Published in final edited form as:

Stat Probab Lett. 2017 April ; 123: 122–127. doi:10.1016/j.spl.2016.12.003.

A New Variance Component Score Test for Testing Distributed Lag Functions with Applications in Time Series Analysis

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Abstract

We propose to test a given constrained distributed lag model (DLM) of the form $\beta = C\theta$ against an unconstrained alternative using a variance component score test (VCST) and show that VCST is more powerful than the standard likelihood ratio test in a simulation study.

1 Introduction

Distributed lag models (DLMs), first introduced in the econometrics literature, are often used to model the current value of a response variable at time t in association with both the current value and the lagged values of an independent variable in a time series analysis. For example, environmental epidemiologists model the current day mortality counts in association with daily air pollution related exposure, such as PM_{10} , up to several days prior to the event day (Schwartz, 2000; Welty et al., 2009). Economists study the long-term effects of macroeconomic variables on stock returns using distributed lag models (Majid and Yusof, 2009; Hsu, 2015). Unconstrained DLMs entail the potential problem of multicollinearity among the various lagged values of the independent variable and the number of parameters to be estimated can be large. Constrained DLMs assume some functional relationship between lag coefficients and lag indices (in the form of a distributed lag function) and serve as a potential solution to the problem. Common constraints include a polynomial (Almon, 1965), a spline (Corradi, 1977), and a natural cubic spline (Hastie and Tibshirani, 1993). There exists extensive literature on characterization of the distributed lag function and inference, assuming the constraints are correctly specified. Very few strategies exist for testing given distributed lag (DL) constraints. In this letter, we propose a new simple and efficient framework for testing a constrained DLM against an unconstrained DLM. In Section 2, we briefly introduce DLMs and present the proposed variance component score test (VCST) procedure. In Section 3, we conduct a simulation study to compare the statistical power of the standard likelihood ratio test (LRT) and VCST and illustrate both of the approaches using the National Mortality, Morbidity, and Air Pollution Study (NMMAPS) data. We conclude with discussions in Section 4.

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2 Method

Let x_t denote the independent variable measured at time t , y_t denote the response variable measured at time t , z_t denote the other covariates obtained at time t , T be the length of the time series, and L be the pre-determined maximum number of lags. Without loss of generality, we leave out intercept and covariates in the rest of the presentation and consider the generalized linear model $\eta_t = g[\mu_t] = g[E(y_t | x_t, x_{t-1}, \dots, x_{t-L})] = \mathbf{X}_t^T \boldsymbol{\beta}$ where $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_L)^T$ is the vector of the lag effects, $\mathbf{X}_t = (x_t, x_{t-1}, \dots, x_{t-L})^T$, η_t is the canonical (natural) parameter, $g(\cdot)$ is the link function, y_t is a random variable generated from a distribution $\mathcal{F}$ in canonical exponential family with probability density

$$f(y_t) = \exp \left\{ \frac{y_t \eta_t - b(\eta_t)}{a(\phi)} + c(y_t; \phi) \right\}, \quad (1)$$

ϕ is the dispersion parameter, and $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ are known functions. It is well-known that the exponential family possess the properties of $\mu_t = b'(\mathbf{X}_t^T \boldsymbol{\beta}) = g^{-1}(\mathbf{X}_t^T \boldsymbol{\beta})$ and $V(y_t) = b''(\mathbf{X}_t^T \boldsymbol{\beta}) a(\phi) = \nu(\mathbf{X}_t^T \boldsymbol{\beta}) a(\phi)$ where $\nu(\cdot)$ is the variance function.

2.1 Constrained DLM

Constrained DLM imposes a pre-specified structure to constrain the lag coefficients to be a smooth function of the lags (i.e. $\beta_\ell = f(\ell)$ for $\ell = 0, \dots, L$). Denote the p basis functions that generate the class of functions in which $\boldsymbol{\beta}$ can lie as $B_1(\cdot), \dots, B_p(\cdot)$. The transformation matrix C as defined by Gasparrini et al. (Gasparrini et al., 2010) is given by

$$\mathbf{C} = \begin{bmatrix} B_1(0) & B_2(0) & \cdots & B_p(0) \\ B_1(1) & B_2(1) & \cdots & B_p(1) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(L) & B_2(L) & \cdots & B_p(L) \end{bmatrix}_{(L+1) \times p}$$

The constrained DLM estimator can be expressed in the form of $\boldsymbol{\beta} = \mathbf{C}\boldsymbol{\theta}$ where $\boldsymbol{\theta}$ is a vector of p free parameters to be estimated in $\mathbb{R}^p$. The maximum likelihood estimate of $\boldsymbol{\theta}$ is obtained as

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}} \sum_{t=1}^T \left[y_t \mathbf{X}_t^T \mathbf{C} \boldsymbol{\theta} - b(\mathbf{X}_t^T \mathbf{C} \boldsymbol{\theta}) \right].$$

The estimation of $\boldsymbol{\theta}$ does not involve the dispersion parameter ϕ . The constrained DLM estimator is given by $\hat{\boldsymbol{\beta}}_{\text{CDLM}} = \mathbf{C} \hat{\boldsymbol{\theta}}$. The $\mathbf{C}$ corresponding to an unconstrained DLM is a $(L+1) \times (L+1)$ identity matrix.

2.2 Hypothesis Testing

$\hat{\beta}_{\text{CDLM}}$ can be alternatively obtained by maximizing log-likelihood with respect to β subject to $R\beta = 0$ where R is a $(L+1-p) \times (L+1)$ constraint matrix corresponding to the transformation matrix C (Chen et al., 2016). The basis functions in C span the solution space of $R\beta = 0$. R can be obtained from C via the following procedure. Define C_e as a $(L+1) \times (L+1)$ matrix $[C \ 0_{(L+1) \times (L+1-p)}]$ where $0_{(L+1) \times (L+1-p)}$ is a $(L+1) \times (L+1-p)$ matrix with zero entries. Applying singular value decomposition (SVD) $C_e^T = U_C D_C V_C^T$ where U_C is the $(L+1) \times (L+1)$ unitary matrix with left-singular column vectors, V_C is the $(L+1) \times (L+1)$ unitary matrix with right-singular column vectors, and D_C is a $(L+1) \times (L+1)$ diagonal matrix with singular values of C_e^T along the diagonal, the constraint matrix R can then be obtained as the last $(L+1-p)$ rows of V_C^T .

Testing a particular DLM structure against an unconstrained alternative can now be formulated as testing $H_0: R\beta = 0$ against $H_1: R\beta \neq 0$. A standard likelihood ratio test (LRT), Wald test, and score test can be conducted and the test statistics asymptotically follow a χ^2 distribution with $L+1-p$ degrees of freedom and large sample inference can be obtained. We propose a VCST approach to this problem. Consider a generalized ridge regression estimator (Chen et al., 2016) that minimizes the penalized negative log-likelihood function

$$\ell_p(\beta) = [S(\mathbf{X}\beta) - \mathbf{Y}^T \mathbf{X}\beta] + \lambda \beta^T \mathbf{R}^T \mathbf{R} \beta \quad (2)$$

where $\mathbf{Y} = (y_1, \dots, y_T)^T$, $\mathbf{X}$ is a $T \times (L+1)$ matrix with $\mathbf{X}_t^T$ as the t -th row for $t = 1, \dots, T$, $S(\cdot)$ is the $\mathbf{R}^T \rightarrow \mathbf{R}^1$ cumulant function such that $S(\mathbf{a}) = \sum_{t=1}^T b(a_t)$ with $\mathbf{a} = (a_1, \dots, a_T)^T$, and λ is the tuning parameter. We can rewrite

$$\mathbf{R}^T \mathbf{R} = \mathbf{U} \mathbf{D} \mathbf{U}^T \quad (3)$$

where $\mathbf{U}$ is a $(L+1) \times (L+1)$ matrix with orthogonal columns and $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{R}^T \mathbf{R}$ using singular value decomposition (SVD). Since $\mathbf{R}^T \mathbf{R}$ is not of

full rank and has rank $L+1-p$, we can write $\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where $\mathbf{D}_1$ is a $(L+1-p) \times (L+1-p)$ diagonal matrix of full rank. Let $\mathbf{U} = [\mathbf{U}_1 \ \mathbf{U}_2]$ where $\mathbf{U}_1$ is a $(L+1) \times (L+1-p)$ matrix with columns of singular vectors corresponding to nonzero eigenvalues in $\mathbf{D}$ and $\mathbf{U}_2$ is a $(L+1) \times p$ matrix with columns of singular vectors corresponding to the eigenvalues of 0. One can rewrite the penalized negative log-likelihood as

$$\ell_p(\beta) = [S(\mathbf{X} * \beta * + \mathbf{Z} * \mathbf{u}) - \mathbf{Y}^T (\mathbf{X} * \beta * + \mathbf{Z} * \mathbf{u})] + \lambda \mathbf{u}^T \mathbf{u} \quad (4)$$

where $\mathbf{X}^* = \mathbf{X}\mathbf{U}_2$, $\boldsymbol{\beta}^* = \mathbf{U}_2^T \boldsymbol{\beta}$, $\mathbf{Z}^* = \mathbf{X}\mathbf{U}_1\mathbf{D}_1^{-1/2}$, and $\mathbf{u} = \mathbf{D}_1^{1/2}\mathbf{U}_1^T \boldsymbol{\beta}$. We rewrite (2) as (4) to divide $\mathbf{X}$ into two parts, namely $\mathbf{X}^*$ and $\mathbf{Z}^*$. The first part $\mathbf{X}^*$ represents the orthogonal projection of $\mathbf{X}$ onto the space (say, $\mathcal{W}$) spanned by the p basis functions. The second part $\mathbf{Z}^*$ contains the transformed variables in the orthogonal complement of $\mathcal{W}$. The notion is to transform $\boldsymbol{\beta}$ and separate the component representing the specified DLM ($\boldsymbol{\beta}^*$) from the component representing the departure from the DLM ($\mathbf{u}$). Testing $H_0: \mathbf{R}\boldsymbol{\beta} = \mathbf{0}$ against $H_1: \mathbf{R}\boldsymbol{\beta} \neq \mathbf{0}$ is the same as testing whether $\mathbf{u}$ varies significantly from $\mathbf{0}$ or not.

The expressions in (4) can be viewed as the joint log-likelihood of $f(\mathbf{Y}, \mathbf{u}|\mathbf{X}^*) = f(\mathbf{Y}|\mathbf{u}, \mathbf{X}^*)f(\mathbf{u}|\mathbf{X}^*) = f(\mathbf{Y}|\mathbf{u}, \mathbf{X}^*)f(\mathbf{u})$ in a typical generalized linear mixed model of the form

$$\mathbf{Y}|\mathbf{u}, \mathbf{X}^* \sim \mathcal{F}(\mathbf{X}^* \boldsymbol{\beta}^* + \mathbf{Z}^* \mathbf{u}), \mathbf{u} \sim N(\mathbf{0}, \sigma_u^2) \quad (5)$$

where $\mathcal{F}$ is the canonical distribution belonging to the exponential family defined in (1) and σ_u^2 is the random effect variance, inversely proportional to λ in (4), with $\mathbf{u}$ independent of $\mathbf{X}^*$ as a result of the definition of $\mathbf{u}$ and $\mathbf{X}^*$ by performing the SVD as (3).

As λ in (4) becomes larger, $\mathbf{R}\boldsymbol{\beta}$ is forced to approach $\mathbf{0}$. Since $\sigma_u^2 \propto \frac{1}{\lambda}$ and $\sigma_u^2 \rightarrow 0$ as $\lambda \rightarrow \infty$, testing $H_0: \mathbf{R}\boldsymbol{\beta} = \mathbf{0}$ against $H_1: \mathbf{R}\boldsymbol{\beta} \neq \mathbf{0}$ is equivalent to testing $H_0: \sigma_u^2 = 0$ against

$H_1: \sigma_u^2 > 0$. Let $\mu_t^{(H_0)} = E_{H_0}(y_t)$, the expected value of y_t under the null and let $\boldsymbol{\mu}^{(H_0)} = (\mu_1^{(H_0)}, \dots, \mu_T^{(H_0)})^T$, $\mathbf{g}'(\boldsymbol{\mu}) = \text{diag}[g'(\mu_\rho)]$, and $\mathbf{V} = \text{diag}\{a(\boldsymbol{\phi}) \boldsymbol{\nu}(\boldsymbol{\mu}) [g'(\boldsymbol{\mu})]^2\}$. The VCST statistic (Lin, 1997; Zhang and Lin, 2003) is given by

$$Q = (\mathbf{Y} - \hat{\boldsymbol{\mu}}^{(H_0)})^T \hat{\boldsymbol{\Delta}} \hat{\mathbf{V}}^{-1} \mathbf{Z}^* \mathbf{Z}^{*T} \hat{\mathbf{V}}^{-1} \hat{\boldsymbol{\Delta}} (\mathbf{Y} - \hat{\boldsymbol{\mu}}^{(H_0)}). \quad (6)$$

where $\hat{\boldsymbol{\mu}}^{(H_0)}$, $\hat{\boldsymbol{\Delta}}$, and $\hat{\mathbf{V}}$ are REML estimates of $\boldsymbol{\mu}^{(H_0)}$, $\boldsymbol{\Delta}$, and $\mathbf{V}$, respectively, under H_0 . The test statistic Q asymptotically follows a mixture $\sum_{i=1}^{L+1-p} \gamma_i \chi_{1,i}^2$ where $\gamma_1, \dots, \gamma_{L+1-p}$ are the eigenvalues of $\hat{\mathbf{V}}^{-1/2} \mathbf{Z}^{*T} \mathbf{Z}^* \hat{\mathbf{V}}^{-1/2}$ and $\chi_{1,i}^2$ are independent χ_1^2 random variables. The Davies exact method (Davies, 1980) based on inverting the characteristic function can be used to calculate the distribution of any quadratic form in normal random variables. The details are provided in the Supplementary Material A.1. We make use of the method to obtain the threshold of a significance level and p -value corresponding to Q via R package `CompQuadForm` for simulation and data analysis in later sections.

Remark—The VCST approach can be applied for testing $H_0: \mathbf{R}_0 \boldsymbol{\beta} = \mathbf{0}$ against $H_1: \mathbf{R}_1 \boldsymbol{\beta} \neq \mathbf{0}$ where the DLM represented by $\mathbf{R}_0$ is nested within the DLM represented by $\mathbf{R}_1$. Let $\mathbf{C}_0$ and $\mathbf{C}_1$ denote the $(L+1) \times p_0$ transformation matrix and $(L+1) \times p_1$ transformation matrix corresponding to $\mathbf{R}_0$ and $\mathbf{R}_1$, respectively. The penalized negative log-likelihood function analogous to (2) is given by

$$\ell_p(\theta) = \left[S(\mathbf{X}\mathbf{C}_1\theta) - \mathbf{Y}^T \mathbf{X}\mathbf{C}_1\theta \right] + \lambda \theta^T \mathbf{C}_1^T \mathbf{R}_0^T \mathbf{R}_0 \mathbf{C}_1 \theta. \quad (7)$$

Similarly, (7) can be rewritten as the joint log-likelihood function of a generalized linear mixed model and the hypothesis testing can be conducted subsequently. However, the power gain from VCST as opposed to LRT is less in this situation because the degrees of freedom of LRT, namely $p_1 - p_0$, is generally small. For example, for testing a cubic DLM versus a linear DLM, we will have $p_1 - p_0 = 2$. We considered a simulation setting to illustrate the potential power gain when testing a nested DLM against a more general one for small to moderate $p_1 - p_0$. The results are provided in Supplementary Material Section A.2. One can observe that the LRT and VCST have similar power when the difference in the degree of the two DLMs is small and there is modest power gain by using VCST when this difference is large.

3 Results

3.1 Simulation

We conducted a simulation study to compare the power of the standard LRT and VCST for testing $H_0: \mathbf{R}\boldsymbol{\beta} = \mathbf{0}$ against $H_1: \mathbf{R}\boldsymbol{\beta} \neq \mathbf{0}$. We generated a predictor series of length 2000 with mean 0 and first order autocorrelation $\rho = 0.6$ from the model $x_t = \rho x_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim \text{i.i.d } N(0, 1)$ for $t = 1, \dots, 2000$ for the independent variable. We then set $L = 30$ and generated

the outcome series $\mathbf{Y}$ from the model $y_t = \delta \sum_{\ell=0}^L \beta_{\ell} x_{t-\ell} + \varepsilon_t$ where $\boldsymbol{\beta} = (\beta_0, \dots, \beta_L)^T$ denote the true coefficients and $\varepsilon_t \sim \text{i.i.d } N(0, 1)$ for $t = 1, \dots, 2000$. δ is used to control the signal-to-noise ratio. We generated 1000 data sets and calculated the empirical power at level $\alpha = 0.05$ as the observed proportion of times that a test yields a p -value less than α .

We first examined the type I error rate and power of the two testing procedures across different combinations of true DLMs and fitted DLMs. The results with moderate signal-to-noise level ($\delta = 1$) are presented in Table 1 and the results with stronger ($\delta = 1.25$) and weaker ($\delta = 0.8$) signal-to-noise levels are provided in the Supplementary Material A.3. When the true DLM coincides with the fitted DLM, the numbers in italics are estimated type I error rates. When the true DLM is different from the fitted DLM, the numbers are estimated powers. As we can observe, both LRT and VCST properly maintain the type I error rate at level $\alpha = 0.05$. In addition, VCST is more powerful than LRT irrespective of the signal-to-noise level across all combinations of true DLM and fitted DLM. We also examined the gain in statistical power to detect the departure from a DLM using VCST as opposed to using LRT with different autocorrelation levels for predictor series ρ and different maximum number of lags L . The left panel of Figure 1 displays the power curves of VCST against LRT with three different autocorrelation levels. The three curves are all above the 45° line indicating that VCST is more powerful than LRT across the board. Moreover, the higher the correlation in the predictor series is, the more advantageous using VCST is. The justification is that LRT treats L correlated predictors as L independent variables and it does not make use of the serial correlation when constructing the test statistic, whereas VCST orthogonalizes the design matrix by removing the correlation and

increases power through testing on a single variance component by estimating the true underlying degrees of freedom (df). The right panel of Figure 1 exhibits the power curves of VCST against LRT with three different maximum numbers of lags L . Again, the curves are all above the 45° line illustrating that VCST is more powerful than LRT across different L . We can also notice that the larger the L , the more power gain from using VCST. The explanation is that large L leads to loss of power at the expense of more degrees of freedom for LRT. The gain in power by using VCST via estimating the effective df is more appreciable when L is large and the correlation in x_t is strong.

3.2 Application to NMMAPS Data

We illustrated LRT and VCST using NMMAPS data. The details describing NMMAPS data are available at <http://www.ihapss.jhsph.edu/data/NMMAPS/>. We follow Welty et al. (Welty et al., 2009) for the choice of $L = 14$ and the specification of covariates to associate daily particular matter, with aerodynamic diameter less than 10 microns (PM_{10}), with (1) cardiovascular death count and (2) daily non-accidental mortality counts. We consider polynomial DLMs from degrees 1 to 5 for the lagged effects of exposure on each health outcome. We conducted hypothesis testing to evaluate whether the specified DLMs depart from the underlying unconstrained DLM. Since the outcomes are counts, we specify the first term of the penalized log-likelihood function (2) as the negative log-likelihood of Poisson distribution with mean $e^{X\beta}$. Similar testing procedure as described in Section 2.2 follows.

Table 2 presents the p -values obtained from LRT and VCST. In general, the two testing procedures yield similar results for each combination of a specified DLM and a health outcome, with the p -values from VCST slightly smaller than those obtained from LRT. We resist to interpret that smaller p -values indicate higher power although the smaller p -values from VCST provide stronger evidence against the null. For this example, both tests suggest that none of the DL functions considered is adequate for modeling cardiovascular death in association with PM_{10} . For mortality, it is suggested that a 5-degree polynomial is more appropriate to characterize the DL function than the lower-order polynomial.

The estimated distributed lag functions are displayed in Figure 2. The interquartile range of PM_{10} is $21.49\mu g/m^3$. The quantity $100[\exp(21.49\sum_{j=0}^{\ell}\beta_j) - 1]$ represents to the percentage change in daily cardiovascular death/non-accidental mortality associated with an IQR increase in PM_{10} ($21.49\mu g/m^3$) across lag 0- ℓ . For cardiovascular death, the estimated cumulative lag effects up to lag 4, lag 7, and lag 14 are 1.61% (95%CI: [0.34%, 2.87%]), 1.27% (95%CI: [-0.30%, 2.84%]), and 0.53% (95%CI: [-1.74%, 2.80%]) based on the unconstrained DLM. For non-accidental mortality, the estimated cumulative lag effects up to lag 4, lag 7, and lag 14 are 0.69% (95%CI: [-0.11%, 1.49%]), 0.27% (95%CI: [-0.74%, 1.27%]), and -0.80% (95%CI: [-2.29%, 0.69%]) based on the 5-degree polynomial DLM.

4 Discussion

The simulation study indicates that VCST procedure is more powerful than a standard LRT for testing a constrained DLM against an unconstrained alternative across different scenarios. The power gain from VCST over LRT is greater when the predictor series auto-

correlation is larger and when the maximal lag is larger. One caveat is that the VCST framework only applies to the constrained DLMs for which the estimator can be written in the form of $C\theta$, so that the corresponding constraint matrix R can be obtained and the hypothesis testing can be constructed as $H_0: R\beta = 0$ against $H_1: R\beta \neq 0$. Most commonly used DLMs fall into this category. Bayesian DLM (Welty et al., 2009) is an exception since all the lag effect coefficients are related through the specification of variance-covariance matrix rather than the mean. Although we illustrate the testing procedure using a data set in environmental epidemiology, the new test is potentially useful for time series analysis in economics, finance, ecology, and a wide range of applications.

Supplementary Material

Refer to Web version on PubMed Central for supplementary material.

Acknowledgments

Funding for the project was provided by awards NSF DMS 1406712 and NIH ES 20811.

References

- Shirley, Almon. The distributed lag between capital appropriations and expenditures. *Econometrica: Journal of the Econometric Society*. 1965:178–196.
- Yin-Hsiu, Chen, Mukherjee, Bhramar, Adar, Sara D., Berrocal, Veronica J., Coull, Brent A. Robust distributed lag models using data adaptive shrinkage. *Biostatistics (Under Revision)*. 2016
- Corrado, Corradi. Smooth distributed lag estimators and smoothing spline functions in hilbert spaces. *Journal of Econometrics*. 1977; 5(2):211–219.
- Davies, Robert B. Algorithm as 155: The distribution of a linear combination of χ^2 random variables. *Applied Statistics*. 1980:323–333.
- Gasparrini A, Armstrong B, Kenward MG. Distributed lag non-linear models. *Statistics in medicine*. 2010; 29(21):2224–2234. [PubMed: 20812303]
- Hastie, Trevor, Tibshirani, Robert. Varying-coefficient models. *Journal of the Royal Statistical Society Series B (Methodological)*. 1993:757–796.
- Tzu-Kuang, Hsu. Exploring relationship between the stock price of taiwan and the exchange rate: An autoregressive distributed lag model with a quantile regression. *International Journal of Economics and Finance*. 2015; 8(1):72.
- Lin, Xihong. Variance component testing in generalised linear models with random effects. *Biometrika*. 1997; 84(2):309–326.
- Majid M, Yusof Rosylin Mohd. Long-run relationship between islamic stock returns and macroeconomic variables: An application of the autoregressive distributed lag model. *Humanomics*. 2009; 25(2):127–141.
- Schwartz, Joel. The distributed lag between air pollution and daily deaths. *Epidemiology*. 2000; 11(3): 320–326. [PubMed: 10784251]
- Welty, Leah J., Peng, RD., Zeger, SL., Dominici, F. Bayesian distributed lag models: estimating effects of particulate matter air pollution on daily mortality. *Biometrics*. 2009; 65(1):282–291. [PubMed: 18422792]
- Zhang, Daowen, Lin, Xihong. Hypothesis testing in semiparametric additive mixed models. *Biostatistics*. 2003; 4(1):57–74. [PubMed: 12925330]

Figure 1.

Plots of the power of variance component score test (VCST) against the power of likelihood ratio test (LRT) for testing a constrained distributed lag model (DLM) against an unconstrained alternative with three different first order autocorrelation levels (ρ) for predictor series (left panel) and three different maximum number of lags (L) for predictor series (right panel) based on 1000 repetitions.

Figure 2.

Plots of estimated distributed lag functions for cardiovascular death (left panel) and non-accidental mortality (right panel) in association with PM_{10} in Chicago, Illinois from 1987 to 2000 using the National Mortality, Morbidity, and Air Pollution Study (NMMAPS) data.

Empirical type I error and power of likelihood ratio test (LRT) and variance component score test (VCST) for testing a constrained distributed lag model (DLM) against an unconstrained alternative based on 1000 repetitions when signal-to-noise level is moderate ($\delta = 1$) with significance level 0.05.

Table 1

Fitted DLM	Linear		Quadratic		Cubic	
	LRT	VCST	LRT	VCST	LRT	VCST
True DLM						
Linear	0.057	0.040	–	–	–	–
Quadratic	0.234	0.584	0.038	0.039	–	–
Cubic	0.282	0.611	0.276	0.612	0.045	0.047
Unstructured	0.317	0.640	0.312	0.641	0.319	0.572

* Italicized numbers are type I error rates at level 0.05.

Table 2

P -values obtained from likelihood ratio test (LRT) (with $L + 1 - p$ degrees of freedom) and variance component score test (VCST) for testing a specified distributed lag model (DLM) against an unconstrained DLM in association of daily PM_{10} measurements with cardiovascular death and non-accidental mortality in Chicago, Illinois from 1987 to 2000 using the National Mortality, Morbidity, and Air Pollution Study (NMMAPS) data where the maximum number of lags L is fixed at 14 days and p denote the number of basis functions of a DLM.

DLM	CVD ^I		Mortality	
	LRT	VCST	LRT	VCST
Linear ($p = 2$)	0.002	0.002	0.009	0.005
Quadratic ($p = 3$)	0.003	0.007	0.017	0.015
Cubic ($p = 4$)	0.020	0.019	0.055	0.013
4DF Polynomial ($p = 5$)	0.040	0.032	0.084	0.019
5DF Polynomial ($p = 6$)	0.038	0.033	0.286	0.150

^ICVD: cardiovascular deaths