insulin therapy the first time their A1c drifts above 8.5%?* The monitoring interventions of interest are now restricted to only enforcing monitoring as soon as the patient's last A1c measurement is 6-month (or 12-month) old and A1c monitoring is not intervened upon in between such monitoring interventions (i.e., monitoring follows the natural decision process of the setting in which the study is conducted). Finally, we also consider here monitoring interventions that are defined by uniformly increasing or decreasing the probability of A1c monitoring at each time point defined by the natural decision process of the study. More specifically, we also formalize here the definition of counterfactual outcomes involved in the evaluation of the following types of diabetes CER questions: (Q4) *What is the difference in the risk of onset DPN if at each time point 10% versus 20% of patients who would otherwise not be monitored had their A1c tested and all patients initiate insulin therapy the first time their A1c drifts above 8.5%?*

The effect of any one of the intervention regimens described above on all other endogenous variables in the causal model is defined by the modified NPSEM with the following structural equations for the action variables $\mathcal{X} = \{A_1(K), A_2(K), N(K)\}$:

- $A_2^*(t) = f_{A_2(t)}^*(P_{A_2^*(t)}^0, U_{A_2(t)}, U_{A_2(t)}^*) = Y^*(t) f_{A_2(t)}(P_{A_2^*(t)}^0, U_{A_2(t)}) + (1 - Y^*(t)) a_2^*(t)$
 - $A_1^*(t) = f_{A_1(t)}^*(P_{A_1^*(t)}^0, U_{A_1(t)}, U_{A_1(t)}^*) = Y^*(t) f_{A_1(t)}(P_{A_1^*(t)}^0, U_{A_1(t)}) + (1 - Y^*(t)) \times d_x(t) (P_{A_1^*(t)}^0)$
 - $N^*(t) = f_{N(t)}^*(P_{N^*(t)}^0, U_{N(t)}, U_{N(t)}^*) = Y^*(t) f_{N(t)}(P_{N^*(t)}^0, U_{N(t)}) + (1 - Y^*(t)) h_{N(t)}^*(P_{N^*(t)}, U_{N(t)}^*, f_{N(t)}(P_{N^*(t)}^0))$
- where $P_{N^*(t)}$ is defined as $P_{N^*(t)}^0$ with the difference that the variables $\bar{I}_0^*(t)$ are replaced with the variables $\bar{I}_0^*(t)$, i.e.,
- $$P_{N^*(t)} \equiv (\bar{Y}^*(t), \bar{Z}^*(t), \bar{I}^*(t), \bar{A}_2^*(t), \bar{A}_1^*(t), \bar{N}^*(t-1)),$$

for a particular choice of distribution $P_{U^*|U}$ and functions $h_{N(t)}^*$. The argument

$f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})$ of the function $h_{N(t)}^*$ represents the monitoring variable (denoted by $N^{*0}(t)$) that would be observed if the monitoring intervention at time t was not carried out (i.e., what we referred to as the natural monitoring decision process in the examples above). Thus, except when $t = 0$, this monitoring variable is not the observed monitoring variable $N(t)$

because $P_{N^*(t)}^0 \neq P_{N(t)}^0$. As a result, the definition of the function $h_{N(t)}^*$ above allows the specification of a monitoring intervention at time t that can be defined as a perturbation of the counterfactual monitoring variable $N^{*0}(t)$ defined by the previous treatment, monitoring, and censoring interventions. We note that the conditional probability (denoted by g) of the counterfactual variable $N^*(t)$ above given its parents will not typically be degenerate for a general choice of functions $h_{N(t)}^*$ and distribution $P_{U^*|U}$. The notation g thus symbolizes the stochastic intervention on the monitoring variables which is entirely defined

by the choice of distribution $P_{U^*|U}$ and function $h_{N(t)}^*$. Consequently, we also denote the counterfactual variables $Y^*(t)$, $Z^*(t)$, $I_0^*(t)$, $I^*(t)$, $A_2^*(t)$, $A_1^*(t)$ and $N^*(t)$ defined by the modified NPSEM just described by $Y_{d_{X,g}}(t)$, $Z_{d_{X,g}}(t)$, $I_{0,d_{X,g}}(t)$, $I_{d_{X,g}}(t)$, $A_{2,d_{X,g}}(t)$, $A_{1,d_{X,g}}(t)$ and $N_{d_{X,g}}(t)$, respectively. Note that here again, we omit the use of the subscript $\bar{a}_2^*(K)$ and thus make the intervention on the right-censoring variables implicit in the notation of these counterfactual variables. Similarly, we also denote the counterfactual variable $N^{*0}(t) \equiv f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})$ by $N_{d_{X,g}}^0(t)$. We denote the support of the counterfactual treatment and monitoring processes $1_{d_{X,g}}(K)$ and $\bar{N}_{d_{X,g}}(K)$ by $\bar{\mathcal{A}}_{d_{X,g}}$ and $\bar{\mathcal{N}}_{d_{X,g}}$, respectively.

We now illustrate how particular choices for the function $h_{N(t)}^*$ in the modified NPSEM above allow the definition of each of the four types of stochastic monitoring interventions described in the four examples of CER questions outlined earlier. The effect of the stochastic monitoring intervention from CER question (Q1) that requires A1c testing on average every 3 months can be formalized using $h_{N(t)}^* : (U_{N(t)}^*, \bar{N}^*(t-1)) \mapsto I(U_{N(t)}^* \leq u_{\bar{N}^*(t-1)}^*)$ with

$$\begin{aligned} u_{\bar{N}^*(t-1)}^* &= 0 \text{ if } N^*(t-1)=1 \\ &= \frac{1}{4} \text{ if } N^*(t-2)=1 \text{ and } N^*(t-1)=0 \\ &= \frac{2}{3} \text{ if } N^*(t-3)=1 \text{ and } N^*(t-2)=N^*(t-1)=0 \\ &= 1 \text{ if } N^*(t-4)=1 \text{ and } N^*(t-3)=N^*(t-2)=N^*(t-1)=0 \\ &= 0 \text{ otherwise,} \end{aligned}$$

where the disturbance $U_{N(t)}^*$ is distributed uniformly between 0 and 1 and independent of all other disturbances and where $I(\cdot)$ denotes the indicator function. The effect of the stochastic adaptive monitoring interventions from CER question (Q2) can similarly be formalized

using $h_{N(t)}^* : (U_{N(t)}^*, \bar{N}^*(t-1), \bar{I}^*(t)) \mapsto I(U_{N(t)}^* \leq u_{\bar{N}^*(t-1), \bar{I}^*(t)}^*)$ with the differences that the values $u_{\bar{N}^*(t-1), \bar{I}^*(t)}^*$ are now also defined based on the past observed A1c measurements $\bar{I}^*(t)$. The effect of the adaptive monitoring intervention from CER question (Q3) that precludes a gap between A1c testing greater than 6 months can be formalized using

$h_{N(t)}^* : (\bar{N}^*(t-1), f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})) \mapsto f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})^{I(\bar{N}^*(t-6, t-1) \neq 0)}$. Finally, the monitoring interventions from CER question (Q4) can be formalized using

$h_{N(t)}^* : (U_{N(t)}^*, f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})) \mapsto I(U_{N(t)}^* \leq \delta^{1-f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})})$ with $\delta=0.1$ or 0.2 .

4.5 Static intervention on the right-censoring variables and a subset of the monitoring variables combined with a dynamic intervention on the treatment variables

In this section, we formalize the definition of counterfactual variables involved in the two identifiability results presented later in this paper. They are defined by the interventions introduced in Section 4.3 with the difference that the interventions on the monitoring

variables $N(t)$ are now restricted to the subset of monitoring variables identified by the time points t when the static monitoring interventions $n^*(t)$ are equal to 1.

The effect of such an intervention regimen on all other endogenous variables in the causal model is defined by the modified NPSEM with the following structural equations for the action variables $\mathcal{X} = \{A_1(K), A_2(K), \bar{N}(K)\}$:

$$\begin{aligned} \bullet \quad A_2^*(t) &= f_{A_2(t)}^*(P_{A_2^*(t)}^0, U_{A_2(t)}, U_{A_2(t)}^*) = Y^*(t) f_{A_2(t)}(P_{A_2^*(t)}^0, U_{A_2(t)}) + (1 - Y^*(t)) a_2^*(t) \\ \bullet \quad A_1^*(t) &= f_{A_1(t)}^*(P_{A_1^*(t)}^0, U_{A_1(t)}, U_{A_1(t)}^*) = Y^*(t) f_{A_1(t)}(P_{A_1^*(t)}^0, U_{A_1(t)}) + (1 - Y^*(t)) \times d_x^*(t) (P_{A_1^*(t)}^0) \\ \bullet \quad N^*(t) &= f_{N(t)}^*(P_{N^*(t)}^0, U_{N(t)}, U_{N(t)}^*) = Y^*(t) f_{N(t)}(P_{N^*(t)}^0, U_{N(t)}) + (1 - Y^*(t)) f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})^{1-n^*(t)} \end{aligned}$$

for any given distribution $P_{U^*|U}$ and a particular choice of dynamic treatment intervention $d_x^*(t): P_{A_1^*(t)} \mapsto a_1^*(t)$ for $t = 0, \dots, K$. We note that the conditional probability (denoted by g^*) of the counterfactual variable $N^*(t)$ above given its parents will not typically be degenerate for a general choice of static monitoring regimen n^* . By analogy with the effect definition described in the previous section, one may thus view the intervention on monitoring above as a stochastic intervention on the monitoring process $\bar{N}(t)$ such that the distribution of the resulting counterfactual monitoring variable $N^*(t)$ is defined as degenerate at 1 for time points t when $n^*(t) = 1$ and defined by the distribution P_U and functions $f_{N(t)}$ otherwise. Consequently, we also denote the counterfactual variables $Y^*(t)$, $Z^*(t)$, $I_0^*(t)$, $P^*(t)$, $A_2^*(t)$, $A_1^*(t)$ and $N^*(t)$ defined by the modified NPSEM just described by $Y_{d_x^*, g^*}^*(t)$, $Z_{d_x^*, g^*}^*(t)$, $I_{0_{d_x^*, g^*}}^*(t)$, $I_{d_x^*, g^*}^*(t)$, $A_{2_{d_x^*, g^*}}^*(t)$, $A_{1_{d_x^*, g^*}}^*(t)$ and $N_{d_x^*, g^*}^*(t)$, respectively. Note that here again, we omit the use of the subscript $\bar{a}_2^*(K)$ and thus make the intervention on the right-censoring variables implicit in the notation of these counterfactual variables. We denote the support of the counterfactual treatment and monitoring processes $\bar{A}_{1_{d_x^*, g^*}}^*(t)$ and $\bar{N}_{d_x^*, g^*}^*(K)$ by $\bar{\mathcal{A}}_{d_x^*, g^*}^*$ and $\bar{\mathcal{N}}_{d_x^*, g^*}^*$, respectively.

5 No Direct Effect (NDE) assumption

We denote, respectively, by $\bar{\mathcal{A}}^*$ and $\bar{\mathcal{N}}^*$, the union of the supports of the counterfactual treatment processes $\bar{A}_1^*(K)$ and counterfactual monitoring processes $\bar{N}^*(K)$ introduced in Section 4.2 for any choices of static treatment and monitoring interventions $\bar{a}_1^*$ and n^* :

$$\bar{\mathcal{A}}^* \equiv \bigcup_{\bar{a}_1^*, n^*} \bar{\mathcal{A}}_{\bar{a}_1^*, n^*}^* \quad (1)$$

$$\bar{\mathcal{N}}^* \equiv \bigcup_{\bar{a}_1^*, n^*} \bar{\mathcal{N}}_{\bar{a}_1^*, n^*}^* \quad (2)$$

We note that these sets contain the supports of, not only, the observed treatment and monitoring processes, but also the supports of the various counterfactual treatment and

monitoring processes defined in Sections 4 based on specific choices for $\bar{a}_1^*$, $\bar{\pi}^*$, d_x , d_x^* , and $h_{N(t)}^*$, i.e.,

$$\bar{\mathcal{A}}^* \supset \bar{\mathcal{A}} \cup \bar{\mathcal{A}}_{\bar{a}^*} \cup \bar{\mathcal{A}}_{\bar{a}^*, \bar{n}^*} \cup \bar{\mathcal{A}}_{d_x, \bar{n}^*} \cup \bar{\mathcal{A}}_{d_x, g} \cup \bar{\mathcal{A}}_{d_x^*, g^*}$$

$$\bar{\mathcal{N}}^* \supset \bar{\mathcal{N}} \cup \bar{\mathcal{N}}_{\bar{a}^*} \cup \bar{\mathcal{N}}_{\bar{a}^*, \bar{n}^*} \cup \bar{\mathcal{N}}_{d_x, \bar{n}^*} \cup \bar{\mathcal{N}}_{d_x, g} \cup \bar{\mathcal{N}}_{d_x^*, g^*}.$$

We define the NDE assumption as the following set of equalities:

$$\left. \begin{aligned} f_{Y(t+1)}(P_{Y_{\bar{a}^*, \bar{n}^*}(t+1)}^0, U_{Y(t+1)}) &= f_{Y(t+1)}(P_{Y_{\bar{a}^*, \bar{n}^*}(t+1), \bar{n}^{*'}}^0, U_{Y(t+1)}) \\ f_{Z(t+1)}(P_{Z_{\bar{a}^*, \bar{n}^*}(t+1)}^0, U_{Z(t+1)}) &= f_{Z(t+1)}(P_{Z_{\bar{a}^*, \bar{n}^*}(t+1), \bar{n}^{*'}}^0, U_{Z(t+1)}) \\ f_{I_0(t+1)}(P_{I_{0\bar{a}^*, \bar{n}^*}(t+1)}^0, U_{I_0(t+1)}) &= f_{I_0(t+1)}(P_{I_{0\bar{a}^*, \bar{n}^*}(t+1), \bar{n}^{*'}}^0, U_{I_0(t+1)}) \end{aligned} \right\} \quad \text{for all } t=0, \dots, K,$$

$\bar{a}_1^* \in \bar{\mathcal{A}}^*$, and
 $(\bar{n}^*, \bar{n}^{*'}) \in \bar{\mathcal{N}}^{*2}$

(3)

where we use the generic notation $P_{X^*(t+1), \bar{n}^{*'}}^0$ to denote the vector of variables defined by the parents $P_{X^*(t+1)}^0$ of the counterfactual variable $X^*(t+1)$ in which the monitoring variables $\bar{N}^*(t)$ are replaced by the fixed values $\bar{\pi}^{*'}(t)$. The equalities above encode the assumption that the NPSEM functions that generate the covariates $Y(t+1)$, $Z(t+1)$, and $I_0(t+1)$ for $t = 0, \dots, K$ return the same values whether their input arguments are set to the vector of their respective counterfactual parents under the static right-censoring, treatment, and monitoring interventions $\bar{a}_2^*=0$, $\bar{a}_1^* \in \bar{\mathcal{A}}^*$, $\bar{\pi}^* \in \bar{\mathcal{N}}^*$ from Section 4.2 or set to the same vector in which the counterfactual monitoring history $\bar{N}^*(t)$ is replaced with any other monitoring history $\bar{\pi}^{*'} \in \bar{\mathcal{N}}^*$.

We note that the NDE assumption is implied by the assumption that the functions $f_{Y(t)}$, $f_{Z(t)}$ and $f_{I_0(t)}$ do not depend on the monitoring process $\bar{N}^*(t-1)$ for $t \geq 1$. As illustrated in Figure 1, the latter exclusion restrictions can be encoded graphically in a Directed Acyclic Graph (DAG) with the absence of arrows from a node $N(t)$ to any subsequent covariate nodes $Y(j)$, $Z(j)$, and $I_0(j)$ for $j > t$. Such exclusion restrictions do not preclude however the existence of directed paths from a node $N(t)$ to any subsequent covariate nodes $Y(j)$, $Z(j)$, and $I_0(j)$ but requires that all such paths be intercepted by treatment or censoring nodes $A_2(j)$ or $A_1(j)$ for $j > t$.

In Appendix B, we show that the NDE assumption above is equivalent to the following equalities between the counterfactuals introduced in Sections 4.1 and 4.2:

$$L_{0_{\bar{a}^*, \bar{n}^*}}(t) = L_{0_{\bar{a}^*}}(t) \text{ for all } t=1, \dots, K+1, \bar{a}_1^* \in \bar{\mathcal{A}}^*, \text{ and } \bar{n}^* \in \bar{\mathcal{N}}^* \quad (4)$$

We note that this latter formulation of the NDE was introduced in prior published work (Robins et al., 2008) which was not based on an NPSEM and in which counterfactuals were instead introduced as primitives (Pearl, 2010b).

We also define the weak NDE assumption (which is implied by the NDE assumption above) as the upholding of one of the following two sets of equalities defined by the same choice of interventions $\bar{m}^*$, d_x , and d_x^* :

$$\left. \begin{aligned} f_{Y(t+1)}(P_{Y_{d_x^*, g^*}(t+1)}^0, U_{Y(t+1)}) &= f_{Y(t+1)}(P_{Y_{d_x^*, g^*}(t+1), \bar{N}_{d_x, \bar{n}^*}(t)}^0, U_{Y(t+1)}) \\ f_{Z(t+1)}(P_{Z_{d_x^*, g^*}(t+1)}^0, U_{Z(t+1)}) &= f_{Z(t+1)}(P_{Z_{d_x^*, g^*}(t+1), \bar{N}_{d_x, \bar{n}^*}(t)}^0, U_{Z(t+1)}) \\ f_{I_0(t+1)}(P_{I_{0_{d_x^*, g^*}(t+1)}}^0, U_{I_0(t+1)}) &= f_{I_0(t+1)}(P_{I_{0_{d_x^*, g^*}(t+1), \bar{N}_{d_x, \bar{n}^*}(t)}}^0, U_{I_0(t+1)}) \end{aligned} \right\} \text{ for all } t=0, \dots, K \text{ or}$$

(5)

$$\left. \begin{aligned} f_{Y(t+1)}(P_{Y_{d_x, \bar{n}^*}(t+1)}^0, U_{Y(t+1)}) &= f_{Y(t+1)}(P_{Y_{d_x, \bar{n}^*}(t+1), \bar{N}_{d_x^*, g^*}(t)}^0, U_{Y(t+1)}) \\ f_{Z(t+1)}(P_{Z_{d_x, \bar{n}^*}(t+1)}^0, U_{Z(t+1)}) &= f_{Z(t+1)}(P_{Z_{d_x, \bar{n}^*}(t+1), \bar{N}_{d_x^*, g^*}(t)}^0, U_{Z(t+1)}) \\ f_{I_0(t+1)}(P_{I_{0_{d_x, \bar{n}^*}(t+1)}}^0, U_{I_0(t+1)}) &= f_{I_0(t+1)}(P_{I_{0_{d_x, \bar{n}^*}(t+1), \bar{N}_{d_x^*, g^*}(t)}}^0, U_{I_0(t+1)}) \end{aligned} \right\} \text{ for all } t=0, \dots, K.$$

(6)

The equalities (5) (resp. (6)) above encode the assumption that the NPSEM functions that generate the covariates $Y(t+1)$, $Z(t+1)$, and $I_0(t+1)$ for $t=0, \dots, K$ return the same values whether their input arguments are set to the vector of their respective counterfactual parents under the interventions from Section 4.5 (resp. Section 4.3) or set to the same vector in which the counterfactual monitoring history $\bar{N}_{d_x^*, g^*}(t)$ (resp. $\bar{N}_{d_x, \bar{m}^*}(t)$) is replaced with the counterfactual monitoring history $\bar{N}_{d_x, \bar{m}^*}(t)$ from Section 4.3 (resp. $\bar{N}_{d_x^*, g^*}(t)$ from Section 4.5).

6 Identifiability result 1

In Appendix C, we prove the following equalities of counterfactuals.

Theorem 1

For any given dynamic treatment intervention d_x defined in Section 4.3 and any given static monitoring intervention $\bar{m}^*$, if the dynamic treatment intervention d_x^* is defined as the dynamic treatment intervention d_x applied using only the information from the past covariate process (t) available under both the actual observed monitoring process and the static monitoring regimen $\bar{m}^*$, i.e., for $t = 0, \dots, K$:

$$d_x^*(t)(P_{A_1(t)}) \equiv d_x(t) \left(\Gamma_{\bar{m}^*}(P_{A_1(t)}) \right) \quad (7)$$

with the function $\Gamma_{\bar{m}^*}: P_{A_1(t)} \mapsto P_{A_1(t)}^{\bar{m}^*} \equiv (\bar{L}'(t), \bar{A}_2(t), \bar{A}_1(t-1), \bar{N}'(t-1))$ defined by

$$\begin{aligned} N'(j) &\equiv Y(j)N'(j-1) + (1-Y(j))n^*(j)N(j) \text{ for } j=0, \dots, t-1 \\ L'(j) &\equiv (Y(j), Z(j), I'(j)) \text{ such that } I'(j) \equiv Y(j)I'(j-1) + (1-Y(j)) \times \\ &n^*(j-1)I(j) \text{ for } j=0, \dots, t(N'(-1) \text{ and } n^*(-1) \text{ are nil by convention}) \end{aligned} \quad (8)$$

then, if the (weak) NDE assumption also holds, the counterfactual outcomes defined in Section 4.3 by a static intervention on the monitoring variables $\bar{m}^*$ combined with the dynamic treatment intervention d_x equal the counterfactual outcomes defined in Section 4.5 by a stochastic intervention g^* on the monitoring combined with the dynamic treatment intervention d_x^* defined above, i.e.:

$$Y_{d_x, \bar{n}^*}(t+1) = Y_{d_x^*, g^*}(t+1) \text{ for } t=0, \dots, K, \quad (9)$$

and the following counterfactual equalities also hold:

$$\begin{aligned} \bar{Z}_{d_x, \bar{n}^*}(t) &= \bar{Z}_{d_x^*, g^*}(t) \\ \bar{I}_{0, d_x, \bar{n}^*}(t) &= \bar{I}_{0, d_x^*, g^*}(t) \\ \bar{I}_{d_x, \bar{n}^*}(t) &= \bar{I}_{d_x^*, g^*}(t) \\ \bar{A}_{2, d_x, \bar{n}^*}(t) &= \bar{A}_{2, d_x^*, g^*}(t) \\ \bar{A}_{1, d_x, \bar{n}^*}(t) &= \bar{A}_{1, d_x^*, g^*}(t) \\ \bar{N}_{d_x, \bar{n}^*}(t) &= \bar{N}_{d_x^*, g^*}(t) \end{aligned} \quad \text{for } t=0, \dots, K, \quad (10)$$

where we recall (see Section 4.5 for details) that the stochastic intervention g^* on the monitoring process is entirely defined by a static intervention on a subset of the monitoring variables before failure (setting them to 1 whenever $n^*(t) = 1$ while all other monitoring variables are not intervened upon).

7 Identifiability result 2

In Appendix D, we prove the following results.

Theorem 2

The following equality (13) holds for each $t = 1, \dots, K + 1$ and any intervention regimen on the action variables $A_1(t), A_2(t), N(t)$ as defined in Section 4.4 that satisfies both the following conditional independences:

$$\begin{aligned}
 A_2^*(t) &\perp \left(U_{A_1(j)}, U_{N(j)}, U_{L_0(j+1)}, U_{A_2(j+1)} \right)_{j=t, \dots, K} \left| P_{A_2^*(t)}^0, U_{A_2(t)}, \right. \\
 A_1^*(t) &\perp \left(\left(U_{N(j)}, U_{L_0(j+1)}, U_{A_2(j+1)}, U_{A_1(j+1)} \right)_{j=t, \dots, K-1}, U_{N(K)}, U_{L_0(K+1)}, U_{A_2(K+1)} \right) \left| P_{A_1^*(t)}^0, U_{A_1(t)}, \right. \\
 N^*(t) &\perp \left(\left(U_{L_0(j)}, U_{A_2(j)}, U_{A_1(j)}, U_{N(j)} \right)_{j=t+1, \dots, K}, U_{L_0(K+1)}, U_{A_2(K+1)} \right) \left| P_{N^*(t)}^0, U_{N(t)}, \right. \quad (11)
 \end{aligned}$$

for $t = 0, \dots, K$ with $U_{L_0(j)} \equiv (U_{Y(j)}, U_{Z(j)}, U_{I_0(j)})$, and the constraint that the functions $h_{N(t)}^*$ are not dependent on the counterfactual monitoring variables, $N^{*0}(t) = f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})$, i.e.:

$$h_{N(t)}^* \left(P_{N^*(t)}, U_{N(t)}^*, N^{*0}(t) \right) = l_{N(t)}^* \left(P_{N^*(t)}, U_{N(t)}^* \right) \quad (12)$$

for a function $l_{N(t)}^*$ of only $P_{N^*(t)}$ and $U_{N(t)}^*$:

$$\begin{aligned}
 E(Y_{d_x, g}(t)) &= \sum_{\bar{n}^*(t-1)} E \left(Y_{d_x, \bar{n}^*(t)} \prod_{j=0}^{t-1} g \left(N_{d_x, \bar{n}^*(j)} | P_{N_{d_x, \bar{n}^*(j)}} \right) I(N_{d_x, \bar{n}^*(j)} = n^*(j))^{Y_{d_x, \bar{n}^*(j)}} \right), \\
 &\quad (13)
 \end{aligned}$$

where $\bar{n}^*$ is defined by an arbitrary choice for $\bar{n}^*(t, K)$, $I(\cdot)$ denotes the indicator function, and the functions g are defined by the distribution of the counterfactual variables introduced in Section 4.4 as follows:

$$\begin{aligned}
 g(n(j) | p_{n(j)}) &= P \left(N_{d_x, g}(j) = n(j) | P_{N_{d_x, g}(j)} = p_{n(j)} \right) \\
 &= P_{U, U^*} \left((U_{N(j)}, U_{N(j)}^*) \in f_{p_{n(j)}^0}^{*-1}(n(j)) | P_{N_{d_x, g}(j)}^0 = p_{n(j)}^0 \right),
 \end{aligned}$$

when the conditional probabilities on the right-hand side are well-defined (i.e., the conditional event has non-zero probability) and otherwise (e.g., when $\pi_2(j) = 0$), by convention,

$$g(n(j)|p_{n(j)}) = P_{U, U^*} \left((U_{N(j)}, U_{N(j)}^*) \in f_{p_{n(j)}^0}^{*-1}(n(j)) \right)$$

where we use $p_n(j) \equiv (\bar{l}(j), \pi_2(j), \pi_1(j), \pi(j-1))$ with $\bar{l}(j) \equiv (\bar{y}(j), \bar{z}(j), \bar{i}(j))$, $p_{n(j)}^0 \equiv (\bar{l}_0(j), \bar{a}_2(j), \bar{a}_1(j), \bar{n}(j-1))$ with $\bar{l}_0(j) \equiv (\bar{y}(j), \bar{z}(j), \bar{i}_0(j))$ for any given $i_0(k)$ such that $i(k) = n(k-1)i_0(k)$, and $f_{p_{n(j)}^0}^{*-1}(n(j)) = \{(u_{N(j)}, u_{N(j)}^*) : f_{N(j)}^*(p_{n(j)}^0, u_{N(j)}, u_{N(j)}^*) = n(j)\}$.

Alternatively, the previous equality (13) also holds even when the functions $h_{N(t)}^*$ are dependent on the counterfactual monitoring variables, $N^{*0}(t) = f_{N(t)}(P_{N(t)}^0, U_{N(t)})$, if, not only, the intervention regimen on the action variables $A_1(t)$, $A_2(t)$, $N(t)$ defined in Section 4.4 satisfies the previous conditional independences (11), but also if both the following conditional independences hold

$$U_{N(t)} \perp (U_{L_0(j)})_{j=t+1, \dots, K} \left| (U_{L_0(j)}, U_{A_2(j)}, U_{A_1(j)})_{j=0, \dots, t}, (U_{N(j)})_{j=0, \dots, t-1} \text{ for } t=0, \dots, K. \right. \quad (14)$$

and the functions $f_{N(t)}$ are not dependent on the latent variables $i_0(t)$ for $t=0, \dots, K$:

$$f_{N(t)}(P_{N(t)}^0, U_{N(t)}) = f'_{N(t)}(P_{N(t)}, U_{N(t)}) \quad (15)$$

for a function $f'_{N(t)}$ of only the observed parents $P_{N(t)}$ and $U_{N(t)}$.

In addition, if both the previous equality (13) and the (weak) NDE assumption hold then the following equality also holds:

$$E(Y_{d_x, g}(t)) = \sum_{\bar{n}^*(t-1)} E \left(Y_{d_x, g^*}^*(t) \prod_{j=0}^{t-1} g \left(N'_{d_x, g^*}(j) | P_{N'_{d_x, g^*}(j)} \right) I \left(N'_{d_x, g^*}(j) = n^*(j) \right)^{Y_{d_x, g^*}(j)} \right), \quad (16)$$

where we use the definition for the interventions d_x^* and g^* given in Theorem 1,

$$P'_{N'_{d_x^*, g^*}(j)} \equiv (\bar{L}'_{d_x^*, g^*}(j), \bar{A}'_{2, d_x^*, g^*}(j), \bar{A}'_{1, d_x^*, g^*}(j), \bar{N}'_{d_x^*, g^*}(j-1)) \text{ and we recall the definitions}$$

$$L'_{d_x^*, g^*}(j) \equiv (Y_{d_x^*, g^*}(j), Z_{d_x^*, g^*}(j), I'_{d_x^*, g^*}(j)), I'_{d_x^*, g^*}(j) \equiv Y_{d_x^*, g^*}(j)I'_{d_x^*, g^*}(j-1) + (1 - Y_{d_x^*, g^*}(j))n^*(j-1)I_{d_x^*, g^*}(j)$$

, and $N'_{d_x^*, g^*}(j) \equiv Y_{d_x^*, g^*}(j)N'_{d_x^*, g^*}(j-1) + (1 - Y_{d_x^*, g^*}(j))n^*(j)N_{d_x^*, g^*}(j)$.

Equality (16) corresponds to equality (13) with the difference that all counterfactual variables resulting from the joint treatment and monitoring interventions d_x and $\bar{n}^*$ on the right-hand side of equality (13) are replaced with counterfactual variables resulting from the joint treatment and monitoring interventions d_x^* and g^* (using the counterfactual equalities from Theorem 1).

We note that while the conditional independence assumptions (11) and condition (12) only put constraints on the definition of intervention regimens (specified by the analyst) for which equalities (13) and (16) can hold, the conditional independences (14) and exclusion restriction (15) can be viewed instead as causal assumptions since they restrict the class of NPSEM (not under the analyst's control) for which these equalities can hold.

In practice and as illustrated with the four monitoring intervention types (Q1–Q4) described in Section 4.4, the conditional independence assumptions (11) will typically hold since interventions that have been commonly considered in real-data analyses can be defined using disturbances U^* (specified by the analyst) that are independent of each other and of all disturbances U . However, condition (12) precludes stochastic monitoring interventions that are defined based on the monitoring decisions ($N^{*0}(t)$) that would naturally occur under no monitoring interventions such as two of the four types of interventions illustrated in Section 4.4, more specifically, the interventions involved in the CER questions (Q3) and (Q4). In particular, this assumption also excludes monitoring interventions of the type “intention-to-treat” where monitoring variables are only intervened upon up to occurrence of some events (e.g., start of treatment).

The conditional independences (14) holds in practice if it can be assumed that all backdoor paths from any given monitoring node $N(t)$ to all subsequent covariates including the latent variables $I_0(j)$ are blocked by conditioning on past treatment, right-censoring, monitoring and covariates (including the past latent variables $I_0(j)$). The exclusion restriction (15) corresponds with the assumption of no arrows from any node $I_0(j)$ to any subsequent monitoring node $N(k)$ for $k > j$ in a DAG (past latent covariates can have however an indirect effect on monitoring through past observed covariates).

8 Practical relevance

Although the exploitation of the two identifiability results in this article to derive alternate NDE-based estimators of the joint effect of a dynamic treatment intervention and a general monitoring intervention will be detailed in separate work, we briefly illustrate below the practical implications of the formal results presented above with a simple example introduced in Section 4.3.

To address the diabetes CER question “*What is the difference in the risk of onset DPN if patients have their A1c monitored every 3 months versus every 6 months and initiate insulin therapy the first time their A1c drifts above 8.5%?*”, we would aim to evaluate the following causal estimands for $t = 1, \dots, K + 1$:

$$E(Y_{d_x, \bar{n}_1^*}(t)) - E(Y_{d_x, \bar{n}_2^*}(t)),$$

where t is expressed in months from study entry, d_x represents the dynamic A1c-based insulin initiation strategy of interest, and $\bar{n}_1^*$ and $\bar{n}_2^*$ denote the two static monitoring interventions corresponding with A1c testing every 3 and 6 months respectively, i.e., $\bar{n}_1^* = (1, 0, 0, 1, 0, 0, \dots)$ and $\bar{n}_2^* = (1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, \dots)$.

At each time point t , each of the two counterfactual cumulative risks of interest $\psi_{d_x, \bar{n}^*} \equiv E(Y_{d_x, \bar{n}^*}(t))$ for $\bar{n}^* = \bar{n}_1^*, \bar{n}_2^*$ can be evaluated using the following estimating function derived from a straightforward extension of prior results (Murphy et al., 2001, Hernan et al., 2006, van der Laan and Petersen, 2007, Robins et al., 2008, Cain et al., 2010, Petersen et al., 2014) on dynamic marginal structural modeling by IPW estimation:

$$D_{d_x, \bar{n}^*}^{\text{bd ipw}}(O|g_0, \psi) = \frac{\prod_{j=0}^{\tilde{T}(t)} I(A_2(j)=0) I(A_1(j)=d_x(j)(P_{A_1(j)})) I(N(j)=n^*(j))}{\prod_{j=0}^{\tilde{T}(t)} g_0(A_2(j)|P_{A_2(j)}) g_0(A_1(j)|P_{A_1(j)}) g_0(N(j)|P_{N(j)})} (Y(t+1) - \psi),$$

where the notation g_0 in the denominator refers to the conditional probabilities (defined by P_0) of each action variable given the subset of its *observed* parents and the notation $\tilde{T}(t)$ represents the minimum of time t and the patient's follow-up time, i.e., $\tilde{T}(t) = \min(t, T, C)$, with T denoting the failure time when it occurs, i.e., $T \equiv \min\{j = 0, \dots, K: Y(j+1) = 1\}$ and is nil if $\bar{Y}(K+1) = 0$, and C denoting the right-censoring time when it occurs, i.e., $C \equiv \min\{j = 0, \dots, K: A_2(j) = 1\}$ and is nil if $A_2(K+1) = 0$. Thus, the following solution of the estimating equations associated with this estimating function defines a consistent and asymptotically linear estimator (van der Laan and Robins, 2003) of $\psi_{d_x, \bar{n}^*}$ referred to as the standard bounded (Robins, Sued, Lei-Gomez, and Rotnitzky, 2007) IPW estimator and denoted by $\hat{\psi}_{d_x, \bar{n}^*}^{\text{bd ipw}}$:

$$\sum_{i=1}^n \frac{w(O_i)}{\sum_{j=1}^n w(O_j)} Y_i(t+1), \quad (17)$$

with

$$w(O_i) \equiv \frac{\prod_{j=0}^{\tilde{T}_i(t)} I(A_{2,i}(j)=0) I(A_{1,i}(j)=d_x(j)(P_{A_1(j),i})) I(N_i(j)=n^*(j))}{\prod_{j=0}^{\tilde{T}_i(t)} g_0(A_{2,i}(j)|P_{A_2(j),i}) g_0(A_{1,i}(j)|P_{A_1(j),i}) g_0(N_i(j)|P_{N(j),i})}, \quad (18)$$

and where the subscript i is used to identify each of the n independent and identically distributed copies of the observed data process O (or, by extension, any components of O) defined in Section 2.

If the NDE assumption holds, we can also make use of the equality

$E(Y_{d_x, \bar{n}^*}(t)) = E(Y_{d_x^*, g^*}(t))$ implied by Theorem 1 to indirectly develop another IPW estimator of $\Psi_{d_x, \bar{n}^*}$ through the evaluation of the causal estimand $\psi_{d_x^*, g^*} \equiv E(Y_{d_x^*, g^*}(t))$ using the same general IPW estimation methodology just described but applied now to the joint dynamic treatment and stochastic monitoring intervention (d_x^*, g^*) instead of the joint dynamic treatment and static monitoring intervention $(d_x, \bar{n}^*)$. We recall that the stochastic monitoring intervention g^* consists of a static intervention on a subset of the monitoring variables before failure (setting them to 1 whenever $\bar{n}^*(t) = 1$ while all other monitoring variables are not intervened upon). More specifically at each time point t , each of the counterfactual cumulative risks $\psi_{d_x^*, g^*}$ with $\bar{n}^* = \bar{n}_1^*, \bar{n}_2^*$ can be evaluated using the bounded IPW estimator denoted by $\hat{\psi}_{d_x^*, g^*}^{\text{bd ipw}}$ and defined by expression (17) with the difference that the inverse probability (IP) weights (18) are now defined by:

$$w(O_i) = \frac{\prod_{j=0}^{\tilde{T}_i(t)} I(A_{2,i}(j)=0) I(A_{1,i}(j)=\mathbf{d}_x^*(j)(P_{A_1(j),i})) \prod_{j \leq \tilde{T}_i(t): \mathbf{n}^*(j)=1} I(N_i(j)=1)}{\prod_{j=0}^{\tilde{T}_i(t)} g_0(A_{2,i}(j)|P_{A_2(j),i}) g_0(A_{1,i}(j)|P_{A_1(j),i}) \prod_{j \leq \tilde{T}_i(t): \mathbf{n}^*(j)=1} g_0(N_i(j)|P_{N(j),i})}, \quad (19)$$

where we use bold font to highlight the difference with the IP weights (18) that define the standard IPW estimator $\hat{\psi}_{d_x, \bar{n}^*}^{\text{bd ipw}}$. The IPW estimator $\hat{\psi}_{d_x^*, g^*}^{\text{bd ipw}}$ is referred to as the NDE-based bounded IPW estimator of $\Psi_{d_x, \bar{n}^*}$ because, although it is foremost an estimator of $\psi_{d_x^*, g^*}$, Theorem 1 implies that it is also an indirect estimator of $\Psi_{d_x, \bar{n}^*}$ when the NDE assumption holds.

We note that the alternate NDE-based IPW estimator $\hat{\psi}_{d_x^*, g^*}^{\text{bd ipw}}$ of $\Psi_{d_x, \bar{n}^*}$ derived above (i.e., estimator (17) with $w(O_i)$ defined by equality (19)) is the bounded, time-to-event analog of the NDE-based unbounded IPW estimator given by expression (30) in Section 6.3.2 of prior published work (Robins et al., 2008). Here, we have formalized results that demonstrate that this previously proposed alternate IPW estimator can simply be viewed as a *standard* IPW estimator for the mean counterfactual outcome $Y_{d_x^*, g^*}(t)$ which, under the NDE, is equal to the mean counterfactual outcome of interest $Y_{d_x, \bar{n}^*}(t)$. An important consequence of this formalization is that, even when the NDE assumption is violated, the inference from the IPW estimator $\hat{\psi}_{d_x^*, g^*}^{\text{bd ipw}}$ above can remain causally interpretable as the effect of the dynamic treatment intervention d_x^* and stochastic monitoring intervention g^* . When the NDE assumption also holds, the IPW estimator $\hat{\psi}_{d_x^*, g^*}^{\text{bd ipw}}$ may not only provide valid inference for the causal estimand originally of interest (i.e., $E(Y_{d_x, \bar{n}^*})$) as could the standard IPW estimator (17, 18) but, unlike that latter estimator, it relies on a possibly weaker positivity

assumption which can translate into improved estimation properties as discussed further below. Another important practical consequence of the identifiability results in this paper is the simplification of the development of NDE-based doubly robust and locally efficient estimators of the effect of joint dynamic treatment and general monitoring interventions. The description of these new estimators is left for future work.

As just noted, consistent estimation of the causal estimand $\Psi_{d_X, \bar{n}^*}$ with either the standard bounded IPW estimator $\hat{\psi}_{d_X, \bar{n}^*}^{\text{bd ipw}}$ or its NDE-based analog $\hat{\psi}_{d_X^*, g^*}^{\text{bd ipw}}$ relies on a positivity assumption about the monitoring decisions $\bar{N}(K)$ in the observed data. More specifically, the standard IPW estimator $\hat{\psi}_{d_X, \bar{n}^*}^{\text{bd ipw}}$ relies on the following positivity assumption about the observed monitoring process:

$$P_0(N(j)=n^*(j) | \bar{Y}(j)=0, \bar{Z}(j), \bar{I}(j), \bar{A}_2(j)=0, \bar{A}_1(j)=d_X, \bar{N}(j-1)=\bar{n}^*(j-1)) > 0 \text{ for all time points } j=0, \dots, K \quad (20)$$

while the NDE-based IPW estimator $\hat{\psi}_{d_X^*, g^*}^{\text{bd ipw}}$ relies on the following alternate positivity assumption about the observed monitoring process:

$$P_0(N(j)=1 | \bar{Y}(j)=0, \bar{Z}(j), \bar{I}(j), \bar{A}_2(j)=0, \bar{A}_1(j)=d_X^*, \bar{N}^0(\bar{n}^*, j-1), \bar{N}^1(\bar{n}^*, j-1)=1) > 0 \text{ only for } j=0, \dots, K \text{ such that } n^*(j)=1$$

(21)

where we use the shorthand notation $\bar{A}_1(j) = d_X$ and $\bar{A}_1(j) = d_X^*$ to represent the conditional events indicating that treatment decisions were made before and at time j according to the dynamic intervention d_X and d_X^* , respectively, and where we also use the notation $\bar{N}^n(\bar{n}^*, j) \equiv (N(k))_{k:j, n^*(k)=n}$ for $n=0,1$ to represent the monitoring decisions up to time $j-1$ that are (when $n=1$) or are not (when $n=0$) intervened upon under intervention g^* . Interpreting the previous two positivity assumptions in the context of the diabetes example introduced at the beginning of this section can clarify the difference between the two sets of conditions. For instance, if we consider the monitoring intervention $\bar{n}_2^* = (1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, \dots)$, the first positivity assumption requires that each subject can possibly experience the exact sequence of monitoring decisions $\bar{n}_2^*$, i.e., be monitored every 6 months with no additional monitoring in between these monitoring events, whereas the second positivity assumption only requires that each subject can possibly be monitored every 6 months and *does not place any constraint* on the monitoring decision processes in between these 6-month periodic monitoring events. Thus, as highlighted by the bold font, a quick examination of these two assumptions suggest that the second positivity assumption is weaker than the first because the constraints of strictly positive monitoring probabilities will generally apply to a smaller set of monitoring variables for the intervention g^* than for the intervention $\bar{n}^*$. The expected

practical consequence of this weakening of the positivity assumption for monitoring decisions is improved estimation performance (bias and precision) for the NDE-based IPW estimator $\hat{\psi}_{d_x^*, g^*}^{\text{bd ipw}}$ compared to that of the standard estimator $\hat{\psi}_{d_x, \bar{g}^*}^{\text{bd ipw}}$ because of the likely more stable IPW weights (19) that result from the exclusion of the terms $\prod_{j=i(t): \pi^*(j)=0} g_0(N_j(j) | P_{N(j), i})$ from the definition of the denominators of the NDE-based IP weights compared to that of the standard IP weights (18). A closer examination of the positivity assumptions above suggests however that, in practice, gains from weakening of the positivity assumption for the monitoring process might be offset by the fact that there might be fewer observations whose treatment history is concordant with following the dynamic intervention

d_x^* or the fact that, for these observations, the treatment factors $\prod_{j=0}^{\tilde{T}_i(t)} g_0(A_{1,i}(j) | P'_{A_1(j)})$ might be closer to 0 than for the observations whose treatment history is concordant with following the dynamic intervention d_x (i.e., for the observations contributing to the standard IPW estimator (17,18)). Future applied work with simulated and real data should provide practical insights into the relevance of NDE-based estimators by studying the potential trade-off between improved weight stability resulting from an intervention g^* on fewer monitoring variables and worsened weight stability resulting from poorer adherence to a treatment strategy d_x^* that requires that treatment decisions be made as if some past observed information about the patient's condition had actually been unobserved. Similarly, such applied work should also illustrate the stronger sequential randomization assumption (SRA) required for consistent estimation of the joint effect of a dynamic treatment intervention and a stochastic (e.g., static) monitoring intervention compared to the weaker SRA required for consistent estimation of the effect of a dynamic treatment intervention (without a monitoring intervention). In particular, it has been argued (Robins et al., 2008, Section 7) that the stronger SRA requirement will impede applications of these estimators in studies based on standard Electronic Health Records databases “[...] unless we collect data at t on the health status of not only those who return to the clinic at t (which we typically do) but also those who do not come to the clinic at t (which we almost never do)”.

9 Discussion

In this article, we utilized the NPSEM framework to formally derive two identifiability results for evaluating the effect of general joint treatment and monitoring interventions on time-to-event outcomes under a no direct effect assumption. In addition, we illustrated the practical relevance of these identifiability results by showing how one of them can be directly exploited to easily derive a new NDE-based estimator by simply applying a common estimation approach. More specifically, we have shown that our first identifiability result quickly leads to the construction of a bounded IPW estimator for the counterfactual cumulative risk under a joint dynamic treatment intervention and a static monitoring intervention. We also noted that a slightly different NDE-based estimator that can similarly be derived from our first identifiability result corresponds to the unbounded (Horvitz-Thompson) IPW estimator presented in prior work. This unbounded estimator had been formally derived based on the missing data framework for problems with end-of-study outcomes using a novel theorem.

With this paper, our intention is not to emphasize that the IPW estimator developed here is novel or better because it is bounded, NPSEM-based, or applicable to time-to-event outcomes compared to the unbounded alternative developed previously for end-of-study outcomes and derived based on the missing data framework. Instead, we want to highlight how formal derivation of NDE-based estimators can be made transparent and simple using the identifiability results presented here. Indeed, our example shows that the unbounded IPW estimator developed in prior work actually corresponds to a usual (i.e., based on the standard G-computation formula from Robins (1986)) IPW estimator that targets a particular causal estimand which happens to equal the intended causal estimand of interest when the NDE assumption holds. If the NDE assumption is violated, our result also directly implies that the estimator can remain causally interpretable. This result is an additional contribution which, to our knowledge, was not formally noted previously.

In short, we propose that results in this paper can, not only, further explicate previously proposed NDE-based IPW estimators but also facilitate the derivation of alternate NDE-based estimators using existing general estimation roadmaps such as those rooted in the standard G-computation formula (e.g., maximum likelihood, IPW, augmented-IPW, and targeted minimum loss based estimators). These results also show that NDE-based estimators can remain causally interpretable even when the NDE assumption is violated.

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Appendix

A Link between the NPSEM and the observed data distribution

For any realization $p_{X(t)}^0$ of the parents $P_{X(t)}^0$ of a given variable $X(t)$, we define the function $f_{p_{X(t)}^0} : U_{X(t)} \mapsto f_{X(t)}(p_{X(t)}^0, U_{X(t)})$ and denote the fiber of any element $x(t)$ in its codomain by $f_{p_{X(t)}^0}^{-1}(x(t)) = \{u_{X(t)} : f_{X(t)}(p_{X(t)}^0, u_{X(t)}) = x(t)\}$. If variable $X(t)$ has no parent, i.e., $P_{X(t)}^0$ is nil, i.e., the following set of realizations of $U_{X(t)}$:

$$f_{p_{X(t)}^0}^{-1}(x(t)) = \{u_{X(t)} : f_{X(t)}(p_{X(t)}^0, u_{X(t)}) = x(t)\}.$$

Under the causal model encoded by the NPSEM above, the distribution of the vector of disturbances $\mathcal{U}$ defines the distribution of the observed data O as follows:

$$\begin{aligned} P_0(\mathbf{O} = o) &\equiv \sum_{\bar{\mathbf{I}}_0(K+1)} I(i_0(0) = i(0)) \prod_{t=1}^{K+1} I(n(t-1)i_0(t) = i(t)) P_U \\ &\left(\left(\mathbf{U}_{Y(t)} \in f_{p_{Y(t)}^0}^{-1}(y(t)), \mathbf{U}_{Z(t)} \in f_{p_{Z(t)}^0}^{-1}(z(t)), \mathbf{U}_{I_0(t)} \in f_{p_{I_0(t)}^0}^{-1}(i_0(t)), \mathbf{U}_{A_2(t)} \in f_{p_{A_2(t)}^0}^{-1}(a_2(t)) \right)_{t=0, \dots, K+1}, \right. \\ &\left. \left(\mathbf{U}_{A_1(t)} \in f_{p_{A_1(t)}^0}^{-1}(a_1(t)), \mathbf{U}_{N(t)} \in f_{p_{N(t)}^0}^{-1}(n(t)) \right)_{t=0, \dots, K} \right), \end{aligned} \quad (22)$$

where bold font is used to underscore the random variables and we recall that $o = (\bar{\mathbf{I}}(K+1), \mathbf{I}_1(K), \bar{\mathbf{A}}(K))$ with $\mathbf{I}(t) = (Y(t), Z(t), I(t))$ and $I_0(t) = (Y(t), Z(t), i_0(t))$ for $t = 0, \dots, K+1$.

B Proof of the equivalence between the NPSEM-based (3) and counterfactual-based formulations (4) of the NDE

Proof that the set of equalities (3) implies the set of equalities (4)

Below, we show that the set of equalities (3) implies the following proposition for any $k = 0, \dots, K$, $\bar{a}_1^* \in \bar{\mathcal{A}}^*$, and $\bar{n}^* \in \bar{\mathcal{N}}^*$:

$$(A) \begin{cases} \bar{L}_{0_{\bar{a}^*, \bar{n}^*}}(k) = \bar{L}_{0_{\bar{a}^*}}(k) \\ \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k) = \bar{A}_{2_{\bar{a}^*}}(k) \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k) = \bar{A}_{1_{\bar{a}^*}}(k) \end{cases} \Rightarrow (B) \begin{cases} \bar{L}_{0_{\bar{a}^*, \bar{n}^*}}(k+1) = \bar{L}_{0_{\bar{a}^*}}(k+1) \\ \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1) = \bar{A}_{2_{\bar{a}^*}}(k+1) \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1) = \bar{A}_{1_{\bar{a}^*}}(k+1), \end{cases}$$

where we define $A_{1*}(\bar{r}^*(K+1))$ and $A_{1*}(K+1)$ as nil by convention.

By definition of the counterfactual variables defined in Section 4.1, we have for any $k = 0, \dots, K$ and any $* \in \bar{\mathcal{A}}^*$:

$$\begin{aligned} Y_{\bar{a}^*}(k+1) &= f_{Y(k+1)}(P_{Y_{\bar{a}^*}(k+1)}^0, U_{Y(k+1)}) \\ &= f_{Y(k+1)}((\bar{Y}_{\bar{a}^*}(k), \bar{Z}_{\bar{a}^*}(k), \bar{I}_{0_{\bar{a}^*}}(k), \bar{A}_{2_{\bar{a}^*}}(k), \bar{A}_{1_{\bar{a}^*}}(k), \bar{N}_{\bar{a}^*}(k)), U_{Y(k+1)}) \end{aligned}$$

where the last equality is derived from the definition of the parents of the counterfactual variable $Y^*(k+1)$. If (A) holds then the last equality becomes for any $\bar{r}^* \in \bar{\mathcal{N}}^*$:

$$Y_{\bar{a}^*}(k+1) = f_{Y(k+1)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k), \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k), \bar{N}_{\bar{a}^*}(k)), U_{Y(k+1)}),$$

Because the set $\bar{\mathcal{N}}^*$ contains the support of $\bar{N}^*(K)$ (by definition (2)), the set of equalities (3) implies that the right-hand side of the previous equality is equal to

$f_{Y(k+1)}(P_{Y_{\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{Y(k+1)})$. The previous equality thus becomes

$$\begin{aligned} Y_{\bar{a}^*}(k+1) &= f_{Y(k+1)}(P_{Y_{\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{Y(k+1)}) \\ &= Y_{\bar{a}^*, \bar{n}^*}(k+1) \end{aligned} \quad (23)$$

where the last equality is directly derived from the definition of the counterfactual outcome $Y^*, \bar{r}^*(k+1)$ in Section 4.2.

By definition of the counterfactual variables defined in Section 4.1, we have for any $k = 0, \dots, K$ and any $* \in \bar{\mathcal{A}}^*$:

$$\begin{aligned} Z_{\bar{a}^*}(k+1) &= f_{Z(k+1)}(P_{Z_{\bar{a}^*}(k+1)}^0, U_{Z(k+1)}) \\ &= f_{Z(k+1)}((\bar{Y}_{\bar{a}^*}(k+1), \bar{Z}_{\bar{a}^*}(k), \bar{I}_{0_{\bar{a}^*}}(k), \bar{A}_{2_{\bar{a}^*}}(k), \bar{A}_{1_{\bar{a}^*}}(k), \bar{N}_{\bar{a}^*}(k)), \\ &\quad U_{Z(k+1)}) \end{aligned}$$

where the last equality is derived from the definition of the parents of the counterfactual variable $Z^*(k+1)$. If (A) holds then the last equality becomes for any $\bar{r}^* \in \bar{\mathcal{N}}^*$:

$$Z_{\bar{a}^*}(k+1) = f_{Z(k+1)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k), \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k), \bar{N}_{\bar{a}^*}(k)), U_{Z(k+1)}),$$

where we also made use of equality (23) (just shown to be implied by equalities (A)).

Because the set $\bar{\mathcal{N}}^*$ contains the support of $\bar{N}^*(K)$ (by definition (2)), the set of equalities (3) implies that the right-hand side of the previous equality is equal to

$f_{Z(k+1)}(P_{Z_{\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{Z(k+1)})$. The previous equality thus becomes

$$\begin{aligned}
Z_{\bar{a}^*}(k+1) &= f_{Z(k+1)}(P_{Z_{\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{Z(k+1)}) \\
&= Z_{\bar{a}^*, \bar{n}^*}(k+1)
\end{aligned} \tag{24}$$

where the last equality is directly derived from the definition of the counterfactual outcome $Z_{*, \bar{n}^*}(k+1)$ in Section 4.2.

By definition of the counterfactual variables defined in Section 4.1, we have for any $k = 0, \dots, K$ and any $* \in \mathcal{A}^*$:

$$\begin{aligned}
I_{0\bar{a}^*}(k+1) &= f_{I_0(k+1)}(P_{I_{0\bar{a}^*}(k+1)}^0, U_{I_0(k+1)}) \\
&= f_{I_0(k+1)}((\bar{Y}_{\bar{a}^*}(k+1), \bar{Z}_{\bar{a}^*}(k+1), \bar{I}_{0\bar{a}^*}(k), \bar{A}_{2\bar{a}^*}(k), \bar{A}_{1\bar{a}^*}(k), \bar{N}_{\bar{a}^*}(k)), U_{I_0(k+1)})
\end{aligned}$$

where the last equality is derived from the definition of the parents of the counterfactual variable $I_{0*}(k+1)$. If (A) holds then the last equality becomes for any $\bar{n}^* \in \bar{\mathcal{N}}^*$:

$$I_{0\bar{a}^*}(k+1) = f_{I_0(k+1)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k), \bar{I}_{0\bar{a}^*, \bar{n}^*}(k), \bar{A}_{2\bar{a}^*, \bar{n}^*}(k), \bar{A}_{1\bar{a}^*, \bar{n}^*}(k), \bar{N}_{\bar{a}^*}(k)), U_{I_0(k+1)}),$$

where we also made use of equalities (23) and (24) (both just shown to be implied by equalities (A)). Because the set $\bar{\mathcal{N}}^*$ contains the support of $\bar{N}_{*}(K)$ (by definition (2)), the set of equalities (3) implies that the right-hand side of the previous equality is equal to

$f_{I_0(k+1)}(P_{I_{0\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{I_0(k+1)})$. The previous equality thus becomes

$$\begin{aligned}
I_{0\bar{a}^*}(k+1) &= f_{I_0(k+1)}(P_{I_{0\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{I_0(k+1)}) \\
&= I_{0\bar{a}^*, \bar{n}^*}(k+1)
\end{aligned} \tag{25}$$

where the last equality is directly derived from the definition of the counterfactual outcome $I_{0*, \bar{n}^*}(k+1)$ in Section 4.2.

We have just shown that equalities (3) imply: (A) $\Rightarrow \bar{L}_{0*, \bar{n}^*}(k+1) = \bar{L}_{0*}(k+1)$.

Moreover, by definition of the counterfactual variables defined in Section 4.1, we also have for $k = 0, \dots, K-1$

$$A_{2\bar{a}^*}(k+1) = Y_{\bar{a}^*}(k+1)f_{A_2(k+1)}(P_{A_{2\bar{a}^*}(k+1)}^0, U_{A_2(k+1)}) + (1 - Y_{\bar{a}^*}(k+1))a_2^*(k+1)$$

$$A_{1\bar{a}^*}(k+1) = Y_{\bar{a}^*}(k+1)f_{A_1(k+1)}(P_{A_{1\bar{a}^*}(k+1)}^0, U_{A_1(k+1)}) + (1 - Y_{\bar{a}^*}(k+1))a_1^*(k+1).$$

We recall from the constraints on $f_{A_2(k+1)}$ and $f_{A_1(k+1)}$ described in Section 3 and imposed by convention (degenerate variable distributions after failure or censoring) that

$f_{A_2(k+1)}(P_{A_2^*(k+1)}^0, U_{A_2(k+1)}) = A_{2\bar{a}^*}(k)$ and $f_{A_1(k+1)}(P_{A_1^*(k+1)}^0, U_{A_1(k+1)}) = A_{1\bar{a}^*}(k)$ if $Y^*(k+1) = 1$. As a result and using equalities (A), (23), and the definition of the counterfactual variables $a_{2^*}^*(k+1)$ and $a_{1^*}^*(k+1)$ defined in Section 4.2, the previous two equalities can be expressed as

$$\begin{aligned} A_{2\bar{a}^*}(k+1) &= Y_{\bar{a}^*, \bar{n}^*}(k+1) A_{2\bar{a}^*, \bar{n}^*}(k) + (1 - Y_{\bar{a}^*, \bar{n}^*}(k+1)) a_2^*(k+1) \\ &= A_{2\bar{a}^*, \bar{n}^*}(k+1), \end{aligned}$$

and

$$\begin{aligned} A_{1\bar{a}^*}(k+1) &= Y_{\bar{a}^*, \bar{n}^*}(k+1) A_{1\bar{a}^*, \bar{n}^*}(k) + (1 - Y_{\bar{a}^*, \bar{n}^*}(k+1)) a_1^*(k+1) \\ &= A_{1\bar{a}^*, \bar{n}^*}(k+1). \end{aligned}$$

We have just shown that equalities (3) imply: (A) $\Rightarrow \begin{cases} \bar{A}_{2\bar{a}^*, \bar{n}^*}(k+1) = \bar{A}_{2\bar{a}^*}(k+1) \\ \bar{A}_{1\bar{a}^*, \bar{n}^*}(k+1) = \bar{A}_{1\bar{a}^*}(k+1) \end{cases}$. Thus, we have now shown the proposition that, for any $k = 0, \dots, K$, $\bar{a}_1^* \in \bar{\mathcal{A}}^*$, and $\bar{n}^* \in \bar{\mathcal{N}}^*$, if equalities (A) hold then they imply equalities (B) when equalities (3) hold. We now show the proposition that, for any $\bar{a}_1^* \in \bar{\mathcal{A}}^*$ and $\bar{n}^* \in \bar{\mathcal{N}}^*$, equalities (A) hold at $k = 0$ when equalities (3) hold.

By definition of the counterfactuals defined in Sections 4.1 and 4.2, we have for any

$\bar{a}_1^* \in \bar{\mathcal{A}}^*$ and $\bar{n}^* \in \bar{\mathcal{N}}^*$:

$$\begin{cases} Y_{\bar{a}^*}(0) = 0 \\ Z_{\bar{a}^*}(0) = f_{Z(0)}((0), U_{Z(0)}) \\ I_{0\bar{a}^*}(0) = f_{I_0(0)}((0), Z_{\bar{a}^*}(0), U_{I_0(0)}) \\ A_{2\bar{a}^*}(0) = a_2^*(0) \\ A_{1\bar{a}^*}(0) = f_{A_1(0)}((0), Z_{\bar{a}^*}(0), I_{0\bar{a}^*}(0), a_2^*(0), U_{A_1(0)}) \end{cases}$$

and

$$\begin{cases} Y_{\bar{a}^*, \bar{n}^*}(0) = 0 \\ Z_{\bar{a}^*, \bar{n}^*}(0) = f_{Z(0)}((0), U_{Z(0)}) \\ I_{0\bar{a}^*, \bar{n}^*}(0) = f_{I_0(0)}((0), Z_{\bar{a}^*, \bar{n}^*}(0), U_{I_0(0)}) \\ A_{2\bar{a}^*, \bar{n}^*}(0) = a_2^*(0) \\ A_{1\bar{a}^*, \bar{n}^*}(0) = f_{A_1(0)}((0), Z_{\bar{a}^*, \bar{n}^*}(0), I_{0\bar{a}^*, \bar{n}^*}(0), a_2^*(0), U_{A_1(0)}), \end{cases}$$

where we used the constraint on $f_{Y(0)}$ and parent definitions detailed in Section 3.

These two sets of equalities clearly imply:

$$\begin{cases} Y_{\bar{a}^*}(0) = Y_{\bar{a}^*, \bar{n}^*}(0) \\ Z_{\bar{a}^*}(0) = Z_{\bar{a}^*, \bar{n}^*}(0) \\ I_{0\bar{a}^*}(0) = I_{0\bar{a}^*, \bar{n}^*}(0) \\ A_{2\bar{a}^*}(0) = A_{2\bar{a}^*, \bar{n}^*}(0) \\ A_{1\bar{a}^*}(0) = A_{1\bar{a}^*, \bar{n}^*}(0) \end{cases}$$

By induction, the previous two propositions imply that equalities (4) hold when equalities (3) hold.

Proof that the set of equalities (4) implies the set of equalities (3)

Below, we show that the set of equalities (4) implies the following proposition for any $k = 0, \dots, K-1$, $\bar{a}_1^* \in \bar{\mathcal{A}}^*$, and $(\bar{n}^*, \bar{n}^{*'}) \in \bar{\mathcal{N}}^{*2}$:

$$(A) \begin{cases} (E(t))_{t=0, \dots, k} \\ \bar{A}_{2\bar{a}^*, \bar{n}^*}(k) = \bar{A}_{2\bar{a}^*, \bar{n}^{*'}}(k) \\ \bar{A}_{1\bar{a}^*, \bar{n}^*}(k) = \bar{A}_{1\bar{a}^*, \bar{n}^{*'}}(k) \end{cases} \Rightarrow (B) \begin{cases} (E(t))_{t=0, \dots, k+1} \\ \bar{A}_{2\bar{a}^*, \bar{n}^*}(k+1) = \bar{A}_{2\bar{a}^*, \bar{n}^{*'}}(k+1) \\ \bar{A}_{1\bar{a}^*, \bar{n}^*}(k+1) = \bar{A}_{1\bar{a}^*, \bar{n}^{*'}}(k+1) \end{cases}$$

where, for $t = 0, \dots, K$, $E(t)$ denotes the following three equalities:

$$\begin{cases} f_{Y(t+1)}(P_{Y_{\bar{a}^*, \bar{n}^*}(t+1)}^0, U_{Y(t+1)}) = f_{Y(t+1)}(P_{Y_{\bar{a}^*, \bar{n}^{*'}}(t+1), \bar{n}^{*'}}^0, U_{Y(t+1)}) \\ f_{Z(t+1)}(P_{Z_{\bar{a}^*, \bar{n}^*}(t+1)}^0, U_{Z(t+1)}) = f_{Z(t+1)}(P_{Z_{\bar{a}^*, \bar{n}^{*'}}(t+1), \bar{n}^{*'}}^0, U_{Z(t+1)}) \\ f_{I_0(t+1)}(P_{I_{0\bar{a}^*, \bar{n}^*}(t+1)}^0, U_{I_0(t+1)}) = f_{I_0(t+1)}(P_{I_{0\bar{a}^*, \bar{n}^{*'}}(t+1), \bar{n}^{*'}}^0, U_{I_0(t+1)}). \end{cases}$$

By definition of the counterfactual variables defined in Section 4.2, we have for any $k = 0, \dots, K-1$, $\bar{a}_1^* \in \bar{\mathcal{A}}^*$, and $\bar{n}^* \in \bar{\mathcal{N}}^*$:

$$A_{2\bar{a}^*, \bar{n}^*}(k+1) = Y_{\bar{a}^*, \bar{n}^*}(k+1) f_{A_2(k+1)}(P_{A_{2\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{A_2(k+1)}) + (1 - Y_{\bar{a}^*, \bar{n}^*}(k+1)) a_2^*(k+1)$$

$$A_{1\bar{a}^*, \bar{n}^*}(k+1) = Y_{\bar{a}^*, \bar{n}^*}(k+1) f_{A_1(k+1)}(P_{A_{1\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{A_1(k+1)}) + (1 - Y_{\bar{a}^*, \bar{n}^*}(k+1)) a_1^*(k+1).$$

We recall from the constraints on $f_{A_2(k+1)}$ and $f_{A_1(k+1)}$ described in Section 3 and imposed by convention (degenerate variable distributions after failure or censoring) that

$$f_{A_2(k+1)}(P_{A_{2\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{A_2(k+1)}) = A_{2\bar{a}^*, \bar{n}^*}(k) \text{ and}$$

$$f_{A_1(k+1)}(P_{A_{1\bar{a}^*, \bar{n}^*}(k+1)}^0, U_{A_1(k+1)}) = A_{1\bar{a}^*, \bar{n}^*}(k) \text{ if } Y_{\bar{n}^*, \bar{n}^*}(k+1) = 1. \text{ As a result and using equalities (A), the previous two equalities become}$$

$$A_{2_{\bar{a}^*, \bar{n}^*}}(k+1) = Y_{\bar{a}^*, \bar{n}^*}(k+1)A_{2_{\bar{a}^*, \bar{n}^{*'}}}(k) + (1 - Y_{\bar{a}^*, \bar{n}^*}(k+1))a_2^*(k+1)$$

$$A_{1_{\bar{a}^*, \bar{n}^*}}(k+1) = Y_{\bar{a}^*, \bar{n}^*}(k+1)A_{1_{\bar{a}^*, \bar{n}^{*'}}}(k) + (1 - Y_{\bar{a}^*, \bar{n}^*}(k+1))a_1^*(k+1).$$

From the set of equalities (4), we have for $m = 1, \dots, K+1$: $L_{0^*, \bar{n}^*}(m) = L_{0^*}(m)$ and $L_{0^*, \bar{n}^{*'}}(m) = L_{0^*}(m)$ and thus $L_{0^*, \bar{n}^*}(m) = L_{0^*, \bar{n}^{*'}}(m)$. Because k ranging from 0 to $K-1$ implies $k+1$ ranges from 1 to K , the latter equality thus implies $Y_{*, \bar{n}^*}(k+1) = Y_{*, \bar{n}^{*'}}(k+1)$ and we have:

$$A_{2_{\bar{a}^*, \bar{n}^*}}(k+1) = Y_{\bar{a}^*, \bar{n}^{*'}}(k+1)A_{2_{\bar{a}^*, \bar{n}^{*'}}}(k) + (1 - Y_{\bar{a}^*, \bar{n}^{*'}}(k+1))a_2^*(k+1)$$

$$A_{1_{\bar{a}^*, \bar{n}^*}}(k+1) = Y_{\bar{a}^*, \bar{n}^{*'}}(k+1)A_{1_{\bar{a}^*, \bar{n}^{*'}}}(k) + (1 - Y_{\bar{a}^*, \bar{n}^{*'}}(k+1))a_1^*(k+1).$$

Using the definitions of the counterfactual variables defined in Section f these three terms are defined as follows 4.2 and recalling again that the constraints on $f_{A_{2(k+1)}}$ and $f_{A_{1(k+1)}}$

described in Section 3 imply $f_{A_{2(k+1)}}(P_{A_{2_{\bar{a}^*, \bar{n}^{*'}}}^0(k+1)}, U_{A_{2(k+1)}}) = A_{2_{\bar{a}^*, \bar{n}^{*'}}}(k)$ and

$f_{A_{1(k+1)}}(P_{A_{1_{\bar{a}^*, \bar{n}^{*'}}}^0(k+1)}, U_{A_{1(k+1)}}) = A_{1_{\bar{a}^*, \bar{n}^{*'}}}(k)$. if $Y_{*, \bar{n}^{*'}}(k+1) = 1$, we obtain

$$\begin{aligned} A_{2_{\bar{a}^*, \bar{n}^*}}(k+1) &= A_{2_{\bar{a}^*, \bar{n}^{*'}}}(k+1) \\ A_{1_{\bar{a}^*, \bar{n}^*}}(k+1) &= A_{1_{\bar{a}^*, \bar{n}^{*'}}}(k+1) \end{aligned} \quad (26)$$

We have just shown that equalities (4) imply:

$$(A) \Rightarrow \begin{cases} \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1) = \bar{A}_{2_{\bar{a}^*, \bar{n}^{*'}}}(k+1) \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1) = \bar{A}_{1_{\bar{a}^*, \bar{n}^{*'}}}(k+1). \end{cases}$$

Moreover, by definition of the counterfactual variables in Section 4.2, we have for any $k = 0$,

$\dots, K-1$, $\bar{a}_1^* \in \bar{\mathcal{A}}^*$, and $\bar{n}^* \in \bar{\mathcal{N}}^*$:

$$(i) \left\{ \begin{array}{l} Y_{\bar{a}^*, \bar{n}^*}(k+2) = f_{Y(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}(k+1)), U_{Y(k+2)}) \\ Z_{\bar{a}^*, \bar{n}^*}(k+2) = f_{Z(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}(k+1)), U_{Z(k+2)}) \\ I_{0_{\bar{a}^*, \bar{n}^*}}(k+2) = f_{I_0(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}(k+1)), U_{I_0(k+2)}) \end{array} \right.$$

where for $t = 0, \dots, k+1$

$$(j) \begin{cases} Y_{\bar{a}^*, \bar{n}^*}(t) = f_{Y(t)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(t-1), \bar{Z}_{\bar{a}^*, \bar{n}^*}(t-1), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(t-1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(t-1), \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(t-1), \\ \bar{N}_{\bar{a}^*, \bar{n}^*}(t-1)), U_{Y(t)}) \\ Z_{\bar{a}^*, \bar{n}^*}(t) = f_{Z(t)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(t), \bar{Z}_{\bar{a}^*, \bar{n}^*}(t-1), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(t-1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(t-1), \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(t-1), \\ \bar{N}_{\bar{a}^*, \bar{n}^*}(t-1)), U_{Z(t)}) \\ I_{0_{\bar{a}^*, \bar{n}^*}}(t) = f_{I_0(t)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(t), \bar{Z}_{\bar{a}^*, \bar{n}^*}(t), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(t-1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(t-1), \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(t-1), \\ \bar{N}_{\bar{a}^*, \bar{n}^*}(t-1)), U_{I_0(t)}). \end{cases}$$

Similarly, for any $\bar{n}^{*'} \in \bar{\mathcal{N}}^*$, we have:

$$(ii) \begin{cases} Y_{\bar{a}^*, \bar{n}^{*'}}(k+2) = f_{Y(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^{*'}}(k+1), \bar{Z}_{\bar{a}^*, \bar{n}^{*'}}(k+1), \bar{I}_{0_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^{*'}}(k+1)), U_{Y(k+2)}) \\ Z_{\bar{a}^*, \bar{n}^{*'}}(k+2) = f_{Z(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^{*'}}(k+2), \bar{Z}_{\bar{a}^*, \bar{n}^{*'}}(k+1), \bar{I}_{0_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^{*'}}(k+1)), U_{Z(k+2)}) \\ I_{0_{\bar{a}^*, \bar{n}^{*'}}}(k+2) = f_{I_0(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^{*'}}(k+2), \bar{Z}_{\bar{a}^*, \bar{n}^{*'}}(k+2), \bar{I}_{0_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^{*'}}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^{*'}}(k+1)), U_{I_0(k+2)}) \end{cases}$$

where for $t = 0, \dots, k+1$

$$(jj) \begin{cases} Y_{\bar{a}^*, \bar{n}^{*'}}(t) = f_{Y(t)}((\bar{Y}_{\bar{a}^*, \bar{n}^{*'}}(t-1), \bar{Z}_{\bar{a}^*, \bar{n}^{*'}}(t-1), \bar{I}_{0_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \bar{A}_{2_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \bar{N}_{\bar{a}^*, \bar{n}^{*'}}(t-1)), U_{Y(t)}) \\ Z_{\bar{a}^*, \bar{n}^{*'}}(t) = f_{Z(t)}((\bar{Y}_{\bar{a}^*, \bar{n}^{*'}}(t), \bar{Z}_{\bar{a}^*, \bar{n}^{*'}}(t-1), \bar{I}_{0_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \bar{A}_{2_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \bar{A}_{1_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \\ \bar{N}_{\bar{a}^*, \bar{n}^{*'}}(t-1)), U_{Z(t)}) \\ I_{0_{\bar{a}^*, \bar{n}^{*'}}}(t) = f_{I_0(t)}((\bar{Y}_{\bar{a}^*, \bar{n}^{*'}}(t), \bar{Z}_{\bar{a}^*, \bar{n}^{*'}}(t), \bar{I}_{0_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \bar{A}_{2_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \bar{A}_{1_{\bar{a}^*, \bar{n}^{*'}}}(t-1), \\ \bar{N}_{\bar{a}^*, \bar{n}^{*'}}(t-1)), U_{I_0(t)}). \end{cases}$$

From the set of equalities (4), we have for $m = 1, \dots, K+1$: $L_{0^*, \bar{n}^*}(m) = L_{0^*}(m)$ and $L_{0^*, \bar{n}^{*'}}(m) = L_{0^*}(m)$ and thus $L_{0^*, \bar{n}^*}(m) = L_{0^*, \bar{n}^{*'}}(m)$. We also note that equalities (j) and (jj) evaluated at $t = 0$ clearly imply $L_{0^*, \bar{n}^*}(0) = L_{0^*, \bar{n}^{*'}}(0)$. The previous two equalities thus imply $\bar{L}_{0^*, \bar{n}^*}(m) = \bar{L}_{0^*, \bar{n}^{*'}}(m)$ for $m = 1, \dots, K+1$. Because k ranging from 0 to $K-1$ implies $k+2$ ranges from 2 to $K+1$, the previous equality thus implies $\bar{L}_{0^*, \bar{n}^*}(k+2) = \bar{L}_{0^*, \bar{n}^{*'}}(k+2)$. Using this last result, equalities (ii) become

$$\begin{cases} Y_{\bar{a}^*, \bar{n}^*}(k+2) = f_{Y(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}(k+1)), U_{Y(k+2)}) \\ Z_{\bar{a}^*, \bar{n}^*}(k+2) = f_{Z(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}(k+1)), U_{Z(k+2)}) \\ I_{0_{\bar{a}^*, \bar{n}^*}}(k+2) = f_{I_0(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}(k+1)), U_{I_0(k+2)}) \end{cases}$$

Using equalities (A) and equality (26) (just shown to be itself implied by equalities (A) and (4)), we obtain

$$\begin{cases} Y_{\bar{a}^*, \bar{n}^*}(k+2) = f_{Y(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}'(k+1)), U_{Y(k+2)}) \\ Z_{\bar{a}^*, \bar{n}^*}(k+2) = f_{Z(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+1), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}'(k+1)), U_{Z(k+2)}) \\ I_{0_{\bar{a}^*, \bar{n}^*}}(k+2) = f_{I_0(k+2)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{Z}_{\bar{a}^*, \bar{n}^*}(k+2), \bar{I}_{0_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{A}_{2_{\bar{a}^*, \bar{n}^*}}(k+1), \\ \bar{A}_{1_{\bar{a}^*, \bar{n}^*}}(k+1), \bar{N}_{\bar{a}^*, \bar{n}^*}'(k+1)), U_{I_0(k+2)}) \end{cases}$$

We note that if $Y_{*, \bar{n}^*}(k+1) = 0$, then the definition of the counterfactual monitoring variables in Section 4.2 implies that all previous counterfactual outcomes are also equal to 0, i.e., $\bar{Y}_{*, \bar{n}^*}(k+1) = 0$, and therefore $\bar{N}_{*, \bar{n}^*}'(k+1) = \bar{n}^{*'}(k+1)$. Otherwise, we note that this equality may not hold. However, if $Y_{*, \bar{n}^*}(k+1) = 1$, then the constraints on $f_{Y(k+2)}$, $f_{Z(k+2)}$ and $f_{I_0(k+2)}$ described in Section 3 imply that the counterfactual monitoring process $\bar{N}_{*, \bar{n}^*}'(k+1)$ can nevertheless be replaced by $\bar{n}^{*'}(k+1)$ in the previous three equalities since the values of the functions $f_{Y(k+2)}$, $f_{Z(k+2)}$ and $f_{I_0(k+2)}$ is then equal to $Y_{*, \bar{n}^*}(k+1)$, $Z_{*, \bar{n}^*}(k+1)$, and $I_{0_{*, \bar{n}^*}}(k+1)$, respectively, whatever the value of past covariates (including the monitoring process). The previous three equalities can thus be expressed as

$$\begin{cases} Y_{\bar{a}^*, \bar{n}^*}(k+2) = f_{Y(k+2)}(P_{Y_{\bar{a}^*, \bar{n}^*}(k+2), \bar{n}^*}'^{(k+2)}, U_{Y(k+2)}) \\ Z_{\bar{a}^*, \bar{n}^*}(k+2) = f_{Z(k+2)}(P_{Z_{\bar{a}^*, \bar{n}^*}(k+2), \bar{n}^*}'^{(k+2)}, U_{Z(k+2)}) \\ I_{0_{\bar{a}^*, \bar{n}^*}}(k+2) = f_{I_0(k+2)}(P_{I_{0_{\bar{a}^*, \bar{n}^*}(k+2), \bar{n}^*}'^{(k+2)}, U_{I_0(k+2)}}^{(k+2)}) \end{cases}$$

where we use the generic notation $P_{X^*(t), \bar{n}^*}'^{(t)}$ to denote the vector of variables defined by the parents $P_{X^*(t+1)}^0$ of the counterfactual variable $X^*(t+1)$ in which the monitoring variables $\bar{N}^*(t)$ are replaced by the fixed values $\bar{n}^{*'}(t)$. By transitivity, the combination of the latter equalities with equalities (i) implies that the equalities $E(k+1)$ hold. We have just shown that equalities (4) imply: $A \Rightarrow (E(t))_{t=0, \dots, k+1}$.

Above, we have shown the proposition that for any $k = 0, \dots, K-1$, $\bar{a}_1^* \in \bar{\mathcal{A}}^*$, and $(\bar{n}^*, \bar{n}^{*'}) \in \bar{\mathcal{N}}^{*2}$, if equalities (A) hold then they imply equalities (B) when equalities (4) hold. We now show the proposition that, for any $\bar{a}_1^* \in \bar{\mathcal{A}}^*$ and $(\bar{n}^*, \bar{n}^{*'}) \in \bar{\mathcal{N}}^{*2}$ equalities (A) hold at $k = 0$ when equalities (4) hold.

By definition of the counterfactuals described in Section 4.2 and the convention that $f_{Y(0)}(U_{Y_0}) = 0$, we have $Y_{*, \bar{n}^*}(0) = Y_{*, \bar{n}^{*'}}(0) = 0$. As a result, the definition of the counterfactuals described in Section 4.2 also implies the equalities $Z_{*, \bar{n}^*}(0) = Z_{*, \bar{n}^{*'}}(0)$, $I_{0_{*, \bar{n}^*}}(0) = I_{0_{*, \bar{n}^{*'}}}(0)$, $A_{2_{\bar{a}^*, \bar{n}^*}}(0) = A_{2_{\bar{a}^*, \bar{n}^{*'}}}(0) = a_2^*(0)$, $A_{1_{\bar{a}^*, \bar{n}^*}}(0) = A_{1_{\bar{a}^*, \bar{n}^{*'}}}(0) = a_1^*(0)$, $N_{*, \bar{n}^*}(0) = \bar{n}^*(0)$, and $N_{*, \bar{n}^{*'}}(0) = \bar{n}^{*'}(0)$. We also note that equalities (4) imply the equalities: $L_{*, \bar{n}^*}(1) = L_{*, \bar{n}^{*'}}(1)$. Using all the previous equalities and the definition of the counterfactuals $L_{*, \bar{n}^*}'(1)$ described in Section 4.2, we obtain:

$$\begin{aligned}
Y_{\bar{a}^*, \bar{n}^{*'}}(1) &\equiv f_{Y(1)}((Y_{\bar{a}^*, \bar{n}^{*'}}(0), Z_{\bar{a}^*, \bar{n}^{*'}}(0), I_{0_{\bar{a}^*, \bar{n}^{*'}}}(0), A_{2_{\bar{a}^*, \bar{n}^{*'}}}(0), A_{1_{\bar{a}^*, \bar{n}^{*'}}}(0), N_{\bar{a}^*, \bar{n}^{*'}}(0)), U_{Y(1)}) \\
&= f_{Y(1)}((Y_{\bar{a}^*, \bar{n}^*}(0), Z_{\bar{a}^*, \bar{n}^*}(0), I_{0_{\bar{a}^*, \bar{n}^*}}(0), A_{2_{\bar{a}^*, \bar{n}^*}}(0), A_{1_{\bar{a}^*, \bar{n}^*}}(0), n^{*'}(0)), U_{Y(1)}) \\
&= f_{Y(1)}(P_{Y_{\bar{a}^*, \bar{n}^*}(1), \bar{n}^{*'}}^0, U_{Y(1)}), \\
Z_{\bar{a}^*, \bar{n}^{*'}}(1) &\equiv f_{Z(1)}((\bar{Y}_{\bar{a}^*, \bar{n}^{*'}}(1), Z_{\bar{a}^*, \bar{n}^{*'}}(0), I_{0_{\bar{a}^*, \bar{n}^{*'}}}(0), A_{2_{\bar{a}^*, \bar{n}^{*'}}}(0), A_{1_{\bar{a}^*, \bar{n}^{*'}}}(0), N_{\bar{a}^*, \bar{n}^{*'}}(0)), U_{Z(1)}) \\
&= f_{Z(1)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(1), Z_{\bar{a}^*, \bar{n}^*}(0), I_{0_{\bar{a}^*, \bar{n}^*}}(0), A_{2_{\bar{a}^*, \bar{n}^*}}(0), A_{1_{\bar{a}^*, \bar{n}^*}}(0), n^{*'}(0)), U_{Z(1)}) \\
&= f_{Z(1)}(P_{Z_{\bar{a}^*, \bar{n}^*}(1), \bar{n}^{*'}}^0, U_{Z(1)}), \\
I_{0_{\bar{a}^*, \bar{n}^{*'}}}(1) &\equiv f_{I_0(1)}((\bar{Y}_{\bar{a}^*, \bar{n}^{*'}}(1), \bar{Z}_{\bar{a}^*, \bar{n}^{*'}}(1), I_{0_{\bar{a}^*, \bar{n}^{*'}}}(0), A_{2_{\bar{a}^*, \bar{n}^{*'}}}(0), A_{1_{\bar{a}^*, \bar{n}^{*'}}}(0), N_{\bar{a}^*, \bar{n}^{*'}}(0)), U_{I_0(1)}) \\
&= f_{I_0(1)}((\bar{Y}_{\bar{a}^*, \bar{n}^*}(1), \bar{Z}_{\bar{a}^*, \bar{n}^*}(1), I_{0_{\bar{a}^*, \bar{n}^*}}(0), A_{2_{\bar{a}^*, \bar{n}^*}}(0), A_{1_{\bar{a}^*, \bar{n}^*}}(0), n^{*'}(0)), U_{I_0(1)}) \\
&= f_{I_0(1)}(P_{I_{0_{\bar{a}^*, \bar{n}^*}(1), \bar{n}^{*'}}}^0, U_{I_0(1)}).
\end{aligned}$$

(27)

From the definition of the counterfactuals $L^*, \bar{n}^{*'}(1)$ described in Section 4.2, we also have

$$\begin{aligned}
Y_{\bar{a}^*, \bar{n}^*}(1) &= f_{Y(1)}(P_{Y_{\bar{a}^*, \bar{n}^*}(1)}^0, U_{Y(1)}) \\
Z_{\bar{a}^*, \bar{n}^*}(1) &= f_{Z(1)}(P_{Z_{\bar{a}^*, \bar{n}^*}(1)}^0, U_{Z(1)}) \\
I_{0_{\bar{a}^*, \bar{n}^*}}(1) &= f_{I_0(1)}(P_{I_{0_{\bar{a}^*, \bar{n}^*}(1)}}^0, U_{I_0(1)}).
\end{aligned} \tag{28}$$

Using the equality $L^*, \bar{n}^{*'}(1) = L^*, \bar{n}^{*'}(1)$ implied by equalities (4), the previous two sets of equalities (27) and (28) imply that the equalities $E(0)$ hold. We have just shown the proposition that equalities (A) hold at $k = 0$ when equalities (4) hold.

By induction, the previous two propositions imply that equalities (3) hold when equalities (4) hold.

C Proof of the first identifiability result

Below, we show that the (weak) NDE assumption (i.e., equalities (3), (5) or (6)) implies the following proposition for any $t = 1, \dots, K$:

$$(A) \left\{ \begin{array}{l} \bar{L}_{0_{d_x, \bar{n}^*}}(t) = \bar{L}_{0_{d_x^*, g^*}}(t) \\ \bar{I}_{d_x, \bar{n}^*}(t-1) = \bar{I}_{d_x^*, g^*}(t-1) \\ \bar{A}_{2_{d_x, \bar{n}^*}}(t-1) = \bar{A}_{2_{d_x^*, g^*}}(t-1) \\ \bar{A}_{1_{d_x, \bar{n}^*}}(t-1) = \bar{A}_{1_{d_x^*, g^*}}(t-1) \\ \bar{N}_{d_x, \bar{n}^*}(t-1) = \bar{N}_{d_x^*, g^*}(t-1) \end{array} \right. \Rightarrow (B) \left\{ \begin{array}{l} \bar{L}_{0_{d_x, \bar{n}^*}}(t+1) = \bar{L}_{0_{d_x^*, g^*}}(t+1) \\ \bar{I}_{d_x, \bar{n}^*}(t) = \bar{I}_{d_x^*, g^*}(t) \\ \bar{A}_{2_{d_x, \bar{n}^*}}(t) = \bar{A}_{2_{d_x^*, g^*}}(t) \\ \bar{A}_{1_{d_x, \bar{n}^*}}(t) = \bar{A}_{1_{d_x^*, g^*}}(t) \\ \bar{N}_{d_x, \bar{n}^*}(t) = \bar{N}_{d_x^*, g^*}(t), \end{array} \right.$$

where the definition of the dynamic intervention d_x^* is given by (7) for any given dynamic treatment intervention d_x defined in Section 4.3 and any given static monitoring intervention

$\bar{H}^*$ and where the monitoring intervention g^* is defined in Section 4.5 using the same choice for $\bar{H}^*$.

From the definition of the variables in the causal model given in Section 3, their $\bar{H}^*$ -specific transformation defined by (8), and the definition of their counterfactuals analogs given in Section 4, we have

$$I_{d_x, \bar{n}^*}(t) = Y_{d_x, \bar{n}^*}(t) I_{d_x, \bar{n}^*}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) N_{d_x, \bar{n}^*}(t-1) I_{0_{d_x}, \bar{n}^*}(t) \quad (29)$$

$$I'_{d_x, g^*}(t) = Y_{d_x, g^*}(t) I'_{d_x, g^*}(t-1) + (1 - Y_{d_x, g^*}(t)) N'_{d_x, g^*}(t-1) I_{0_{d_x}, g^*}(t) \quad (30)$$

If equalities (A) hold, we have

$Y_{d_x, \bar{n}^*}(t) = Y_{d_x, g^*}(t)$, $I_{d_x, \bar{n}^*}(t-1) = I'_{d_x, g^*}(t-1)$, $N_{d_x, \bar{n}^*}(t-1) = N'_{d_x, g^*}(t-1)$, and $I_{0_{d_x}, \bar{n}^*}(t) = I_{0_{d_x}, g^*}(t)$. As a result, equality (29) becomes

$$I_{d_x, \bar{n}^*}(t) = Y_{d_x, g^*}(t) I'_{d_x, g^*}(t-1) + (1 - Y_{d_x, g^*}(t)) N'_{d_x, g^*}(t-1) I_{0_{d_x}, g^*}(t).$$

By transitivity and using equality (30), we obtain

$$I_{d_x, \bar{n}^*}(t) = I'_{d_x, g^*}(t). \quad (31)$$

We have just shown: $(A) \Rightarrow \bar{I}_{d_x, \bar{n}^*}(t) = \bar{I}'_{d_x, g^*}(t)$.

From the definition of the counterfactuals variables given in Sections 4.3 and 4.5, we have

$$A_{2_{d_x}, \bar{n}^*}(t) = Y_{d_x, \bar{n}^*}(t) A_{2_{d_x}, \bar{n}^*}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) a_2^*(t) \quad (32)$$

$$A_{2_{d_x}, g^*}(t) = Y_{d_x, g^*}(t) A_{2_{d_x}, g^*}(t-1) + (1 - Y_{d_x, g^*}(t)) a_2^*(t). \quad (33)$$

If equalities (A) hold, equality (32) becomes

$$A_{2_{d_x}, \bar{n}^*}(t) = Y_{d_x, g^*}(t) A_{2_{d_x}, g^*}(t-1) + (1 - Y_{d_x, g^*}(t)) a_2^*(t).$$

By transitivity and using equality (33), we obtain

$$A_{2_{d_x}, \bar{n}^*}(t) = A_{2_{d_x}, g^*}(t). \quad (34)$$

We have just shown: $(A) \Rightarrow \bar{A}_{d_x, \bar{n}^*}(t) = \bar{A}_{d_x, g^*}(t)$.

From the definition of the counterfactuals variables given in Sections 4.3 and 4.5, we have

$$A_{1_{d_x, \bar{n}^*}}(t) = Y_{d_x, \bar{n}^*}(t) A_{1_{d_x, \bar{n}^*}}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) d_x(t) (P_{A_{1_{d_x, \bar{n}^*}}(t)}) \quad (35)$$

$$A_{1_{d_x, g^*}}(t) = Y_{d_x, g^*}(t) A_{1_{d_x, g^*}}(t-1) + (1 - Y_{d_x, g^*}(t)) d_x^*(t) (P_{A_{1_{d_x, g^*}}(t)}). \quad (36)$$

From the definition of the dynamic interventions d_x^* given by (7), equality (36) becomes:

$$A_{1_{d_x, g^*}}(t) = Y_{d_x, g^*}(t) A_{1_{d_x, g^*}}(t-1) + (1 - Y_{d_x, g^*}(t)) d_x(t) ((\bar{Y}_{d_x, g^*}(t), \bar{Z}_{d_x, g^*}(t), \bar{I}'_{d_x, g^*}(t), \bar{A}_{2_{d_x, g^*}}(t), \bar{A}_{1_{d_x, g^*}}(t-1), \bar{N}'_{d_x, g^*}(t-1))).$$

If equalities (A) hold, the previous equality becomes

$$A_{1_{d_x, g^*}}(t) = Y_{d_x, \bar{n}^*}(t) A_{1_{d_x, \bar{n}^*}}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) d_x(t) ((\bar{Y}_{d_x, \bar{n}^*}(t), \bar{Z}_{d_x, \bar{n}^*}(t), \bar{I}_{d_x, \bar{n}^*}(t), \bar{A}_{2_{d_x, \bar{n}^*}}(t), \bar{A}_{1_{d_x, \bar{n}^*}}(t-1), \bar{N}_{d_x, \bar{n}^*}(t-1))).$$

where we also made use of equalities (31) and (34) (just shown to hold if equalities (A) hold). Using the parent notation, the previous equality can be expressed as

$$A_{1_{d_x, g^*}}(t) = Y_{d_x, \bar{n}^*}(t) A_{1_{d_x, \bar{n}^*}}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) d_x(t) (P_{A_{1_{d_x, \bar{n}^*}}(t)}).$$

By transitivity and using equality (35), we obtain

$$A_{1_{d_x, \bar{n}^*}}(t) = A_{1_{d_x, g^*}}(t). \quad (37)$$

We have just shown: $(A) \Rightarrow \bar{A}_{d_x, \bar{n}^*}(t) = \bar{A}_{d_x, g^*}(t)$.

From the definition of the variables in the causal model given in Section 3, their $\bar{n}^*$ -specific transformation defined by (8), and the definition of their counterfactuals analogs given in Section 4, we have

$$N^{*'}(t) = Y^*(t) N^{*'}(t-1) + (1 - Y^*(t)) n^*(t) N^*(t) \quad (38)$$

$$N_{d_x, \bar{n}^*}(t) = Y_{d_x, \bar{n}^*}(t) N_{d_x, \bar{n}^*}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) n^*(t) \quad (39)$$

$$N_{d_x, g^*}^*(t) = Y_{d_x, g^*}^*(t) N_{d_x, g^*}^*(t) + (1 - Y_{d_x, g^*}^*(t)) f_{N(t)}(P_{N_{d_x, g^*}^*(t)}^0, U_{N(t)})^{1-n^*(t)}. \quad (40)$$

These equalities imply:

- If $n^*(t) = 1$ and $Y_{d_x, g^*}(t) = 0$ then $N_{d_x, g^*}'(t) = N_{d_x, g^*}^*(t)$ from (38), $N_{d_x, g^*}^*(t) = 1$ from equality (40) and thus $N_{d_x, g^*}'(t) = 1$ by transitivity. If equalities (A) hold, we also have $Y_{d_x, \bar{n}^*}(t) = Y_{d_x, g^*}(t)$ and therefore $N_{d_x, \bar{n}^*}(t) = 1$ from (39). As a result, we obtain $N_{d_x, \bar{n}^*}(t) = N_{d_x, g^*}'(t)$.
- If $n^*(t) = 0$ and $Y_{d_x, g^*}(t) = 0$ then $N_{d_x, g^*}'(t) = 0$ from equality (38). If equalities (A) hold, we also have $Y_{d_x, \bar{n}^*}(t) = Y_{d_x, g^*}(t)$ and therefore $N_{d_x, \bar{n}^*}(t) = 0$ from equality (39). As a result, we also obtain $N_{d_x, \bar{n}^*}(t) = N_{d_x, g^*}'(t)$.
- If $Y_{d_x, g^*}(t) = 1$ then we obtain, respectively, from equalities (A), (38) and (39):

$$N_{d_x, g^*}'(t) = N_{d_x, g^*}'(t-1) \quad (41)$$

$$N_{d_x, \bar{n}^*}(t) = N_{d_x, \bar{n}^*}(t-1). \quad (42)$$

If equalities (A) hold then $N_{d_x, \bar{n}^*}(t-1) = N_{d_x, g^*}'(t-1)$. By transitivity using the latter equality and equalities (41) and (42), we also obtain $N_{d_x, \bar{n}^*}(t) = N_{d_x, g^*}'(t)$.

We have just shown: $(A) \Rightarrow \bar{N}_{d_x, \bar{n}^*}(t) = \bar{N}_{d_x, g^*}'(t)$.

By definition of the counterfactuals in Section 4.5, we have:

$$\begin{aligned} Y_{d_x, g^*}(t+1) &= f_{Y(t+1)}(P_{Y_{d_x, g^*}(t+1)}^0, U_{Y(t+1)}) \\ Z_{d_x, g^*}(t+1) &= f_{Z(t+1)}(P_{Z_{d_x, g^*}(t+1)}^0, U_{Z(t+1)}) \\ I_{0_{d_x, g^*}}(t+1) &= f_{I_0(t+1)}(P_{I_{0_{d_x, g^*}}(t+1)}^0, U_{I_0(t+1)}) \end{aligned}$$

Because the set $\bar{\mathcal{A}}^*$ contains the support of $\bar{A}_{d_x, g^*}^*(K)$ and because $\bar{\mathcal{N}}^*$ contains the supports of both $\bar{N}_{d_x, g^*}^*(K)$ and $\bar{N}_{d_x, \bar{n}^*}^*(K)$ (by definitions (1) and (2)), the right-hand sides of the previous equalities can be modified as follows under the NDE assumption (3):

$$\begin{aligned}
Y_{d_x, g^*}(t+1) &= f_{Y(t+1)}(P_{Y_{d_x, g^*}(t+1), \bar{N}_{d_x, \bar{n}^*}(t)}^0, U_{Y(t+1)}) \\
Z_{d_x, g^*}(t+1) &= f_{Z(t+1)}(P_{Z_{d_x, g^*}(t+1), \bar{N}_{d_x, \bar{n}^*}(t)}^0, U_{Z(t+1)}) \\
I_{0_{d_x, g^*}}(t+1) &= f_{I_0(t+1)}(P_{I_{0_{d_x, g^*}}(t+1), \bar{N}_{d_x, \bar{n}^*}(t)}^0, U_{I_0(t+1)})
\end{aligned} \tag{43}$$

where we use the generic notation $P_{X^*(t+1), \bar{N}^{*'}(t)}^0$ to denote the vector of variables defined by the parents $P_{X^*(t+1)}^0$ of the counterfactual variable $X^*(t+1)$ in which the monitoring variables $\bar{N}^*(t)$ are replaced by $\bar{N}^{*'}(t)$. By expanding the parent notation and then making use of equalities (A), (34), and (37) (the last two were just shown to hold if equalities (A) hold), the first of equalities (43) becomes

$$\begin{aligned}
Y_{d_x, g^*}(t+1) &= f_{Y(t+1)}((\bar{Y}_{d_x, g^*}(t), \bar{Z}_{d_x, g^*}(t), \bar{I}_{0_{d_x, g^*}}(t), \bar{A}_{2_{d_x, g^*}}(t), \bar{A}_{1_{d_x, g^*}}(t), \bar{N}_{d_x, \bar{n}^*}(t)), U_{Y(t+1)}) \\
&= f_{Y(t+1)}((\bar{Y}_{d_x, \bar{n}^*}(t), \bar{Z}_{d_x, \bar{n}^*}(t), \bar{I}_{0_{d_x, \bar{n}^*}}(t), \bar{A}_{2_{d_x, \bar{n}^*}}(t), \bar{A}_{1_{d_x, \bar{n}^*}}(t), \bar{N}_{d_x, \bar{n}^*}(t)), U_{Y(t+1)}) \\
&= f_{Y(t+1)}(P_{Y_{d_x, \bar{n}^*}(t+1)}^0, U_{Y(t+1)}).
\end{aligned}$$

By definition of the counterfactuals in Section 4.3, we also have:

$$Y_{d_x, \bar{n}^*}(t+1) = f_{Y(t+1)}(P_{Y_{d_x, \bar{n}^*}(t+1)}^0, U_{Y(t+1)}).$$

By transitivity, the last two equalities imply:

$$Y_{d_x, \bar{n}^*}(t+1) = Y_{d_x, g^*}(t+1). \tag{44}$$

By expanding the parent notation and then making use of the equality just shown to hold but also equalities (A), (34) and (37), the second of equalities (43) becomes

$$\begin{aligned}
Z_{d_x, g^*}(t+1) &= f_{Z(t+1)}((\bar{Y}_{d_x, g^*}(t+1), \bar{Z}_{d_x, g^*}(t), \bar{I}_{0_{d_x, g^*}}(t), \bar{A}_{2_{d_x, g^*}}(t), \bar{A}_{1_{d_x, g^*}}(t), \bar{N}_{d_x, \bar{n}^*}(t)), \\
&\quad U_{Z(t+1)}) \\
&= f_{Z(t+1)}((\bar{Y}_{d_x, \bar{n}^*}(t+1), \bar{Z}_{d_x, \bar{n}^*}(t), \bar{I}_{0_{d_x, \bar{n}^*}}(t), \bar{A}_{2_{d_x, \bar{n}^*}}(t), \bar{A}_{1_{d_x, \bar{n}^*}}(t), \bar{N}_{d_x, \bar{n}^*}(t)), \\
&\quad U_{Z(t+1)}) \\
&= f_{Z(t+1)}(P_{Z_{d_x, \bar{n}^*}(t+1)}^0, U_{Z(t+1)}).
\end{aligned}$$

By definition of the counterfactuals in Section 4.3, we also have:

$$Z_{d_x, \bar{n}^*}(t+1) = f_{Z(t+1)}(P_{Z_{d_x, \bar{n}^*}(t+1)}^0, U_{Z(t+1)}).$$

By transitivity, the last two equalities imply:

$$Z_{d_x, \bar{n}^*}(t+1) = Z_{d_x^*, g^*}(t+1). \quad (45)$$

By expanding the parent notation and then making use of both equalities (44) and (45) just shown to hold but also equalities (A), (34) and (37), the third of equalities (43) becomes

$$\begin{aligned} I_{0_{d_x, g^*}}(t+1) &= f_{I_0(t+1)}((\bar{Y}_{d_x, g^*}(t+1), \bar{Z}_{d_x, g^*}(t+1), \bar{I}_{0_{d_x, g^*}}(t), \bar{A}_{2_{d_x, g^*}}(t), \bar{A}_{1_{d_x, g^*}}(t), \\ &\quad \bar{N}_{d_x, \bar{n}^*}(t)), U_{I_0(t+1)}) \\ &= f_{I_0(t+1)}((\bar{Y}_{d_x, \bar{n}^*}(t+1), \bar{Z}_{d_x, \bar{n}^*}(t+1), \bar{I}_{0_{d_x, \bar{n}^*}}(t), \bar{A}_{2_{d_x, \bar{n}^*}}(t), \bar{A}_{1_{d_x, \bar{n}^*}}(t), \\ &\quad \bar{N}_{d_x, \bar{n}^*}(t)), U_{I_0(t+1)}) \\ &= f_{I_0(t+1)}(P_{I_{0_{d_x, \bar{n}^*}}(t+1)}^0, U_{I_0(t+1)}). \end{aligned}$$

By definition of the counterfactuals in Section 4.3, we also have:

$$I_{0_{d_x, \bar{n}^*}}(t+1) = f_{I_0(t+1)}(P_{I_{0_{d_x, \bar{n}^*}}(t+1)}^0, U_{I_0(t+1)}).$$

By transitivity, the last two equalities imply:

$$I_{0_{d_x, \bar{n}^*}}(t+1) = I_{0_{d_x, g^*}}(t+1).$$

We have just shown that the NDE assumption (3) implies:

(A) $\Rightarrow \bar{L}_{0_{d_x, \bar{n}^*}}(t+1) = \bar{L}_{0_{d_x, g^*}}(t+1)$. We note that the proof of this result also holds under the weak NDE assumption if equalities (5) hold (because equalities (5) imply equalities (43)). If equalities (6) hold instead, then we have:

$$\begin{aligned} Y_{d_x, \bar{n}^*}(t+1) &= f_{Y(t+1)}(P_{Y_{d_x, \bar{n}^*}(t+1), \bar{N}_{d_x, g^*}(t)}^0, U_{Y(t+1)}) \\ Z_{d_x, \bar{n}^*}(t+1) &= f_{Z(t+1)}(P_{Z_{d_x, \bar{n}^*}(t+1), \bar{N}_{d_x, g^*}(t)}^0, U_{Z(t+1)}) \\ I_{0_{d_x, \bar{n}^*}}(t+1) &= f_{I_0(t+1)}(P_{I_{0_{d_x, \bar{n}^*}}(t+1), \bar{N}_{d_x, g^*}(t)}^0, U_{I_0(t+1)}). \end{aligned} \quad (46)$$

We can then adopt the same reasoning following equalities (43) above to show that

equalities (46) imply $\bar{L}_{0_{d_x, \bar{n}^*}}(t+1) = \bar{L}_{0_{d_x, g^*}}(t+1)$ when equalities (A) hold. We have just shown that the (weak) NDE assumption (i.e., equalities (3), (5), or (6)) implies:

$$(A) \Rightarrow \bar{L}_{0_{d_x, \bar{n}^*}}(t+1) = \bar{L}_{0_{d_x, g^*}}(t+1).$$

Above, we have shown the proposition that for any $t = 1, \dots, K$, if equalities (A) hold then they imply equalities (B) when the (weak) NDE assumption holds. We now show the proposition that, equalities (A) hold at $t = 1$ when the (weak) NDE assumption holds, i.e. equalities (3), (5), or (6) imply:

$$\begin{cases} \bar{L}_{0_{d_x, \bar{n}^*}}(1) = \bar{L}_{0_{d_x^*, g^*}}(1) & (A.1) \\ I_{d_x, \bar{n}^*}(0) = I'_{d_x^*, g^*}(0) & (A.2) \\ A_{2_{d_x, \bar{n}^*}}(0) = A_{2_{d_x^*, g^*}}(0) & (A.3) \\ A_{1_{d_x, \bar{n}^*}}(0) = A_{1_{d_x^*, g^*}}(0) & (A.4) \\ N_{d_x, \bar{n}^*}(0) = N'_{d_x^*, g^*}(0) & (A.5) \end{cases}$$

From the definition of the variables in the causal model given in Section 3, their $\bar{n}^*$ -specific transformation defined by (8), and the definition of their counterfactuals analogs given in Section 4, we have

$$Y^*(0) = 0 \quad (47)$$

$$Z^*(0) = f_{Z(0)}(Y^*(0), U_{Z(0)}) \quad (48)$$

$$I_0^*(0) = f_{I_0(0)}(Y^*(0), Z^*(0), U_{I_0(0)}) \quad (49)$$

$$I_{d_x, \bar{n}^*}(t) = Y_{d_x, \bar{n}^*}(t) I_{d_x, \bar{n}^*}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) N_{d_x, \bar{n}^*}(t-1) I_{0_{d_x, \bar{n}^*}}(t) \quad (50)$$

$$I'_{d_x^*, g^*}(t) = Y_{d_x^*, g^*}(t) I'_{d_x^*, g^*}(t-1) + (1 - Y_{d_x^*, g^*}(t)) N'_{d_x^*, g^*}(t-1) I_{0_{d_x^*, g^*}}(t) \quad (51)$$

From equalities (47), (48), and (49), we have $I_{0_{d_x, \bar{n}^*}}(0) = I_{0_{d_x^*, g^*}}(0)$. From equalities (47), (50), and (51) we also have $I_{d_x, \bar{n}^*}(0) = I_{d_x^*, g^*}(0)$ and $I'_{d_x^*, g^*}(0) = I_{0_{d_x^*, g^*}}(0)$. By transitivity using the last three equalities, we obtain $I_{d_x, \bar{n}^*}(0) = I'_{d_x^*, g^*}(0)$. We have just shown that equality (A.2) holds.

From the definition of the variables in the causal model given in Section 3 and the definition of their counterfactuals analogs given in Section 4, we have

$$Y^*(0) = 0 \quad (52)$$

$$A_{2_{d_x, \bar{n}^*}}(t) = Y_{d_x, \bar{n}^*}(t) A_{2_{d_x, \bar{n}^*}}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) a_2^*(t) \quad (53)$$

$$A_{2_{d_x^*, g^*}}(t) = Y_{d_x^*, g^*}(t) A_{2_{d_x^*, g^*}}(t-1) + (1 - Y_{d_x^*, g^*}(t)) a_2^*(t). \quad (54)$$

These equalities imply $A_{2_{d_x, \bar{n}^*}}(0) = A_{2_{d_x^*, g^*}}(0)$. We have just shown that equality (A.3) holds.

From the definition of the variables in the causal model given in Section 3 and the definition of their counterfactuals analogs given in Section 4, we have

$$Y^*(0) = 0 \quad (55)$$

$$Z^*(0) = f_{Z(0)}(Y^*(0), U_{Z(0)}) \quad (56)$$

$$A_{1_{d_x, \bar{n}^*}}(t) = Y_{d_x, \bar{n}^*}(t) A_{1_{d_x, \bar{n}^*}}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t)) d_x(t) (P_{A_{1_{d_x, \bar{n}^*}}(t)}) \quad (57)$$

$$A_{1_{d_x^*, g^*}}(t) = Y_{d_x^*, g^*}(t) A_{1_{d_x^*, g^*}}(t-1) + (1 - Y_{d_x^*, g^*}(t)) d_x^*(t) (P_{A_{1_{d_x^*, g^*}}(t)}). \quad (58)$$

From equalities (55), (57), and (58), we have

$$A_{1_{d_x, \bar{n}^*}}(0) = d_x(0) (Y_{d_x, \bar{n}^*}(0), Z_{d_x, \bar{n}^*}(0), I_{d_x, \bar{n}^*}(0), A_{2_{d_x, \bar{n}^*}}(0)) \quad (59)$$

$$A_{1_{d_x^*, g^*}}(0) = d_x^*(0) (Y_{d_x^*, g^*}(0), Z_{d_x^*, g^*}(0), I_{d_x^*, g^*}(0), A_{2_{d_x^*, g^*}}(0)). \quad (60)$$

From the definition of the dynamic interventions d_x^* given by (7), equality (60) becomes:

$$A_{1_{d_x^*, g^*}}(0) = d_x^*(0) (Y_{d_x^*, g^*}(0), Z_{d_x^*, g^*}(0), I'_{d_x^*, g^*}(0), A_{2_{d_x^*, g^*}}(0)).$$

From equalities (A.2) and (A.3) (just shown to hold) and because equalities (55) and (56) imply $Y_{d_x, \bar{n}^*}(0) = Y_{d_x^*, g^*}(0)$ and $Z_{d_x, \bar{n}^*}(0) = Z_{d_x^*, g^*}(0)$, the previous equality becomes

$$A_{1_{d_x^*, g^*}}(0) = d_x^*(0) (Y_{d_x, \bar{n}^*}(0), Z_{d_x, \bar{n}^*}(0), I_{d_x, \bar{n}^*}(0), A_{2_{d_x, \bar{n}^*}}(0)).$$

By transitivity, the latter equality and equality (59) imply

$$A_{1_{d_x, \bar{n}^*}}(t) = A_{1_{d_x^*, g^*}}(t).$$

We have just shown that equality (A.4) holds.

From the definition of the variables in the causal model given in Section 3, their $\bar{n}^*$ -specific transformation defined by (8), and the definition of their counterfactuals analogs given in Section 4, we have

$$Y^*(0) = 0$$

$$N^{*'}(t) = Y^*(t)N^{*'}(t-1) + (1 - Y^*(t))n^*(t)N^*(t)$$

$$N_{d_x, \bar{n}^*}(t) = Y_{d_x, \bar{n}^*}(t)N_{d_x, \bar{n}^*}(t-1) + (1 - Y_{d_x, \bar{n}^*}(t))n^*(t)$$

$$N_{d_x^*, g^*}(t) = Y_{d_x^*, g^*}(t)N_{d_x^*, g^*}(t) + (1 - Y_{d_x^*, g^*}(t))f_{N(t)}(P_{N_{d_x^*, g^*}(t)}^0, U_{N(t)})^{1-n^*(t)},$$

and thus, in particular,

$$N^{*'}(0) = n^*(0)N^*(0) \quad (61)$$

$$N_{d_x, \bar{n}^*}(0) = n^*(0) \quad (62)$$

$$N_{d_x^*, g^*}(0) = f_{N(0)}(P_{N_{d_x^*, g^*}(0)}^0, U_{N(0)})^{1-n^*(0)}. \quad (63)$$

These last equalities imply:

- If $n^*(0) = 1$ then $N_{d_x^*, g^*}'(0) = N_{d_x^*, g^*}(0)$ from equality (61), $N_{d_x, \bar{n}^*}(0) = 1$ from equality (62), $N_{d_x^*, g^*}(0) = 1$ from equality (63), and thus $N_{d_x, \bar{n}^*}(0) = N_{d_x^*, g^*}'(0)$ by transitivity.
- If $n^*(0) = 0$ then $N_{d_x^*, g^*}'(0) = 0$ from equality (61) and $N_{d_x, \bar{n}^*}(0) = 0$ from equality (62), and thus $N_{d_x, \bar{n}^*}(0) = N_{d_x^*, g^*}'(0)$ by transitivity.

We have just shown that equality (A.5) holds.

From the definition of the variables in the causal model given in Section 3 and the definition of their counterfactuals analogs given in Section 4, we have

$$Y^*(0)=0$$

$$Z^*(0)=f_{Z(0)}((Y^*(0)), U_{Z(0)})$$

$$I_0^*(0)=f_{I_0(0)}((Y^*(0), Z^*(0)), U_{I_0(0)}).$$

These equalities imply

$$L_{0_{d_x, \bar{n}^*}}(0)=L_{0_{d_x, g^*}}(0). \quad (64)$$

By definition of the counterfactuals in Section 4.5, we have:

$$\begin{aligned} Y_{d_x, g^*}^*(1) &= f_{Y(1)}(P_{Y_{d_x, g^*}^*(1)}^0, U_{Y(1)}) \\ Z_{d_x, g^*}^*(1) &= f_{Z(1)}(P_{Z_{d_x, g^*}^*(1)}^0, U_{Z(1)}) \\ I_{0_{d_x, g^*}}^*(1) &= f_{I_0(t+1)}(P_{I_{0_{d_x, g^*}}^*(1)}^0, U_{I_0(1)}) \end{aligned}$$

Because the set $\bar{\mathcal{A}}^*$ contains the support of $\bar{A}_{d_x, g^*}^*(K)$ and because $\bar{\mathcal{N}}^*$ contains the supports of both $\bar{N}_{d_x, g^*}^*(K)$ and $\bar{N}_{d_x, \bar{n}^*}^*(K)$ (by definitions (1) and (2)), the righth- and sides of the previous equalities can be modified as follows under the NDE assumption (3):

$$\begin{aligned} Y_{d_x, g^*}^*(1) &= f_{Y(1)}(P_{Y_{d_x, g^*}^*(1), \bar{N}_{d_x, \bar{n}^*}^*(0)}^0, U_{Y(1)}) \\ Z_{d_x, g^*}^*(1) &= f_{Z(1)}(P_{Z_{d_x, g^*}^*(1), \bar{N}_{d_x, \bar{n}^*}^*(0)}^0, U_{Z(1)}) \\ I_{0_{d_x, g^*}}^*(1) &= f_{I_0(1)}(P_{I_{0_{d_x, g^*}}^*(1), \bar{N}_{d_x, \bar{n}^*}^*(0)}^0, U_{I_0(1)}) \end{aligned} \quad (65)$$

where we use the generic notation $P_{X^*(t+1), \bar{N}^{*'}(t)}^0$ to denote the vector of variables defined by the parents $P_{X^*(t+1)}^0$ of the counterfactual variable $X^*(t+1)$ in which the monitoring variables $\bar{N}^*(t)$ are replaced by $\bar{N}^{*'}(t)$. By expanding the parent notation and then making use of equalities (64), (A.3), and (A.4) (just shown to hold), the first of equalities (65) becomes

$$\begin{aligned}
Y_{d_x, g^*}(1) &= f_{Y(1)}((Y_{d_x, g^*}(0), Z_{d_x, g^*}(0), I_{0_{d_x, g^*}}(0), A_{2_{d_x, g^*}}(0), A_{1_{d_x, g^*}}(0), N_{d_x, \bar{n}^*}(0)), U_{Y(1)}) \\
&= f_{Y(1)}((Y_{d_x, \bar{n}^*}(0), Z_{d_x, \bar{n}^*}(0), I_{0_{d_x, \bar{n}^*}}(0), A_{2_{d_x, \bar{n}^*}}(0), A_{1_{d_x, \bar{n}^*}}(0), N_{d_x, \bar{n}^*}(0)), U_{Y(1)}) \\
&= f_{Y(1)}(P_{Y_{d_x, \bar{n}^*}}^0(1), U_{Y(1)}).
\end{aligned}$$

By definition of the counterfactuals in Section 4.3, we also have:

$$Y_{d_x, \bar{n}^*}(1) = f_{Y(1)}(P_{Y_{d_x, \bar{n}^*}}^0(1), U_{Y(1)}).$$

By transitivity, the last two equalities imply:

$$Y_{d_x, \bar{n}^*}(1) = Y_{d_x, g^*}(1). \quad (66)$$

By expanding the parent notation and then making use of the equalities (64), (66), (A.3), and (A.4), the second of equalities (65) becomes

$$\begin{aligned}
Z_{d_x, g^*}(1) &= f_{Z(1)}((\bar{Y}_{d_x, g^*}(1), Z_{d_x, g^*}(0), I_{0_{d_x, g^*}}(0), A_{2_{d_x, g^*}}(0), A_{1_{d_x, g^*}}(0), N_{d_x, \bar{n}^*}(0)), U_{Z(1)}) \\
&= f_{Z(1)}((\bar{Y}_{d_x, \bar{n}^*}(1), Z_{d_x, \bar{n}^*}(0), I_{0_{d_x, \bar{n}^*}}(0), A_{2_{d_x, \bar{n}^*}}(0), A_{1_{d_x, \bar{n}^*}}(0), N_{d_x, \bar{n}^*}(0)), U_{Z(1)}) \\
&= f_{Z(1)}(P_{Z_{d_x, \bar{n}^*}}^0(1), U_{Z(1)}).
\end{aligned}$$

By definition of the counterfactuals in Section 4.3, we also have:

$$Z_{d_x, \bar{n}^*}(1) = f_{Z(1)}(P_{Z_{d_x, \bar{n}^*}}^0(1), U_{Z(1)}).$$

By transitivity, the last two equalities imply:

$$Z_{d_x, \bar{n}^*}(1) = Z_{d_x, g^*}(1). \quad (67)$$

By expanding the parent notation and then making use of equalities (64), (66), (67), (A.3), and (A.4), the third of equalities (65) becomes

$$\begin{aligned}
I_{0_{d_x, g^*}}(1) &= f_{I_{0(t)}}((\bar{Y}_{d_x, g^*}(1), \bar{Z}_{d_x, g^*}(1), I_{0_{d_x, g^*}}(0), A_{2_{d_x, g^*}}(0), A_{1_{d_x, g^*}}(0), N_{d_x, \bar{n}^*}(0)), U_{I_{0(1)}}) \\
&= f_{I_{0(1)}}((\bar{Y}_{d_x, \bar{n}^*}(1), \bar{Z}_{d_x, \bar{n}^*}(1), I_{0_{d_x, \bar{n}^*}}(0), A_{2_{d_x, \bar{n}^*}}(0), A_{1_{d_x, \bar{n}^*}}(0), N_{d_x, \bar{n}^*}(0)), U_{I_{0(1)}}) \\
&= f_{I_{0(1)}}(P_{I_{0_{d_x, \bar{n}^*}}}^0(1), U_{I_{0(1)}}).
\end{aligned}$$

By definition of the counterfactuals in Section 4.3, we also have:

$$I_{0_{d_x, \bar{n}^*}}(1) = f_{I_0(1)}(P_{I_{0_{d_x, \bar{n}^*}}(1)}^0, U_{I_0(1)}).$$

By transitivity, the last two equalities imply:

$$I_{0_{d_x, \bar{n}^*}}(1) = I_{0_{d_x, g^*}}(1).$$

We have just shown that the NDE assumption (3) implies equality (A.1) holds. We note that the proof of this result also holds under the weak NDE assumption if equalities (5) hold (because equalities (5) imply equalities (65)). If equalities (6) hold instead, then we have:

$$\begin{aligned} Y_{d_x, \bar{n}^*}(1) &= f_{Y(1)}(P_{Y_{d_x, \bar{n}^*}(1), \bar{N}_{d_x, g^*}(0)}^0, U_{Y(1)}) \\ Z_{d_x, \bar{n}^*}(1) &= f_{Z(1)}(P_{Z_{d_x, \bar{n}^*}(1), \bar{N}_{d_x, g^*}(0)}^0, U_{Z(1)}) \\ I_{0_{d_x, \bar{n}^*}}(1) &= f_{I_0(1)}(P_{I_{0_{d_x, \bar{n}^*}}(1), \bar{N}_{d_x, g^*}(0)}^0, U_{I_0(1)}). \end{aligned} \quad (68)$$

We can then adopt the same reasoning following equalities (65) above to show that equalities (68) imply equality (A.1) holds. We have thus shown that the (weak) NDE assumption (i.e., equalities (3), (5), or (6)) implies equality (A.1). This completes the proof of the proposition that, equalities (A) hold at $t = 1$ when the (weak) NDE assumption holds.

By induction, the previous two propositions imply equalities (9) and (10) of theorem 1.

D Proof of the second identifiability result

D.1 Lemma

Lemma 3—When the NPSEM introduced in Section 3 is modified with an intervention regimen on the action variables in $\mathcal{X} = \{A_2(t), A_1(t), N(t) : t = 0, \dots, K\}$ defined as in Sections 4.2 through 4.5, we define the following conditional independence assumptions on the action variables for $t = 0, \dots, K$:

$$A_2^*(t) \perp \left(U_{A_1(j)}, U_{N(j)}, U_{L_0(j+1)}, U_{A_2(j+1)} \right)_{j=t, \dots, K} \left| P_{A_2^*(t)}^0, U_{A_2(t)}, \quad (69)$$

$$A_1^*(t) \perp \left(\left(U_{N(j)}, U_{L_0(j+1)}, U_{A_2(j+1)}, U_{A_1(j+1)} \right)_{j=t, \dots, K-1}, U_{N(K)}, U_{L_0(K+1)}, U_{A_2(K+1)} \right) \left| P_{A_1^*(t)}^0, U_{A_1(t)}, \quad (70)$$

$$N^*(t) \perp \left(\left(U_{L_0(j)}, U_{A_2(j)}, U_{A_1(j)}, U_{N(j)} \right)_{j=t+1, \dots, K}, U_{L_0(K+1)}, U_{A_2(K+1)} \right) \Big| P_{N^*(t)}^0, U_{N(t)}, \quad (71)$$

where $U_{L_0(j)} \equiv (U_{Y(j)}, U_{Z(j)}, U_{I_0(j)})$.

For any realization $p_{X^*(t)}^0$ of the parents $P_{X^*(t)}^0$ of the counterfactual variable $X^*(t)$ corresponding with a given action variable $X(t) \in \mathcal{X}$, we define the function

$f_{p_{X^*(t)}^0}^*: (U_{X(t)}, U_{X(t)}^*) \mapsto f_{X(t)}^*(p_{X^*(t)}^0, U_{X(t)}, U_{X(t)}^*)$ and denote the fiber of any element $x^*(t)$

in its codomain by $f_{p_{X^*(t)}^0}^{*-1}(x^*(t))$, i.e., the following set of realizations of $(U_{X(t)}, U_{X(t)}^*)$:

$$f_{p_{X^*(t)}^0}^{*-1}(x^*(t)) = \{(u_{X(t)}, u_{X(t)}^*) : f_{X(t)}^*(p_{X^*(t)}^0, u_{X(t)}, u_{X(t)}^*) = x^*(t)\}.$$

In addition, we define the following two sets:

$$f_{p_{X^*(t)}^0, u}^{*-1}(x^*(t)) = \{u_{X(t)} : \exists (u_{X(t)}, u_{X(t)}^*) \in f_{p_{X^*(t)}^0}^{*-1}(x^*(t))\}$$

$$f_{p_{X^*(t)}^0, u^*}^{*-1}(u_{X(t)}, x^*(t)) = \{u_{X(t)}^* : \exists (u_{X(t)}, u_{X(t)}^*) \in f_{p_{X^*(t)}^0}^{*-1}(x^*(t))\},$$

and we thus have:

$$(u_{X(t)}, u_{X(t)}^*) \in f_{p_{X^*(t)}^0}^{*-1}(x^*(t)) \iff u_{X(t)} \in f_{p_{X^*(t)}^0, u}^{*-1}(x^*(t)), u_{X(t)}^* \in f_{p_{X^*(t)}^0, u^*}^{*-1}(u_{X(t)}, x^*(t)).$$

(72)

If the intervention regimen on the action variables $A_2(t)$, $A_1(t)$, and $N(t)$ is defined by a choice of functions f^* and a conditional distribution $P_{L^*|U}$ that satisfy assumptions (69),

(70), and (71) then the conditional distribution of $L_0^*(t)$ given its parents $P_{L_0^*(t)}^0 \equiv P_{Y^*(t)}^0$ does not depend on $P_{L^*|U}$ but depends only on the distribution P_U and the functions f^* . More specifically, the following equality holds for any given time point $t = 0, \dots, K+1$ and any possible realization of the counterfactual variables $(\bar{l}_0^*(K+1), \bar{a}_2^*(K+1), \bar{a}_1^*(K), \bar{n}^*(K))$:

$$P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \mid p_{L_0^*(t)}^0 = p_{L_0^*(t)}^0 \right) = P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \mid \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1} (l_0^*(k)), \right. \right. \\ \left. \left. U_{A_2(k)} \in f_{p_{A_2^*(k)}^0}^{*-1} (a_2^*(k)), U_{A_1(k)} \in f_{p_{A_1^*(k)}^0}^{*-1} (a_1^*(k)), U_{N(k)} \in f_{p_{N^*(k)}^0}^{*-1} (n^*(k)) \right)_{k=0, \dots, t-1} \right) \quad (73)$$

where $f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \equiv f_{p_{Y^*(t)}^0}^{-1} (y^*(t)) \times f_{p_{Z^*(t)}^0}^{-1} (z^*(t)) \times f_{p_{I_0^*(t)}^0}^{-1} (i_0^*(t))$ (see Appendix A for the definition of each set of this cartesian product).

In addition, if the following conditional independence assumptions hold

$$U_{N(t)} \perp (U_{L_0(j)})_{j=t+1, \dots, K} \mid (U_{L_0(j)}, U_{A_2(j)}, U_{A_1(j)})_{j=0, \dots, t}, (U_{N(j)})_{j=0, \dots, t-1} \text{ for } t=0, \dots, K. \quad (74)$$

then the conditional distribution of $L_0^*(t)$ given its parents $P_{L_0^*(t)}^0$ no longer depends on the functions f^* and, more specifically, the following equality holds:

$$P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \mid p_{L_0^*(t)}^0 = p_{L_0^*(t)}^0 \right) = P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \mid \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1} (l_0^*(k)) \right)_{k=0, \dots, t-1} \right). \quad (75)$$

Proof by induction—Below, we show that for any given time point $t=0, \dots, K+1$ and any given possible realization $(\bar{l}_0^*(K+1), \bar{a}_2^*(K+1), \bar{a}_1^*(K), \bar{n}^*(K))$ of one of the counterfactual processes defined in Section 4.2, 4.3, 4.4, or 4.5, the conditional independences (69), (70), and (71) imply the following proposition for any $j=t-2, \dots, 0$:

$$E(j)(A) \Rightarrow E(j-1)(B),$$

where $E(j)$ denotes the following equality:

$$\psi = P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \mid p_{L_0^*(j+1)}^0 = p_{L_0^*(j+1)}^0, \overline{\mathcal{U}}(j+1, t-1) \right)$$

$$\begin{aligned}
 \text{with } \psi &\equiv P_{U, U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) | P_{L_0^*(t)}^0 = P_{L_0^*(t)}^0 \right) \text{ and} \\
 \overline{\mathcal{U}}(j+1, t-1) &\equiv \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1} (l_0^*(k)), U_{A_2(k)} \in f_{p_{A_2^*(k)}^0, u}^{*-1} (a_2^*(k)), U_{A_1(k)} \in f_{p_{A_1^*(k)}^0, u}^{*-1} (a_1^*(k)), U_{N(k)} \in f_{p_{N^*(k)}^0, u}^{*-1} (n) \right) .
 \end{aligned}$$

In the proof below, we use bold font to indicate how each new equality is modified from its previous expression and we also indicate the assumption(s) supporting a new expression above the equal sign. We have for any $j = t - 2, \dots, 0$:

$$\begin{aligned}
\psi &\stackrel{(A)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid P_{L_0^*(j+1)}^0 = p_{L_0^*(j+1)}^0, \mathcal{Z}(j+1, t-1) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid \mathbf{P}_{N^*(j)}^0 = \mathbf{p}_{N^*(j)}^0, \mathbf{N}^*(j) = \mathbf{n}^*(j), \mathcal{Z}(j+1, t-1) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid P_{N^*(j)}^0 = p_{N^*(j)}^0, (\mathbf{U}_{N(j)}, \mathbf{U}_{N(j)}^*) \in \mathbf{f}_{\mathbf{p}_{N^*(j)}^0}^{*-1}(\mathbf{n}^*(j)), \right. \\
&\quad \left. \mathcal{Z}(j+1, t-1) \right) \\
&\stackrel{(72)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid P_{N^*(j)}^0 = p_{N^*(j)}^0, \mathbf{U}_{N(j)} \in \mathbf{f}_{\mathbf{p}_{N^*(j)}^0, \mathbf{u}}^{*-1}(\mathbf{n}^*(j)), (U_{N(j)}, U_{N(j)}^*) \in \right. \\
&\quad \left. f_{p_{N^*(j)}^0}^{*-1}(\mathbf{n}^*(j)), \mathcal{Z}(j+1, t-1) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid P_{N^*(j)}^0 = p_{N^*(j)}^0, U_{N(j)} \in f_{p_{N^*(j)}^0, \mathbf{u}}^{*-1}(\mathbf{n}^*(j)), \mathbf{N}^*(j) = \mathbf{n}^*(j), \right. \\
&\quad \left. \mathcal{Z}(j+1, t-1) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1) \mid P_{N^*(j)}^0 = p_{N^*(j)}^0, U_{N(j)} \in f_{p_{N^*(j)}^0, \mathbf{u}}^{*-1}(\mathbf{n}^*(j)), \right. \\
&\quad \left. N^*(j) = \mathbf{n}^*(j) \right) / \\
&\quad \mathbf{P}_{U,U^*} \left(\mathcal{Z}(j+1, t-1) \mid \mathbf{P}_{N^*(j)}^0 = \mathbf{p}_{N^*(j)}^0, \mathbf{U}_{N(j)} \in \mathbf{f}_{\mathbf{p}_{N^*(j)}^0, \mathbf{u}}^{*-1}(\mathbf{n}^*(j)), \mathbf{N}^*(j) = \mathbf{n}^*(j) \right) \\
&\stackrel{(71)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1) \mid \mathbf{P}_{N^*(j)}^0 = \mathbf{p}_{N^*(j)}^0, \mathbf{U}_{N(j)} \in \mathbf{f}_{\mathbf{p}_{N^*(j)}^0, \mathbf{u}}^{*-1}(\mathbf{n}^*(j)) \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1) \mid \mathbf{P}_{N^*(j)}^0 = \mathbf{p}_{N^*(j)}^0, \mathbf{U}_{N(j)} \in \mathbf{f}_{\mathbf{p}_{N^*(j)}^0, \mathbf{u}}^{*-1}(\mathbf{n}^*(j)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid \mathcal{Z}(j+1, t-1), P_{N^*(j)}^0 = p_{N^*(j)}^0, U_{N(j)} \in f_{p_{N^*(j)}^0, \mathbf{u}}^{*-1}(\mathbf{n}^*(j)) \right)
\end{aligned}$$

$$\begin{aligned}
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{N^*(j)}^0 = p_{N^*(j)}^0 \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{N^*(j)}^0 = p_{N^*(j)}^0 \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, \right. \\
&\quad \left. A_1^*(j) = a_1^*(j) \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, A_1^*(j) = a_1^*(j) \right) \\
&\stackrel{(72)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, \right. \\
&\quad \left. U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)), A_1^*(j) = a_1^*(j) \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)), \right. \\
&\quad \left. A_1^*(j) = a_1^*(j) \right) \\
&\stackrel{(70)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid \right. \\
&\quad \left. p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)) \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)), p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, \right. \\
&\quad \left. U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)), \right. \\
&\quad \left. p_{A_2^*(j)}^0 = p_{A_2^*(j)}^0, A_2^*(j) = a_2^*(j), U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)) \right)
\end{aligned}$$

$$\begin{aligned}
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{N^*(j)}^0 = p_{N^*(j)}^0 \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{N^*(j)}^0 = p_{N^*(j)}^0 \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, \right. \\
&\quad \left. A_1^*(j) = a_1^*(j) \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, A_1^*(j) = a_1^*(j) \right) \\
&\stackrel{(72)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, \right. \\
&\quad \left. U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)), A_1^*(j) = a_1^*(j) \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)), \right. \\
&\quad \left. A_1^*(j) = a_1^*(j) \right) \\
&\stackrel{(70)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid \right. \\
&\quad \left. p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)) \right) / \\
&\quad P_{U,U^*} \left(\mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)) \mid p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)), p_{A_1^*(j)}^0 = p_{A_1^*(j)}^0, \right. \\
&\quad \left. U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \mid \mathcal{Z}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0, u}^{*-1}(n^*(j)), \right. \\
&\quad \left. p_{A_2^*(j)}^0 = p_{A_2^*(j)}^0, A_2^*(j) = a_2^*(j), U_{A_1(j)} \in f_{p_{A_1^*(j)}^0, u}^{*-1}(a_1^*(j)) \right)
\end{aligned}$$

$$\begin{aligned}
\psi &\stackrel{(72)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \overline{\mathcal{W}}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0}^{*-1}(n^*(j)), P_{A_2^*(j)}^0 = p_{A_2^*(j)}^0 \right. \\
&\quad \left. , U_{A_2(j)} \in f_{p_{A_2^*(j)}^0}^{*-1}(\mathbf{a}_2^*(j)), A_2^*(j) = a_2^*(j), U_{A_1(j)} \in f_{p_{A_1^*(j)}^0}^{*-1}(a_1^*(j)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \overline{\mathcal{W}}(\mathbf{j}+1, \mathbf{t}-1), \mathbf{U}_{N(j)} \in f_{p_{N^*(j)}^0}^{*-1}(\mathbf{n}^*(j)), \right. \\
&\quad \left. \mathbf{U}_{A_1(j)} \in f_{p_{A_1^*(j)}^0}^{*-1}(\mathbf{a}_1^*(j)) \middle| P_{A_2^*(j)}^0 = p_{A_2^*(j)}^0, U_{A_2(j)} \in f_{p_{A_2^*(j)}^0}^{*-1}(a_2^*(j)), A_2^*(j) = a_2^*(j) \right) / \\
&P_{U,U^*} \left(\overline{\mathcal{W}}(\mathbf{j}+1, \mathbf{t}-1), \mathbf{U}_{N(j)} \in f_{p_{N^*(j)}^0}^{*-1}(\mathbf{n}^*(j)), \mathbf{U}_{A_1(j)} \in f_{p_{A_1^*(j)}^0}^{*-1}(\mathbf{a}_1^*(j)) \middle| \mathbf{P}_{A_2^*(j)}^0 = \mathbf{p}_{A_2^*(j)}^0, \right. \\
&\quad \left. \mathbf{U}_{A_2(j)} \in f_{p_{A_2^*(j)}^0}^{*-1}(\mathbf{a}_2^*(j)), \mathbf{A}_2^*(j) = \mathbf{a}_2^*(j) \right) \\
&\stackrel{(69)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \overline{\mathcal{W}}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0}^{*-1}(n^*(j)), \right. \\
&\quad \left. U_{A_1(j)} \in f_{p_{A_1^*(j)}^0}^{*-1}(a_1^*(j)) \middle| \mathbf{P}_{A_2^*(j)}^0 = \mathbf{p}_{A_2^*(j)}^0, \mathbf{U}_{A_2(j)} \in f_{p_{A_2^*(j)}^0}^{*-1}(\mathbf{a}_2^*(j)) \right) / \\
&P_{U,U^*} \left(\overline{\mathcal{W}}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0}^{*-1}(n^*(j)), U_{A_1(j)} \in f_{p_{A_1^*(j)}^0}^{*-1}(a_1^*(j)) \middle| \right. \\
&\quad \left. \mathbf{P}_{A_2^*(j)}^0 = \mathbf{p}_{A_2^*(j)}^0, \mathbf{U}_{A_2(j)} \in f_{p_{A_2^*(j)}^0}^{*-1}(\mathbf{a}_2^*(j)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \overline{\mathcal{W}}(\mathbf{j}+1, \mathbf{t}-1), \mathbf{U}_{N(j)} \in f_{p_{N^*(j)}^0}^{*-1}(\mathbf{n}^*(j)), \right. \\
&\quad \left. \mathbf{U}_{A_1(j)} \in f_{p_{A_1^*(j)}^0}^{*-1}(\mathbf{a}_1^*(j)), \mathbf{U}_{A_2(j)} \in f_{p_{A_2^*(j)}^0}^{*-1}(\mathbf{a}_2^*(j)), \mathbf{P}_{A_2^*(j)}^0 = \mathbf{p}_{A_2^*(j)}^0 \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \overline{\mathcal{W}}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0}^{*-1}(n^*(j)), \right. \\
&\quad \left. U_{A_1(j)} \in f_{p_{A_1^*(j)}^0}^{*-1}(a_1^*(j)), U_{A_2(j)} \in f_{p_{A_2^*(j)}^0}^{*-1}(a_2^*(j)), \mathbf{P}_{L_0^*(j)}^0 = \mathbf{p}_{L_0^*(j)}^0, \mathbf{L}_0^*(j) = \mathbf{l}_0^*(j) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \overline{\mathcal{W}}(j+1, t-1), U_{N(j)} \in f_{p_{N^*(j)}^0}^{*-1}(n^*(j)), \right. \\
&\quad \left. U_{A_1(j)} \in f_{p_{A_1^*(j)}^0}^{*-1}(a_1^*(j)), U_{A_2(j)} \in f_{p_{A_2^*(j)}^0}^{*-1}(a_2^*(j)), P_{L_0^*(j)}^0 = p_{L_0^*(j)}^0, \mathbf{U}_{L_0(j)} \in f_{p_{L_0^*(j)}^0}^{*-1}(\mathbf{l}_0^*(j)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{L_0^*(j)}^0 = p_{L_0^*(j)}^0, \overline{\mathcal{W}}(\mathbf{j}, \mathbf{t}-1) \right).
\end{aligned}$$

The last equality corresponds to equality (B).

Thus, we have just shown $E(j) \Rightarrow E(j-1)$.

Above, we have shown the proposition that, for any $j = t - 2 \dots, 0$, if equality (A) holds then it implies equality (B) when the conditional independences (69), (70), and (71) hold. We now show the proposition that equality (A) holds at $j = t - 2$ when the conditional independences (69), (70), and (71) hold:

$$\begin{aligned}
\psi &\equiv P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{L_0^*(t)}^0 = p_{L_0^*(t)}^0 \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \mathbf{P}_{N^*(t-1)}^0 = \mathbf{p}_{N^*(t-1)}^0, \mathbf{N}^*(t-1) = \mathbf{n}^*(t-1) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{N^*(t-1)}^0 = p_{N^*(t-1)}^0, \right. \\
&\quad \left. (\mathbf{U}_{N(t-1)}, \mathbf{U}_{N^*(t-1)}^*) \in \mathbf{f}_{\mathbf{p}_{N^*(t-1)}^0}^{*-1}(\mathbf{n}^*(t-1)) \right) \\
&\stackrel{(72)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{N^*(t-1)}^0 = p_{N^*(t-1)}^0, \mathbf{U}_{N(t-1)} \in \mathbf{f}_{\mathbf{p}_{N^*(t-1)}^0}^{*-1,u}(\mathbf{n}^*(t-1)), \right. \\
&\quad \left. (U_{N(t-1)}, U_{N^*(t-1)}^*) \in f_{p_{N^*(t-1)}^0}^{*-1}(n^*(t-1)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{N^*(t-1)}^0 = p_{N^*(t-1)}^0, U_{N(t-1)} \in f_{p_{N^*(t-1)}^0}^{*-1,u}(n^*(t-1)), \right. \\
&\quad \left. \mathbf{N}^*(t-1) = \mathbf{n}^*(t-1) \right) \\
&\stackrel{(71)}{=} P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \mathbf{P}_{N^*(t-1)}^0 = \mathbf{p}_{N^*(t-1)}^0, \mathbf{U}_{N(t-1)} \in \mathbf{f}_{\mathbf{p}_{N^*(t-1)}^0}^{*-1,u}(\mathbf{n}^*(t-1)) \right) \\
&= P_{U,U^*} \left(\mathbf{U}_{N(t-1)} \in \mathbf{f}_{\mathbf{p}_{N^*(t-1)}^0}^{*-1,u}(\mathbf{n}^*(t-1)), U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{N^*(t-1)}^0 = p_{N^*(t-1)}^0 \right) / \\
&\quad P_{U,U^*} \left(\mathbf{U}_{N(t-1)} \in \mathbf{f}_{\mathbf{p}_{N^*(t-1)}^0}^{*-1,u}(\mathbf{n}^*(t-1)) \middle| P_{N^*(t-1)}^0 = p_{N^*(t-1)}^0 \right) \\
&= P_{U,U^*} \left(U_{N(t-1)} \in f_{p_{N^*(t-1)}^0}^{*-1,u}(n^*(t-1)), U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \mathbf{P}_{\mathbf{A}_1^*(t-1)}^0 = \mathbf{p}_{\mathbf{A}_1^*(t-1)}^0, \right. \\
&\quad \left. \mathbf{A}_1^*(t-1) = \mathbf{a}_1^*(t-1) \right) / \\
&\quad P_{U,U^*} \left(U_{N(t-1)} \in f_{p_{N^*(t-1)}^0}^{*-1,u}(n^*(t-1)) \middle| \mathbf{P}_{\mathbf{A}_1^*(t-1)}^0 = \mathbf{p}_{\mathbf{A}_1^*(t-1)}^0, \mathbf{A}_1^*(t-1) = \mathbf{a}_1^*(t-1) \right)
\end{aligned}$$

$$\begin{aligned}
\psi &\stackrel{(72)}{=} P_{U,U^*} \left(U_{N(t-1)} \in f_{p_{N^*(t-1),u}^0}^{*-1}(n^*(t-1)), U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{A_1^*(t-1)}^0 = p_{A_1^*(t-1)}^0, \mathbf{U}_{A_1(t-1)} \in \right. \\
&\quad \left. \mathbf{f}_{p_{A_1^*(t-1),u}^0}^{*-1}(\mathbf{a}_1^*(t-1)), A_1^*(t-1) = a_1^*(t-1) \right) / \\
&\quad P_{U,U^*} \left(U_{N(t-1)} \in f_{p_{N^*(t-1),u}^0}^{*-1}(n^*(t-1)) \middle| P_{A_1^*(t-1)}^0 = p_{A_1^*(t-1)}^0, \mathbf{U}_{A_1(t-1)} \in \mathbf{f}_{p_{A_1^*(t-1),u}^0}^{*-1}(\mathbf{a}_1^*(t-1)), \right. \\
&\quad \left. A_1^*(t-1) = a_1^*(t-1) \right) \\
&\stackrel{(70)}{=} P_{U,U^*} \left(U_{N(t-1)} \in f_{p_{N^*(t-1),u}^0}^{*-1}(n^*(t-1)), U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \mathbf{P}_{A_1^*(t-1)}^0 = \mathbf{p}_{A_1^*(t-1)}^0, \mathbf{U}_{A_1(t-1)} \in \right. \\
&\quad \left. \mathbf{f}_{p_{A_1^*(t-1),u}^0}^{*-1}(\mathbf{a}_1^*(t-1)) \right) / \\
&\quad P_{U,U^*} \left(U_{N(t-1)} \in f_{p_{N^*(t-1),u}^0}^{*-1}(n^*(t-1)) \middle| \mathbf{P}_{A_1^*(t-1)}^0 = \mathbf{p}_{A_1^*(t-1)}^0, \mathbf{U}_{A_1(t-1)} \in \mathbf{f}_{p_{A_1^*(t-1),u}^0}^{*-1}(\mathbf{a}_1^*(t-1)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \mathbf{P}_{A_1^*(t-1)}^0 = \mathbf{p}_{A_1^*(t-1)}^0, \mathbf{U}_{A_1(t-1)} \in \mathbf{f}_{p_{A_1^*(t-1),u}^0}^{*-1}(\mathbf{a}_1^*(t-1)), \right. \\
&\quad \left. \mathbf{U}_{N(t-1)} \in \mathbf{f}_{p_{N^*(t-1),u}^0}^{*-1}(n^*(t-1)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \mathbf{P}_{A_2^*(t-1)}^0 = \mathbf{p}_{A_2^*(t-1)}^0, \mathbf{A}_2^*(t-1) = \mathbf{a}_2^*(t-1), \right. \\
&\quad \left. U_{A_1(t-1)} \in f_{p_{A_1^*(t-1),u}^0}^{*-1}(a_1^*(t-1)), U_{N(t-1)} \in f_{p_{N^*(t-1),u}^0}^{*-1}(n^*(t-1)) \right) \\
&\stackrel{(72)}{=} P_{U,U^*} \left(U_{A_1(t-1)} \in f_{p_{A_1^*(t-1),u}^0}^{*-1}(a_1^*(t-1)), U_{N(t-1)} \in f_{p_{N^*(t-1),u}^0}^{*-1}(n^*(t-1)), U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \right. \\
&\quad \left. P_{A_2^*(t-1)}^0 = p_{A_2^*(t-1)}^0, \mathbf{U}_{A_2(t-1)} \in \mathbf{f}_{p_{A_2^*(t-1),u}^0}^{*-1}(\mathbf{a}_2^*(t-1)), A_2^*(t-1) = a_2^*(t-1) \right) / \\
&\quad P_{U,U^*} \left(U_{A_1(t-1)} \in f_{p_{A_1^*(t-1),u}^0}^{*-1}(a_1^*(t-1)), U_{N(t-1)} \in f_{p_{N^*(t-1),u}^0}^{*-1}(n^*(t-1)) \middle| P_{A_2^*(t-1)}^0 = p_{A_2^*(t-1)}^0, \right. \\
&\quad \left. \mathbf{U}_{A_2(t-1)} \in \mathbf{f}_{p_{A_2^*(t-1),u}^0}^{*-1}(\mathbf{a}_2^*(t-1)), A_2^*(t-1) = a_2^*(t-1) \right)
\end{aligned}$$

$$\begin{aligned}
\psi &\stackrel{(69)}{=} P_{U,U^*} \left(U_{A_1(t-1)} \in f_{p_{A_1^*(t-1)}^0}^{*-1}(a_1^*(t-1)), U_{N(t-1)} \in f_{p_{N^*(t-1)}^0}^{*-1}(n^*(t-1)), U_{L_0(t)} \in \right. \\
&\quad \left. f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{A_2^*(t-1)}^0 = P_{A_2^*(t-1)}^0, U_{A_2(t-1)} \in f_{p_{A_2^*(t-1)}^0}^{*-1}(a_2^*(t-1)) \right) / \\
&\quad P_{U,U^*} \left(U_{A_1(t-1)} \in f_{p_{A_1^*(t-1)}^0}^{*-1}(a_1^*(t-1)), U_{N(t-1)} \in f_{p_{N^*(t-1)}^0}^{*-1}(n^*(t-1)) \middle| \right. \\
&\quad \left. P_{A_2^*(t-1)}^0 = P_{A_2^*(t-1)}^0, U_{A_2(t-1)} \in f_{p_{A_2^*(t-1)}^0}^{*-1}(a_2^*(t-1)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{A_2^*(t-1)}^0 = P_{A_2^*(t-1)}^0, U_{A_2(t-1)} \in \right. \\
&\quad \left. f_{p_{A_2^*(t-1)}^0}^{*-1}(a_2^*(t-1)), U_{A_1(t-1)} \in f_{p_{A_1^*(t-1)}^0}^{*-1}(a_1^*(t-1)), \right. \\
&\quad \left. U_{N(t-1)} \in f_{p_{N^*(t-1)}^0}^{*-1}(n^*(t-1)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{L_0^*(t-1)}^0 = P_{L_0^*(t-1)}^0, L_0^*(t-1) = l_0^*(t-1), \right. \\
&\quad \left. U_{A_2(t-1)} \in f_{p_{A_2^*(t-1)}^0}^{*-1}(a_2^*(t-1)), U_{A_1(t-1)} \in f_{p_{A_1^*(t-1)}^0}^{*-1}(a_1^*(t-1)), U_{N(t-1)} \in \right. \\
&\quad \left. f_{p_{N^*(t-1)}^0}^{*-1}(n^*(t-1)) \right) \\
&= P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{L_0^*(t-1)}^0 = P_{L_0^*(t-1)}^0, U_{L_0(t-1)} \in f_{p_{L_0^*(t-1)}^0}^{-1}(l_0^*(t-1)), U_{A_2(t-1)} \right. \\
&\quad \left. \in f_{p_{A_2^*(t-1)}^0}^{*-1}(a_2^*(t-1)), U_{A_1(t-1)} \in f_{p_{A_1^*(t-1)}^0}^{*-1}(a_1^*(t-1)), U_{N(t-1)} \in f_{p_{N^*(t-1)}^0}^{*-1}(n^*(t-1)) \right).
\end{aligned}$$

The last equality corresponds to equality (A) at $j = t - 2$. We have just shown the proposition that equality (A) holds at $j = t - 2$ when the conditional independences (69), (70), and (71) hold.

By induction, the previous two propositions imply that equality (73) holds when the conditional independences (69), (70), and (71) hold (because $P_{Y^*(0)}^0$ is nil).

Below, we now show that for any given time point $t = 0, \dots, K + 1$ and any given possible realization $(\bar{l}_0^*(K+1), \bar{a}_2^*(K+1), \bar{a}_1^*(K), \bar{n}^*(K))$ of the counterfactual processes defined in Section 4.2, 4.3, 4.4, or 4.5, the conditional independence (74) implies the following proposition for any $j = t - 2, \dots, 0$:

$$F(j)(C) \Rightarrow F(j-1)(D),$$

where $F(j)$ denotes the following equality:

$$\psi = P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \overline{\mathcal{U}}(0, j), \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1}(l_0^*(k)) \right)_{k=j+1, \dots, t-1} \right)$$

$$\text{with } \psi \equiv P_{U, U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| P_{L_0^*(t)}^0 = p_{L_0^*(t)}^0 \right) \text{ and}$$

$$\overline{\mathcal{U}}(0, j) \equiv \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1}(l_0^*(k)), U_{A_2(k)} \in f_{p_{A_2^*(k)}^{*-1}, u}^{-1}(a_2^*(k)), U_{A_1(k)} \in f_{p_{A_1^*(k)}^{*-1}, u}^{-1}(a_1^*(k)), U_{N(k)} \in f_{p_{N^*(k)}^{*-1}, u}^{-1}(n^*(k)) \right)$$

Indeed, we have for any $j = t - 2, \dots, 0$:

$$\begin{aligned} \psi &\stackrel{(C)}{=} P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \overline{\mathcal{U}}(0, j), \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1}(l_0^*(k)) \right)_{k=j+1, \dots, t-1} \right) \\ &= P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1}(l_0^*(k)) \right)_{k=j+1, \dots, t-1} \middle| \overline{\mathcal{U}}(0, j) \right) / \\ &\quad P_U \left(\left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1}(l_0^*(k)) \right)_{k=j+1, \dots, t-1} \middle| \overline{\mathcal{U}}(0, j) \right) \\ &\stackrel{(74)}{=} P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)), \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1}(l_0^*(k)) \right)_{k=j+1, \dots, t-1} \middle| \overline{\mathcal{U}}(0, j-1), \right. \\ &\quad \left. U_{L_0(j)} \in f_{p_{L_0^*(j)}^0}^{-1}(l_0^*(j)) \right) / \\ &\quad P_U \left(\left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1}(l_0^*(k)) \right)_{k=j+1, \dots, t-1} \middle| \overline{\mathcal{U}}(0, j-1), U_{L_0(j)} \in f_{p_{L_0^*(j)}^0}^{-1}(l_0^*(j)) \right) \\ &= P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1}(l_0^*(t)) \middle| \overline{\mathcal{U}}(0, j-1), \left(U_{L_0(k)} \in f_{p_{L_0^*(k)}^0}^{-1}(l_0^*(k)) \right)_{k=j, \dots, t-1} \right). \end{aligned}$$

We note that the third equality above indeed holds under the conditional independence assumption (74) because, when the conditional probability ψ is defined, the sets

$f_{p_{A_2^*(k)}^0, u}^{*-1}(a_2^*(k))$ and $f_{p_{A_1^*(k)}^0, u}^{*-1}(a_1^*(k))$ are then equal to the support of $U_{A_2(k)}$ and $U_{A_1(k)}$, respectively. The last equality corresponds to equality (D). Thus, we have just shown $R(j) \Rightarrow R(j-1)$.

Above, we have shown the proposition that, for any $j = t - 2, \dots, 0$, if equality (C) holds then it implies equality (D) when the conditional independence (74) holds. We now show the proposition that equality (C) holds at $j = t - 2$ when the conditional independences (69), (70), (71), and (74) hold:

$$\begin{aligned}
\psi &\equiv P_{U,U^*} \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \mid P_{L_0^*(t)}^0 = p_{L_0^*(t)}^0 \right) \\
&\stackrel{(73)}{=} P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \mid \overline{\mathcal{W}}(\mathbf{0}, \mathbf{t} - 1) \right) \\
&\text{(equality (73) was just shown to hold when (69), (70), (71) hold)} \\
&\stackrel{(74)}{=} P_U \left(U_{L_0(t)} \in f_{p_{L_0^*(t)}^0}^{-1} (l_0^*(t)) \mid \overline{\mathcal{W}}(\mathbf{0}, \mathbf{t} - 2), \mathbf{U}_{L_0(\mathbf{t}-1)} \in \mathbf{f}_{p_{L_0^*(\mathbf{t}-1)}^0}^{-1} (l_0^*(\mathbf{t}-1)) \right).
\end{aligned}$$

The last equality corresponds to equality (C) at $j = t - 2$. We have just shown the proposition that equality (C) holds at $j = t - 2$.

By induction, the previous two propositions imply that equality (75) holds when the four conditional independences (69), (70), (71), and (74) hold.

D.2 Proof of theorem 2

Proof—By definition, we have for any $t = 1, \dots, K + 1$:

$$E(Y_{d_x, g}(t)) = \sum_{y(t)} y(t) \left(\sum_{\bar{y}^{-t}(K+1), \bar{z}(K+1), \bar{i}_0(K+1), \bar{a}_2(K+1), \bar{a}_1(K), \bar{n}(K)} P(X_{d_x, g} = x) \right), \quad (76)$$

where $P(X_{d_x, g} = x)$ denotes the probability that the counterfactual data process $X_{d_x, g} = (\bar{L}_0(K+1), \bar{a}_2(K+1), \bar{a}_1(K), \bar{n}(K))$ resulting from the modified NPSEM in Section 4.4 is equal to the realization $x \equiv (\bar{l}_0(K+1), \bar{a}_2(K+1), \bar{a}_1(K), \bar{n}(K))$ and where we use the notation $\bar{y}^{-t}(K+1) = (\bar{y}(t-1), \bar{y}(t+1), \dots, \bar{y}(K+1))$ and $l_0(j) \equiv (y(j), z(j), i_0(j))$ for $j = 0, \dots, K + 1$.

The probability $P(X_{d_x, g} = x)$ is defined as follows based on the distribution of the exogenous variables P_{U, U^*} and the functions f and f^* defined in Section 4.4:

$$\begin{aligned}
P(\mathbf{X}_{d_x, g} = x) &\equiv P_{U, U^*} \left(\left(\mathbf{U}_{Y(t)} \in f_{p_{y(t)}^0}^{-1} (y(t)), \mathbf{U}_{Z(t)} \in f_{p_{z(t)}^0}^{-1} (z(t)), \mathbf{U}_{I_0(t)} \in f_{p_{i_0(t)}^0}^{-1} (i_0(t)) \right)_{t=0, \dots, K+1}, \right. \\
&\mathbf{U}_{A_2(K+1)} \in f_{p_{a_2(K+1)}^0}^{-1} (a_2(K+1)), \left((\mathbf{U}_{A_2(t)}, \mathbf{U}_{A_2^*(t)}) \in f_{p_{a_2(t)}^0}^{*-1} (a_2(t)), \right. \\
&\left. \left. (\mathbf{U}_{A_1(t)}, \mathbf{U}_{A_1^*(t)}) \in f_{p_{a_1(t)}^0}^{*-1} (a_1(t)), (\mathbf{U}_{N(t)}, \mathbf{U}_{N^*(t)}) \in f_{p_{n(t)}^0}^{*-1} (n(t)) \right)_{t=0, \dots, K} \right),
\end{aligned}$$

where bold font is used to highlight the random variables and where we use the generic notation $p_{x(t)}^0$ defined as the realization $p_{X(t)}^0$ of the parents of variable $X(t)$ defined in Section 3 corresponding with x if x were a realization of the process $X = (\bar{L}_0(K+1), \bar{a}_2(K+1), \bar{a}_1(K), \bar{n}(K))$ defined by the unmodified NPSEM. For example, we have

$p_{y(t)}^0 \equiv (\bar{l}_0(t-1), \bar{a}_2(t-1), \bar{a}_1(t-1), \bar{n}(t-1))$. We recall that the generic definition of the set $f_{p_{x(t)}^0}^{-1}$ (resp. $f_{p_{x(t)}^0}^{*-1}$) for a given non-action (resp. action) variable $X(t)$ is given in Appendix A (resp. D.1).

By factorization using the chain rule applied according to the temporal ordering of the variables, the previous expression can be rewritten as follows:

$$P(X_{d_x, g} = x) = \prod_{t=0}^{K+1} Q_t(y(t), z(t), i_0(t) | p_{y(t)}^0) \prod_{t=0}^K g_t(a_2(t), a_1(t), n(t) | p_{a_2(t)}^0) g_{K+1}(a_2(K+1) | p_{a_2(K+1)}^0), \quad (77)$$

where the factors Q_t and g_t are defined recursively for $t = 0, \dots, K+1$ in this order as follows:

- if either $t = 0$ or if both $t > 0$ and

$$\prod_{j=0}^{t-1} Q_j(y(j), z(j), i_0(j) | p_{y(j)}^0) g_j(a_2(j), a_1(j), n(j) | p_{a_2(j)}^0) > 0,$$

$$\begin{aligned} Q_t(y(t), z(t), i_0(t) | p_{y(t)}^0) &\equiv P_{U, U^*} \left(U_{Y(t)} \in f_{p_{y(t)}^0}^{-1}(y(t)), U_{Z(t)} \in f_{p_{z(t)}^0}^{-1}(z(t)), U_{I_0(t)} \in \right. \\ &\quad \left. f_{p_{i_0(t)}^0}^{-1}(i_0(t)) | P_{Y_{d_x, g}(t)}^0 = p_{y(t)}^0 \right) \\ &= P(L_{0, d_x, g}(t) = l_0(t) | P_{Y_{d_x, g}(t)}^0 = p_{y(t)}^0), \end{aligned}$$

and otherwise:

$$\begin{aligned} Q_t(y(t), z(t), i_0(t) | p_{y(t)}^0) &\equiv P_{U, U^*} \left(U_{Y(t)} \in f_{p_{y(t)}^0}^{-1}(y(t)), U_{Z(t)} \in f_{p_{z(t)}^0}^{-1}(z(t)), \right. \\ &\quad \left. U_{I_0(t)} \in f_{p_{i_0(t)}^0}^{-1}(i_0(t)) \right) \\ &\neq P(L_{0, d_x, g}(t) = l_0(t)), \end{aligned}$$

by convention (the previous conditional probability is then undefined). We note that this choice of convention has no impact on the definition of $P(X_{d_x, g} = x)$ since $P(X_{d_x, g} = x) = 0$ whenever at least one of the terms Q_t is not defined by the conditional probability.

- if $\prod_{j=0}^t Q_j(y(j), z(j), i_0(j) | p_{y(j)}^0) \prod_{j=0}^{t-1} g_j(a_2(j), a_1(j), n(j) | p_{a_2(j)}^0) > 0$.

$$\begin{aligned}
g_t \left(a_2(t), a_1(t), n(t) | p_{a_2(t)}^0 \right) &\equiv P_{U,U^*} \left((U_{A_2(t)}, U_{A_2(t)}^*) \in f_{p_{a_2(t)}^0}^{*-1} (a_2(t)), (U_{A_1(t)}, U_{A_1(t)}^*) \in \right. \\
&\quad \left. f_{p_{a_1(t)}^0}^{*-1} (a_1(t)), (U_{N(t)}, U_{N(t)}^*) \in f_{p_{n(t)}^0}^{*-1} (n(t)) | P_{A_{2d_{x,g}(t)}}^0 = p_{a_2(t)}^0 \right) \\
&= P(A_{2d_{x,g}(t)} = a_2(t), A_{1d_{x,g}(t)} = a_1(t), N_{d_{x,g}(t)} = n(t) | \\
&\quad P_{A_{2d_{x,g}(t)}}^0 = p_{a_2(t)}^0),
\end{aligned}$$

and otherwise

$$g_t \left(a_2(t), a_1(t), n(t) | p_{a_2(t)}^0 \right) \equiv g_t \left(a_2(t) | p_{a_2(t)}^0 \right) g_t \left(a_1(t) | p_{a_1(t)}^0 \right) g_t \left(n(t) | p_{n(t)}^0 \right)$$

with:

$$\begin{aligned}
g_t \left(a_2(t) | p_{a_2(t)}^0 \right) &\equiv P_{U,U^*} \left((U_{A_2(t)}, U_{A_2(t)}^*) \in f_{p_{a_2(t)}^0}^{*-1} (a_2(t)) \right) \\
&\neq P(A_{2d_{x,g}(t)} = a_2(t))
\end{aligned}$$

$$\begin{aligned}
g_t \left(a_1(t) | p_{a_1(t)}^0 \right) &\equiv P_{U,U^*} \left((U_{A_1(t)}, U_{A_1(t)}^*) \in f_{p_{a_1(t)}^0}^{*-1} (a_1(t)) \right) \\
&\neq P(A_{1d_{x,g}(t)} = a_1(t))
\end{aligned}$$

$$\begin{aligned}
g_t \left(n(t) | p_{n(t)}^0 \right) &\equiv P_{U,U^*} \left((U_{N(t)}, U_{N(t)}^*) \in f_{p_{n(t)}^0}^{*-1} (n(t)) \right) \\
&\neq P(N_{d_{x,g}(t)} = n(t)),
\end{aligned}$$

by convention (the previous conditional probability is then undefined). We note again that this choice of convention has no impact on the definition of $P(X_{d_{x,g}} = x)$.

$$\bullet \text{ if } \prod_{j=0}^{K+1} Q_j \left(y(j), z(j), i_0(j) | p_{y(j)}^0 \right) \prod_{j=0}^K g_j \left(a_2(j), a_1(j), n(j) | p_{a_2(j)}^0 \right) > 0:$$

$$\begin{aligned}
g_{K+1} \left(a_2(K+1) | p_{a_2(K+1)}^0 \right) &\equiv P_{U,U^*} \left(U_{A_2(K+1)} \in f_{p_{a_2(K+1)}^0}^{-1} (a_2(K+1)) | \right. \\
&\quad \left. P_{A_{2d_{x,g}(K+1)}}^0 = p_{a_2(K+1)}^0 \right) \\
&\neq P(A_{2d_{x,g}(K+1)} = a_2(K+1) | P_{A_{2d_{x,g}(K+1)}}^0 = p_{a_2(K+1)}^0)
\end{aligned}$$

and otherwise:

$$g_{K+1} \left(a_2(K+1) | p_{a_2(K+1)}^0 \right) \equiv P_{U,U^*} \left(U_{A_2(K+1)} \in f_{p_{a_2(K+1)}^0}^{-1} (a_2(K+1)) \right) \\ \neq P(A_{2_{dx,g}}(K+1) = a_2(K+1))$$

by convention (the previous conditional probability is then undefined). In addition, from the definition of $f_{A_2(K+1)}$ and $f_{A_2(K+1)}^*$ given in Sections 3 and 4.4, we have

$$g_{K+1} \left(a_2(K+1) | p_{a_2(K+1)}^0 \right) = I(a_2(K+1) = 1 - y(K+1)). \quad (78)$$

From now on, to simplify notation, we drop the subscript t from the terms g_t and Q_t defined above. Thus, each reference to the functions g and Q below is an implicit reference to one of the functions g_t and Q_t above and we rely on the names of the function arguments to indicate to the reader which function g_t and Q_t is actually referenced. Using this new notation, we note that each factor g above for $t = 0, \dots, K$ can be further factorized as follows when it is defined based on the conditional probability (i.e., if

$$\prod_{j=0}^t Q(y(j), z(j), i_0(j) | p_{y(j)}^0) \prod_{j=0}^{t-1} g(a_2(j), a_1(j), n(j) | p_{a_2(j)}^0) > 0;$$

$$g(a_2(t), a_1(t), n(t) | p_{a_2(t)}^0) = g(a_2(t) | p_{a_2(t)}^0) g(a_1(t) | p_{a_1(t)}^0) g(n(t) | p_{n(t)}^0),$$

where each of these three terms are defined as follows

- $$g(a_2(t) | p_{a_2(t)}^0) \equiv P_{U,U^*} \left((U_{A_2(t)}, U_{A_2(t)}^*) \in f_{p_{a_2(t)}^0}^{*-1} (a_2(t)) | P_{A_{2_{dx,g}}(t)}^0 = p_{a_2(t)}^0 \right) \\ = P(A_{2_{dx,g}}(t) = a_2(t) | P_{A_{2_{dx,g}}(t)}^0 = p_{a_2(t)}^0)$$
- if $g(a_2(t) | p_{a_2(t)}^0) > 0$:

$$g(a_1(t) | p_{a_1(t)}^0) \equiv P_{U,U^*} \left((U_{A_1(t)}, U_{A_1(t)}^*) \in f_{p_{a_1(t)}^0}^{*-1} (a_1(t)) | P_{A_{1_{dx,g}}(t)}^0 = p_{a_1(t)}^0 \right) \\ = P(A_{1_{dx,g}}(t) = a_1(t) | P_{A_{1_{dx,g}}(t)}^0 = p_{a_1(t)}^0)$$

and otherwise:

$$g(a_1(t) | p_{a_1(t)}^0) \equiv P_{U,U^*} \left((U_{A_1(t)}, U_{A_1(t)}^*) \in f_{p_{a_1(t)}^0}^{*-1} (a_1(t)) \right) \\ \neq P(A_{1_{dx,g}}(t) = a_1(t))$$

by convention (the previous conditional probability is then undefined).

- if $g(a_2(t)|p_{a_2(t)}^0)g(a_1(t)|p_{a_1(t)}^0) > 0$:

$$\begin{aligned} g(n(t)|p_{n(t)}^0) &\equiv P_{U,U^*} \left((U_{N(t)}, U_{N(t)}^*) \in f_{p_{n(t)}^0}^{*-1}(n(t)) | P_{N_{dx,g}(t)}^0 = p_{n(t)}^0 \right) \\ &= P(N_{dx,g}(t) = n(t) | P_{N_{dx,g}(t)}^0 = p_{n(t)}^0) \end{aligned}$$

and otherwise:

$$\begin{aligned} g(n(t)|p_{n(t)}^0) &\equiv P_{U,U^*} \left((U_{N(t)}, U_{N(t)}^*) \in f_{p_{n(t)}^0}^{*-1}(n(t)) \right) \\ &\neq P(N_{dx,g}(t) = n(t)), \end{aligned}$$

by convention (the previous conditional probability is then undefined).

From the definition of f^* in Section 4.4, we have for $t = 0, \dots, K$:

$$g(a_2(t)|p_{a_2(t)}^0) = I(a_2(t) = 0) \quad (79)$$

$$g(a_1(t)|p_{a_1(t)}^0) = I(a_1(t) = a_1(t-1))^{y(t)} I(a_1(t) = d_x(t)(p_{a_1(t)}))^{1-y(t)} \quad (80)$$

where we use the generic notation $p_{x(t)}(\neq p_{x(t)}^0)$ defined as the realization $p_{X(t)}$ of the observed parents of variable $X(t)$ defined in Section 3 corresponding with x if x were a realization of the process $X = (\bar{L}_0(K+1), \bar{L}_1(K+1), \bar{L}_1(K), \bar{N}(K))$ defined by the unmodified NPSEM. For example, we have $p_{a_1(t)} \equiv (\bar{L}(t), \bar{L}_1(t), \bar{L}_1(t-1), \bar{n}(t-1))$ with $\bar{L}(k) \equiv (y(k), z(k), \bar{x}(k))$ and $\bar{x}(k) \equiv \bar{n}(k-1)\bar{x}_0(k)$ for $k = 0, \dots, t-1$. We note that equalities (79) and (80) hold even when $g(a_2(t)|p_{a_2(t)}^0)$ and $g(a_1(t)|p_{a_1(t)}^0)$ are defined using the convention introduced earlier.

In addition, from the definition of f^* in Section 4.4, we have

$g(n(t)|p_{n(t)}^0) = I(n(t) = n(t-1))$ when $y(t) = 1$. When $y(t) = 0$ and if either the function $h_{N(t)}^*$ is not dependent on the counterfactual monitoring variable, $f_{N(t)}(P_{N(t)}^0, U_{N(t)})$, or the function $f_{N(t)}$ is not dependent on the variables $p_0(t)$, then the definition of $f_{N(t)}^*$ in Section 4.4 implies that $g(n(t)|p_{n(t)}^0)$ (even when defined by convention) is only a function of the observed past $p_{N(t)}$. More specifically, we have:

$$P(N_{dx,g}(t) = n(t) | P_{N_{dx,g}(t)}^0 = p_{n(t)}^0) = P(N_{dx,g}(t) = n(t) | P_{N_{dx,g}(t)} = p_{n(t)})$$

$$P_{U,U^*} \left((U_{N(t)}, U_{N(t)}^*) \in f_{p_{n(t)}^0}^{*-1}(n(t)) \right) = P_{U,U^*} \left((U_{N(t)}, U_{N(t)}^*) \in f_{p_{n(t)}^{0'}}^{*-1}(n(t)) \right)$$

where $p_{n(t)}^{0'} \equiv (\bar{y}(t), \bar{z}(t), \bar{l}_0(t), \bar{a}_2(t), \bar{a}_1(t), \bar{n}(t-1))$ for any $\bar{l}_0(t)$ such that $i_0'(k) = i_0(k)$ if $n(k-1) = 1$. Thus, from now on, we can also denote $g(n(t)|p_{n(t)}^0)$ by $g(n(t) | p_{n(t)})$ whether $y(t) = 0$ or $y(t) = 1$ and we have:

$$g(n(t)|p_{n(t)}) = g(n(t)|p_{n(t)}) I(n(t) = n(t-1))^{y(t)}, \quad (81)$$

because $I(n(t) = n(t-1))^2 = I(n(t) = n(t-1))$.

Futhermore, from lemma 3, the following equality holds under conditional independences (69), (70), and (71):

$$k=0, \dots, t-1 \quad (82)$$

The use of bold font in this expression makes explicit that these probabilities are invariant to the choice of conditional distribution $P_{U^*|U}$. We note however that these probabilities remain dependent on the choice of functions f^* , i.e., the definition of the intervention regimen on the action variables $A_2(t)$, $A_1(t)$, and $N(t)$. We recall that the factors $Q(y(t), z(t), i_0(t)|p_{y(t)}^0)$ that define the probability $P(X_{d_{x,g}} = x)$ are defined by the probabilities above. To clearly indicate the dependence of these factors on the choice of functions f^* , we now use the notation $Q_{f_{d_{x,g}}^*}(l_0(t)|p_{y(t)}^0)$ for the factors $Q(y(t), z(t), i_0(t)|p_{y(t)}^0)$.

From the results (78), (79), (80), (81), and (82) derived above, the expression (77) of the probability $P(X_{d_{x,g}} = x)$ becomes

$$P(X_{d_{x,g}} = x) = I(a_2(K+1) = 1 - y(K+1)) \prod_{t=0}^{K+1} Q_{f_{d_{x,g}}^*}(l_0(t)|p_{y(t)}^0) \prod_{t=0}^K I(a_2(t) = 0) \times \\ I(a_1(t) = a_1(t-1))^{y(t)} I(a_1(t) = d_x(t)(p_{a_1(t)}))^{1-y(t)} g(n(t)|p_{n(t)}), \quad (83)$$

and we obtain:

$$\sum_{\bar{y}^{-t}(K+1), \bar{z}(K+1), \bar{l}_0(K+1), \bar{a}_2(K+1), \bar{a}_1(K), \bar{n}(K)} P(X_{d_{x,g}} = x) = \sum_{\bar{l}_0(t-1), z(t), i_0(t), \bar{a}_2(t-1), \bar{a}_1(t-1), \bar{n}(t-1)} \\ \prod_{j=0}^t Q_{f_{d_{x,g}}^*}(l_0(j)|p_{y(j)}^0) \prod_{j=0}^{t-1} I(a_2(j) = 0) \times \\ I(a_1(j) = a_1(j-1))^{y(j)} I(a_1(j) = d_x(j)(p_{a_1(j)}))^{1-y(j)} g(n(j)|p_{n(j)}).$$

As a result, equality (76) becomes:

$$E(Y_{d_x,g}(t)) = \sum_{\bar{l}_0(t), \bar{a}_2(t-1), \bar{a}_1(t-1), \bar{n}(t-1)} y(t) \prod_{j=0}^t Q_{f_{d_x,g}}^*(l_0(j)|p_{y(j)}^0) \prod_{j=0}^{t-1} I(a_2(j)=0) \times \\ I(a_1(j)=a_1(j-1))^{y(j)} I(a_1(j)=d_x(j)(p_{a_1(j)}))^{1-y(j)} g(n(j)|p_{n(j)}).$$

Because for any given $\bar{n}(t-1)$, $\sum_{\bar{n}^*(t-1)} I(\bar{n}(t-1) = \bar{n}^*(t-1)) = 1$, we have:

$$E(Y_{d_x,g}(t)) = \sum_{\bar{n}(t-1)} \sum_{\bar{l}_0(t), \bar{a}_2(t-1), \bar{a}_1(t-1)} \sum_{\bar{n}^*(t-1)} y(t) \prod_{j=0}^t Q_{f_{d_x,g}}^*(l_0(j)|p_{y(j)}^0) \prod_{j=0}^{t-1} I(a_2(j)=0) \times \\ I(a_1(j)=a_1(j-1))^{y(j)} I(a_1(j)=d_x(j)(p_{a_1(j)}))^{1-y(j)} g(n(j)|p_{n(j)}) \times \\ I(n(j)=n^*(j))^{y(j)} I(n(j)=n^*(j))^{1-y(j)} \\ E(Y_{d_x,g}(t)) \stackrel{(81)}{=} \sum_{\bar{n}^*(t-1)} \sum_{\bar{n}(t-1)} \sum_{\bar{l}_0(t), \bar{a}_2(t-1), \bar{a}_1(t-1)} y(t) \prod_{j=0}^t Q_{f_{d_x,g}}^*(l_0(j)|p_{y(j)}^0) \prod_{j=0}^{t-1} I(a_2(j)=0) \times \\ I(a_1(j)=a_1(j-1))^{y(j)} I(a_1(j)=d_x(j)(p_{a_1(j)}))^{1-y(j)} g(n(j)|p_{n(j)}) \times \\ I(n(j)=n(j-1))^{y(j)} I(n(j)=n^*(j))^{y(j)} I(n(j)=n^*(j))^{1-y(j)} \\ = \sum_{\bar{n}^*(t-1)} \sum_{\bar{l}_0(t), \bar{a}_2(t-1), \bar{a}_1(t-1), \bar{n}(t-1)} \left(y(t) \prod_{j=0}^{t-1} g(n(j)|p_{n(j)}) I(n(j)=n^*(j))^{y(j)} \right) \times \\ \prod_{j=0}^t Q_{f_{d_x,g}}^*(l_0(j)|p_{y(j)}^0) \prod_{j=0}^{t-1} I(a_2(j)=0) I(a_1(j)=a_1(j-1))^{y(j)} \times \\ I(a_1(j)=d_x(j)(p_{a_1(j)}))^{1-y(j)} I(n(j)=n(j-1))^{y(j)} I(n(j)=n^*(j))^{1-y(j)}.$$

(84)

We now note that, with the particular choice of stochastic monitoring intervention function

$$h_{N(t)}^* \left(P_{N^*(t)}, U_{N(t)}^*, f_{N(t)}(P_{N^*(t)}^0, U_{N(t)}) \right) \equiv n^*(t)$$

in Section 4.4, we have

$$g(n(t)|p_{n(t)}) = I(n(t)=n(t-1))^{y(t)} I(n(t)=n^*(t))^{1-y(t)}$$

and the modified NPSEM in Section 4.4 becomes the same as the modified NPSEM in Section 4.3. Thus, the probability that the counterfactual data process $X_{d_x, \bar{n}^*} = (\bar{L}_{0, d_x, \bar{n}^*}(K$

+ 1), $2_{d_x, \bar{n}^*}(K+1)$, $1_{d_x, \bar{n}^*}(K)$, $\bar{N}_{d_x, \bar{n}^*}(K)$ resulting from the modified NPSEM in Section 4.3 is equal to $\times$ (denoted by $P(X_{d_x, \bar{n}^*} = x)$) can directly be derived from equality (83) as follows:

$$P(X_{d_x, \bar{n}^*} = x) = I(a_2(K+1) = 1 - y(K+1)) \prod_{t=0}^{K+1} Q_{f_{d_x, \bar{n}^*}}^*(l_0(t) | p_{y(t)}^0) \prod_{t=0}^K I(a_2(t) = 0) \times \\ I(a_1(t) = a_1(t-1))^{y(t)} I(a_1(t) = d_x(t)(p_{a_1(t)}))^{1-y(t)} I(n(t) = n(t-1))^{y(t)} I(n(t) = n^*(t))^{1-y(t)}.$$

(85)

In addition, if the functions $h_{N(t)}^*$ are not dependent on the counterfactual monitoring variables, $f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})$, then the functions f^* in Section 4.3 that define $Q_{f_{d_x, \bar{n}^*}}^*$ and the functions f^* in Section 4.4 that define $Q_{f_{d_x, g}}^*$ become independent of the exogenous variables $U_{A_2(t)}$, $U_{A_1(t)}$, and $U_{N(t)}$. As a result, the right-hand side of equality (82) which defines both the factors $Q_{f_{d_x, \bar{n}^*}}^*(l_0(t) | p_{y(t)}^0)$ and $Q_{f_{d_x, g}}^*(l_0(t) | p_{y(t)}^0)$ no longer depends on the choice of functions f^* :

$$= P_U \left(\bigcup_{k=0, \dots, t-1} \left(U_{Y(k)} \in f_{p_{y(k)}}^{-1}(y(k)), U_{Z(k)} \in f_{p_{z(k)}}^{-1}(z(k)), U_{I_0(k)} \in f_{p_{i_0(k)}}^{-1}(i_0(k)) \right) \right) \left(U_{Y(k)} \in f_{p_{y(k)}}^{-1}(y(k)), U_{Z(k)} \in f_{p_{z(k)}}^{-1}(z(k)), U_{I_0(k)} \in f_{p_{i_0(k)}}^{-1}(i_0(k)) \right)$$

because, when these conditional probabilities are defined, the sets

$f_{p_{a_2(k)}, u}^{*-1}(a_2(k))$, $f_{p_{a_1(k)}, u}^{*-1}(a_1(k))$, and $f_{p_{n(k)}, u}^{*-1}(n(k))$ are then equal to the support of $U_{A_2(k)}$, $U_{A_1(k)}$, and $U_{N(k)}$, respectively. We note that, from the second result of lemma 3, the previous equality also holds when the functions $h_{N(t)}^*$ are dependent on the counterfactual monitoring variables, $f_{N(t)}(P_{N^*(t)}^0, U_{N(t)})$, as long as the conditional independences (74) hold.

Therefore, when the previous equality holds, we have $Q_{f_{d_x, \bar{n}^*}}^*(l_0(t) | p_{y(t)}^0) = Q_{f_{d_x, g}}^*(l_0(t) | p_{y(t)}^0)$ and using equality (85), we further obtain:

$$\sum_{\bar{l}_0(t+1, K+1), \bar{a}_2(t, K+1), \bar{a}_1(t, K), \bar{n}(t, K)} P(X_{d_x, \bar{n}^*} = x) \prod_{j=0}^t Q_{f_{d_x, g}}^*(l_0(j) | p_{y(j)}^0) \prod_{j=0}^{t-1} I(a_2(j) = 0) \times \\ I(a_1(j) = a_1(j-1))^{y(j)} I(a_1(j) = d_x(j)(p_{a_1(j)}))^{1-y(j)} I(n(j) = n(j-1))^{y(j)} I(n(j) = n^*(j))^{1-y(j)}$$

where we use the generic notation $\bar{X}(j, j') = (X(j), \dots, X(j'))$ for $j < j'$ ($\bar{X}(j, j')$ is nil if $j > j'$). Thus, equality (84) can be expressed as:

$$E(Y_{d_x,g}(t)) = \sum_{\bar{n}^*(t-1)} \left(\sum_{\bar{l}_0(t), \bar{a}_2(t-1), \bar{a}_1(t-1), \bar{n}(t-1)} \left(y(t) \prod_{j=0}^{t-1} g(n(j)|P_{n(j)}) I(n(j)=n^*(j))^{y(j)} \right) \times \right. \\ \left. \left(\sum_{\bar{l}_0(t+1, K+1), \bar{a}_2(t, K+1), \bar{a}_1(t, K), \bar{n}(t, K)} P(X_{d_x, \bar{n}^*} = x) \right) \right),$$

and can be further simplified as:

$$E(Y_{d_x,g}(t)) = \sum_{\bar{n}^*(t-1)} E \left(Y_{d_x, \bar{n}^*}(t) \prod_{j=0}^{t-1} g(N_{d_x, \bar{n}^*}(j)|P_{N_{d_x, \bar{n}^*}(j)}) I(N_{d_x, \bar{n}^*}(j)=n^*(j))^{Y_{d_x, \bar{n}^*}(j)} \right).$$

From theorem 1 and after expanding $P_{N_{d_x, \bar{n}^*}(j)}$ in the previous equality, we obtain the following equality when the (weak) NDE assumption holds:

$$E(Y_{d_x,g}(t)) = \sum_{\bar{n}^*(t-1)} E \left(Y_{d_x, g^*}(t) \prod_{j=0}^{t-1} g(N'_{d_x, g^*}(j)|\bar{L}'_{d_x, g^*}(j), \bar{A}_{2, d_x, g^*}(j), \bar{A}_{1, d_x, g^*}(j), \bar{N}'_{d_x, g^*}(j-1)) \right. \\ \left. I(N'_{d_x, g^*}(j)=n^*(j))^{Y_{d_x, g^*}(j)} \right),$$

where we use

$$L'_{d_x, g^*}(j) \equiv (Y_{d_x, g^*}(j), Z_{d_x, g^*}(j), I'_{d_x, g^*}(j)), I'_{d_x, g^*}(j) \equiv Y_{d_x, g^*}(j)I'_{d_x, g^*}(j-1) + (1 - Y_{d_x, g^*}(j))n^*(j-1)I_{d_x, g^*}(j) \\ , \text{ and } N'_{d_x, g^*}(j) \equiv Y_{d_x, g^*}(j)N'_{d_x, g^*}(j-1) + (1 - Y_{d_x, g^*}(j))n^*(j)N_{d_x, g^*}(j).$$

Figure 1.

Example of a Directed Acyclic Graph consistent with the general NPSEM introduced in Section 3. This graph assumes, in particular, that all disturbances are mutually independent, that the nodes $Y(t)$ and $Z(t)$ share the same parents (with $t = 1, 2$), and that right-censoring is completely at random. The exclusion of the arrows represented with dashed lines in this diagram graphically encodes the upholding of the NDE assumption introduced in Section 5.