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Does Noise Weighting Matter in CT Iterative Reconstruction?

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Abstract

This paper uses a computer simulation to investigate whether a more accurate noise model always results in less noisy images in CT iterative reconstruction. We start with a hypothetical non-realistic noise model for the CT measurements, by assuming that the attenuation coefficient is energy independent and there is no scattering. A variance formula for this model is derived and presented. Based on this model, computer simulations are conducted with 12 different ad hoc noise weighting methods, and their results are compared. The simple Poisson noise model performs better than other more accurate models, when the projection data are generated with the hypothetical noise model. A more accurate noise model does not necessarily produce a less-noisy image. In this counter example, modeling the system's electronic noise during reconstruction does not help reducing the image noise. A simpler noise model sometimes can outperform the complicated and more accurate noise model.

Index Terms

Image reconstruction; noise model; Poisson distribution; X-ray CT

I. Introduction

In transmission tomography especially in x-ray computed tomography (CT), the iterative image reconstruction algorithms assume a simple Poisson model [1]–[5]. In this Poisson noise model, it is assumed that the number of photons I_0 emitted from the x-ray tube is a constant (not random) because this number is extremely large. After the x-ray photons travel through an attenuating/scattering object, the number of photons escaping from the object is significantly reduced and follows the Poisson noise model.

In fact, the number of photons emitted from the x-ray tube is also characterized as the Poisson distribution [6]. This more accurate noise model is justified in multi-energy x-ray

CT imaging, where the x-ray energy spectrum is considered. The x-ray energy spectrum can be divided into many sub energy windows (or bins); the number of x-ray photons in each window is Poisson. The mean value of the photons in each energy window can be characterized by a spectrum function $\Phi(E)$ [7]. Since the source photon counts are random and Poisson, the detected (after object) x-ray photons are compound Poisson distributed [8], [9] and its probability density function depends on the convolution of the spectrum function $\Phi(E)$. A simple Poisson model may not be accurate enough to exactly model the photon noise.

The purpose of this paper is to investigate whether a more accurate noise model always gives a better (e.g., less noisy with the same contrast) image. A hypothetical noise model is set up in this paper, and it is referred to as the “true model.” However, the exact noise variance for this true model is difficult to find. Many approximate models are presented and used for comparison purposes. It is well known that the linear attenuation coefficient of a material is energy dependent and this fact is the cause of the infamous beam-hardening artifacts in CT. The beam-hardening issues are not in the scope of this paper, and the linear attenuation coefficient is assumed to be energy *independent* in all of our computer simulations.

II. Methods

A. The Hypothetical Noise Model

The x-ray source emits I_0 photons per projection ray. Since I_0 is so large, it is justified to assume I_0 to be a non-random constant. Let the energy spectrum distribution function of the x-ray source be $\Phi(E)$ which is normalized to unity (similar to a probability density function). A typical distribution $\Phi(E)$ is shown in Fig. 1.

The x-ray energy spectrum (before entering the object) can be divided into many sub energy windows (or bins); the number of x-ray photons in each window is Poisson distributed. The mean value of the number of x-ray photons in the k th energy window is $I_0\Phi(E_k)$.

After the x-rays pass through the object, on the detector, each energy window produces a Poisson x-ray measurement p_k , which can be modeled as

$$p_k \sim \text{Poisson} \left(e^{-\int \mu(x) dx} \text{Poisson} (I_0 \Phi(E_k)) \right), \quad (1)$$

where $\mu(x)$ is the linear attenuation coefficient and is *assumed* to be independent of the photon energy in this paper; the line integral $\int \mu(x) dx$ is along the associated projection ray. Eq. (1) is essentially the Beer’s law [11].

If a random variable q follows the Poisson distribution, we symbolically denote it as

$$q \sim \text{Poisson}(\lambda), \quad (2)$$

where λ is the mean value as well as the variance. Then (1) is a special case of

$$p \sim a_1 \text{Poisson}(a_2 \text{Poisson}(\lambda)), \quad (3)$$

where a_1 and a_2 are two non-random constants. The mean and variance of the random variable p can be derived from the law of total expectation and the law of total variance [10]. Thus

$$E_p(p) = E_q[E_{p|q}(p)] = E_q[a_1 a_2 q] = a_1 a_2 \lambda. \quad (4)$$

$$\begin{aligned} \text{Var}_p(p) &= E_q[\text{Var}_{p|q}(p)] + \text{Var}_q[E_{p|q}(p)] = E_q[a_1^2 a_2 q] + \text{Var}_q[a_1 a_2 q] = a_1^2 a_2 \lambda + (a_1 a_2)^2 \lambda = a_1^2 (a_2 + a_2^2) \lambda \\ & \quad (5) \end{aligned}$$

For the random variable p_k in (1), The energy-integrating detection outputs a signal of $aE_k p_k$, where E_k is the photon energy of the energy window and a is the system gain. Using the definition (3) with $a_1 = aE_k$, $a_2 = \exp(-\int \mu(x) dx)$, and $\lambda = I_0 \Phi(E_k)$, by (4) and (5), the mean and variance of the detected energy from p_k are given as:

$$\text{Mean of } aE_k p_k = aE_k I_0 \Phi(E_k) e^{-\int \mu(x) dx}; \quad (6)$$

$$\text{Variance of } aE_k p_k = (aE_k)^2 I_0 \Phi(E_k) \times \left(e^{-\int \mu(x) dx} + e^{-2\int \mu(x) dx} \right). \quad (7)$$

The total signal p received by the energy-integrating system along this ray should include the signals from all energy windows and the system noise d generated in the electronic circuits:

$$p = \sum_k aE_k p_k + d. \quad (8)$$

where d can be assumed to be a zero-mean Gaussian random variable with a variance σ^2 . Using (6) and (7), the mean and variance of the random variable p in (8) can be obtained as

$$\begin{aligned} \text{Mean of } p &= \sum_k aE_k I_0 \Phi(E_k) e^{-\int \mu(x) dx} + 0 = aI_0 e^{-\int \mu(x) dx} \sum_k E_k \Phi(E_k) = aI_0 e^{-\int \mu(x) dx} \bar{E} \\ & \quad (9) \end{aligned}$$

with the first moment of the x-ray source distribution

$$\overline{E} = \sum_k E_k \Phi(E_k); \quad (10)$$

Variance of

$$\begin{aligned} p &= \sum_k (aE_k)^2 I_0 \Phi(E_k) \left(e^{-\int \mu(x) dx} + e^{-2\int \mu(x) dx} \right) + \sigma^2 \\ &= a^2 I_0 \left(e^{-\int \mu(x) dx} + e^{-2\int \mu(x) dx} \right) \sum_k E_k^2 \Phi(E_k) + \sigma^2 \\ &= a^2 I_0 \left(e^{-\int \mu(x) dx} + e^{-2\int \mu(x) dx} \right) \overline{E} + \sigma^2 \end{aligned} \quad (11)$$

with the second moment

$$\overline{\overline{E}} = \sum_k E_k^2 \Phi(E_k). \quad (12)$$

The entire paper assumes that the attenuation coefficient is independent on beam energy. Without this assumption the mathematical derivation in this paper is incorrect.

B. Post-Log Multi-Energy Noise

The mean and variance formulas in Section 2.1 are for the pre-log data. Normally, the post-log data are used for analytical and iterative reconstruction. Let the post-log data be obtained as

$$y = \ln \left(\frac{a\overline{E}I_0}{p} \right). \quad (13)$$

In (13), if p is less than 1, p is set to 1 before taking the logarithm. The exact variance of y is not easy to calculate; an approximation of it can be obtained as

$$\begin{aligned}
\text{Variance of } y &= \sigma_y^2 \approx \left. \left| \frac{dy}{dp} \right|^2 \right|_{p=\text{Mean of } p} \times (\text{Variance of } p) \\
&= \frac{1}{\left(a I_0 e^{-\int \mu(x) dx} \bar{E} \right)^2} \\
&\quad \times \left[a^2 I_0 \left(e^{-\int \mu(x) dx} + e^{-2 \int \mu(x) dx} \right) \bar{E} + \sigma^2 \right] \\
&= \frac{1}{\left(a I_0 e^{-\bar{y}} \bar{E} \right)^2} [a^2 I_0 (e^{-\bar{y}} + e^{-2\bar{y}}) \bar{E} + \sigma^2].
\end{aligned} \tag{14}$$

The approximation in the first line of (14) is based on the truncated Taylor expansion of (13). The approximation error can be large when the quanta are low.

The Gauss–Markov theorem [12], [13] shows that in a linear regression model in which the errors have expectation zero and are uncorrelated and have equal variances, the best linear unbiased estimator of the coefficients is given by the ordinary least squares estimator. Here “best” means giving the lowest variance of the estimate, as compared to other unbiased, linear estimators. The errors do not need to be Gaussian, nor do they need to be independent and identically distributed. If, however, the measurements have different uncertainties, Aitken showed that each weight should be equal to the reciprocal of the variance of the measurement [14]. Thus the optimal noise weighting in this case is the reciprocal of the noise variance:

$$\text{Weighting for ray } y = \frac{1}{\sigma_y^2} = \frac{(a I_0 e^{-\bar{y}} \bar{E})^2}{a^2 I_0 (e^{-\bar{y}} + e^{-2\bar{y}}) \bar{E} + \sigma^2}. \tag{15}$$

Hereafter, the noise weighting (15) will be referred to as the mathematical noise model weighting. However, in practice, the mean value $\bar{y}$ of the post-log data is not available. The one-time measurement y is usually used instead.

C. Computer Simulations

A gradient descent algorithm will be used to minimize the noise-weighted post-log maximum likelihood objective function:

$$\text{Objective Function} = \sum_j w_j \left(\sum_i a_{ij} \mu_i - y_j \right)^2, \tag{16}$$

where μ_i is the image pixel whose value is the linear attenuation coefficient, y_j is the post-log measurement (i.e., the noisy Radon transform), a_{ij} is the contribution from the i th pixel μ_i to the j th measurement y_j , and w_j is the weighting factor which is chosen to be the reciprocal of the noise variance of y_j . The gradient descent algorithm is expressed as

$$\mu_i^{(k+1)} = \mu_i^{(k)} - \alpha \frac{\sum_j a_{ij} w_j \left(\sum_n a_{nj} \mu_n^{(k)} - y_j \right)}{\sum_j a_{ij} w_j \sum_n a_{nj}}, \quad (17)$$

where $\mu_i^{(k)}$ is the estimation of μ_i at the k^{th} iteration and the step size α is a small positive constant to prevent the algorithm from divergence. The value of α is set to 0.2 in this paper.

The purpose of the denominator $\sum_j a_{ij} w_j \sum_n a_{nj}$ is to normalize the weighting factors w_j so that the step size of the algorithm is almost the same for any chosen weighting factor $w_j > 0$. The summation over the index n is the projector and the summation over the index j is the backprojector.

The computer simulations in this paper are based on a scaled-down x-ray CT fan-beam imaging geometry with a curved detector. The energy spectrum $\Phi(E)$ was provided by Toshiba and shown in Fig. 1. The image array was 256×256 , the pixel size was $1.52 \text{ mm} \times 1.52 \text{ mm}$, the number of views was 400 over 360° , the number of detection channels was 400, the distance from the x-ray focal spot to the isocenter was 240 mm, and the virtual detector is at the isocenter. The x-ray source flux had $I_0 = 10^4$ counts per ray, which corresponds to a low-dose imaging setup. The phantom shown in Fig. 2 is a $355 \text{ mm} \times 187 \text{ mm}$ ellipse with water background ($\mu = 0.02/\text{mm}$), 3 high contrast regions ($\mu = 0.032/\text{mm}$) with diameter 48 mm, 2 low-contrast regions ($\mu = 0.0194/\text{mm}$) with diameter 36 mm, surrounded by outer layers of fat ($\mu = 0.019/\text{mm}$) and skin ($\mu = 0.021/\text{mm}$), in which ROI 1 (high contrast object) and ROI 2 (water background) are used to evaluate the image quality. The detailed phantom parameters are listed in Table 1.

Noisy projection data generation procedure is as follows using MATLAB (The MathWorks, Inc., Natick, MA, USA). First, for each ray, a noiseless line integral of the phantom is calculated. Second, using exponential function, the line integral value is converted into noiseless transmission counts. Third, the noiseless transmission data at each energy bin is incorporated with nested Poisson noise according to (1). Random number was generated by MATLAB's built-in noise Poisson generator — `poissrnd`. Forth, the summation of the noisy data from all energy bins is calculated and the system Gaussian noise is added to the summation as (8). Fifth, this noisy pre-log value is transformed into noisy post-log value according to (13) for image reconstruction. The parameter a was set to 0.1 and σ was set to 6.3. The value of 6.3 is a typical value in a Toshiba CT system and was obtained from experimental measurements.

In the entire data generation and reconstruction, the attenuation coefficients were assumed to be energy independent. The iterative algorithm was implemented according to (17), and the iterative algorithm stops when a pre-specified image contrast is reached. This value is set up for 2 cases: $(ROI1 - ROI2)/ROI2 = 0.55$ and $(ROI1 - ROI2)/ROI2 = 0.60$, respectively. The reconstructed images with different methods are compared with the normalized standard

deviation value in ROI 2. The normalized standard deviation value is the standard deviation value divided by the mean value.

This paper will use 12 methods of implementing and approximating the weighting factor w_j , which is the reciprocal of the noise variance sinogram. These 12 methods of calculating the noise weighting are listed below.

Method 1 (Mathematical)—The weighting factor w_j is calculated using (15), with $a = 0.1$, $I_0 = 10^4$, $\sigma^2 = 6.3^2$, and $\bar{y}_j$ being the noiseless post-log projection. The x-ray tube energy distribution function $\Phi(E)$ is measured in a Toshiba CT scanner as shown in Fig. 1. The energy spectrum is subdivided into 120 energy windows from 0 to 120 keV. The first and second moments of the spectrum $\bar{E}$ and $\overline{E^2}$ are calculated using (10) and (12); their numerical values are 67.8776 keV and 5036.9 (keV)², respectively.

Method 2 (Ignoring electronic noise)—This method is the same as Method 1, except that the electronic noise is ignored during image reconstruction. In other words, $\sigma^2 = 6.3^2$ is used in data generation, but $\sigma^2 = 0$ is assumed in reconstruction.

Method 3 (Statistical approach)—This method uses 1000 realizations of the noisy post-log data sets. The ensemble variance “sinogram” is then calculated from these 1000 realizations, and the weighting function is the reciprocal of the variance “sinogram.” We believe that this method is the most accurate one among our 12 methods, while Method 1 may not be accurate when the line integral value is large.

Method 4 (Practical approach)—Almost the same as Method 1, except that the mean value $\bar{y}_j$ is replaced by one-time (i.e., one noise-realization) post-log measurement y_j . Electronic noise is modeled by σ^2 .

$$\text{Weighting for ray } y = \frac{1}{\sigma_y^2} = \frac{(aI_0 e^{-y} \bar{E})^2}{a^2 I_0 (e^{-y} + e^{-2y}) \bar{E} + \sigma^2}. \quad (18)$$

Method 5 (Modified version of Method 1)—We replace the variance by a power function of the variance.

$$\text{Weighting for ray } y = \frac{1}{\sigma_y^{2\gamma}} = \left[\frac{(aI_0 e^{-y} \bar{E})^2}{a^2 I_0 (e^{-y} + e^{-2y}) \bar{E} + \sigma^2} \right]^\gamma \quad (19)$$

Method 6 (Popular model)—This is a simplified version of (18). The expression is

$$\text{Weighting for ray } y = \frac{1}{\sigma_y^2} = \frac{(aI_0 e^{-y})^2}{a^2 I_0 e^{-y} + \sigma^2}. \quad (20)$$

Method 7 (Practical Poisson approximation)—Instead of using the more accurate true model, this method uses the more practical (but less accurate) Poisson model and ignores electronic noise, which leads to the variance sinogram being $\exp(\bar{y}_j)$. To make it practical, the mean value $\bar{y}_j$ is replaced by one-time measurement value y_j .

$$\text{Weighting for ray } y = \frac{1}{\sigma_y^2} = e^{-y}. \quad (21)$$

Method 8 (Constant weights, i.e., No weights)—The variance sinogram is set to a constant for all projection rays.

$$\text{Weighting for ray } y = 1. \quad (22)$$

Method 9 (Totally wrong)—This will show what damages wrong weighting factors can do to the reconstruction. The totally wrong variance sinogram is chosen to be $\exp(-y_j)$.

$$\text{Weighting for ray } y = e^y. \quad (23)$$

Method 10 (Method 7 with the noiseless $\bar{y}$)—This is more accurate than Method 7, where $\bar{y}$ is the noiseless true value.

$$\text{Weighting for ray } y = \frac{1}{\sigma_y^2} = e^{-\bar{y}}. \quad (24)$$

Method 11 (Method 10, plus a power γ)—This method uses the same idea of Method 5, but uses the model in Method 10:

$$\text{Weighting for ray } y = \left(\frac{1}{\sigma_y^2} \right)^\gamma = e^{-\gamma \bar{y}}. \quad (25)$$

Method 12 (Method 11, 1 realization, 5-point average)—Method 11 is not practical, because the noiseless measurement is never available. On the other hand, the one realization measurement is too noisy. This method replaces $\bar{y}$ in (25) by a 5-point running averaged (i.e., lowpass filtered) value of y in the detector channel direction. This method is practical.

III. Results

A. Verification of Eq. (14)

The theoretical variance formula (14) for the noise (8) is verified by ensemble variance of 1000 noise realizations, with $a = 0.1$, $I_0 = 10^4$, and $\sigma^2 = 6.3^2$. The results are summarized in Fig. 3, using various values of $\bar{y} = \int \mu(x) dx$. It is observed from Fig. 3 that when the line-

integrals $\bar{y} = \int \mu(x) dx$ are small, the variance formula (14) and the ensemble variance agree quite well. However, when $\bar{y}$ becomes large, the number of photons $I_0 \exp(-\bar{y})$ is extremely small and can be zero. It is numerically unstable to take the logarithm of a value that is close to zero. Therefore, the noise variance estimation is unreliable when the line-integrals of the object are large (e.g., when the object contains high density materials).

B. Iterative Reconstruction

The reconstructed images with all 12 methods are shown in Fig. 4 and Fig. 5 for contrast 0.55 and 0.60, respectively. A rectangular uniform region-of-interest, ROI 2 (shown in Fig. 2), is selected to compare the normalized noise standard deviations in the reconstructed image. The noise weighting methods are compared in terms of the noise in ROI 2, and the results are listed in Table 2. The noise in the image with Method 9 is the worst, as expected. Surprisingly, the simple exponential method with 5-point running average lowpass filter and a power γ (Method #12) outperforms all other methods.

It is interesting to observe that the most accurate model using 1000 noise realizations (Method 3) does not give the best image. The result is almost the same if the electronic noise is ignored (Method 1 vs. Method 2). Method 1 and Method 2 are not practical, because the true noise variance cannot be obtained by one noise realization in practice. In reality, the noise variance is estimated by measured data (i.e., one noise realization) and this estimation is noisy. The noise from the weighting function propagates into the reconstruction, generating a noisier image (Method 4) than the ensemble approach (Method 3). Method 7 that approximates the mathematical noise model by simple Poisson and ignores electronic noise generates better result than Method 4 that uses a more accurate noise model.

Noise in the weighting function can contribute to the noise in the reconstruction. This can also be observed by comparing the results of Method 7 and Method 10. The most accurate noise method (Method 3) does not necessary give the best image. In fact, the most accurate noise model, as well as many other approximate methods (Methods 1, 2, 4, 6, 7, and 10), gives low-frequency shadowing artifacts.

The weighting functions with power ($\gamma < 1$) (Methods 5, 11 and 12) do not show any low-frequency shadowing artifacts. When γ is small (method 5 with $\gamma = 0.2$), the weighting difference between different projection bins get smaller and its image quality is closer to uniform weighting (Method 8) where streaking noise artifacts starts to appear. A moderate $\gamma = 0.4$ to 0.5 in Methods 11 and 12 seems to be a good compromise between low-frequency shadowing and streaking noise artifacts.

A much simpler model (Method 12) outperforms all other methods. Method 12 is a practical method that uses a lowpass filtered version of the measurement to reduce the noise in the weighting function. The filter is a 5-point running average of the noisy data. We must point out that the smoothed data are only used for forming the weighting function w_j in (17); the projections y_j in (17) are un-smoothed.

Figure 6 illustrates that as the iteration number increases, the image contrast increases. Figure 7 illustrates that as the iteration number increases, the image noise (in terms of normalized standard deviation in a constant region) increases.

IV. Conclusions

Computer simulations in Section 2 show that almost the same noise variance results from the more accurate 1000-realization noise model and from the less accurate Poisson noise model.

This paper investigates whether the more accurate noise model always gives better (meaning: less noisy) images than the less accurate Poisson noise model. This paper gives a counter example. A more accurate noise model does not necessarily give a better image.

Computer simulations using Methods 1, 2, 3, 4, 6, 7, and 10 give similar results, even though they are *ad hoc*. This implies that it is reasonable to approximate the more accurate noise model by the less accurate Poisson model. Another observation is that the use of the one-time measurement to approximate the mean value of the measurement can introduce some noise to the output image. Smoothing the projection measurements may be a remedy when they are used to form the weighting factors.

Methods 1, 2, 3, 4, 6, 7, and 10 can generate some low-frequency shadowing artifacts. If we modify the weighting function by a power function of the weighting function this low-frequency shadowing artifacts can be reduced without degrading the image contrast, see Methods 5, 11, and 12. A thorough study of the weighting function with an exponent parameter is outside the scope of this paper, and will be conducted in a different paper.

Finding the optimal maximum likelihood solution was well established a long time ago. By assuming uncorrelated noise model, the optimal noise weighting is the reciprocal of the noise variance. However, this principle does not apply in medical imaging because the maximum likelihood solution usually is too noisy to be useful. In practice, the iterative algorithms terminate early before convergence is reached. If we stop the algorithm early not to search for the noisy “maximum likelihood” solution, the strategy of using the reciprocal of the noise variance as the noise weighting factor may not be optimal. It is still an open problem how to select the noise weighting factor that can lead to the “minimum noise” solution with a pre-specified image contrast.

The noise weighting does matter. The weighting functions can be categorized in 4 types:

1. Over weighted: Some low-frequency shadowing artifacts can be seen. Many commonly used weighting functions are tend to be over weighted.
2. Properly weighted: Images have the least artifacts and lowest noise. An over weighted weighting function can be tuned down to a properly weighted weighting function by using an exponential factor γ that is less than 1.
3. Under weighted: Almost like no weighting at all. Some streaking artifacts can be seen. One could improve the weighting effectiveness by using an exponential factor γ that is greater than 1.

4. Wrongly weighted: The weighting is in the wrong direction. It emphasizes the rays that should be deemphasized.

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Fig. 1.
Energy spectrum $\Phi(E)$ of an x-ray source.

Fig. 2.

A computer generated torso phantom is used for transmission CT data generation. The linear attenuation coefficients are labeled in the unit of per mm. Two regions of interest (ROIs) are defined for image evaluation and for the image contrast calculation.

Fig. 3.

Evaluation of the noise variance given by Eq. (14) by using 1000 noise realizations

	Method 1 (mathematical)	Method 2 (ignores electronic noise)	Method 3 (1000 realizations)
Recon.
ROI2 stdev	0.0212960, (Rank = 5)	0.0210200, (Rank = 4)	0.0226584, (Rank = 8)
	Method 4 (1 realization)	Method 5 (with a power $\gamma = 0.2$)	Method 6 (popular)
Recon.
ROI2 stdev	0.0418469, (Rank = 11)	0.0188045, (Rank = 3)	0.0413390, (Rank = 10)
	Method 7 (exponential, 1 realization)	Method 8 (constant)	Method 9 (totally wrong, 1 realization)
Recon.
ROI2 stdev	0.0221364, (Rank = 7)	0.0281837, (Rank = 9)	0.2266475, (Rank = 12)
	Method 10 (exponential, true)	Method 11 (exponential, power $\gamma = 0.5$, true)	Method 12 (exponential, power $\gamma = 0.5$, 1 realization, 5-point average)
Recon.
ROI2 stdev	0.0213091, (Rank = 6)	0.0181552, (Rank = 2)	0.0180149, (Rank = 1)

Fig. 4.
Reconstructed images with Methods 1-12, using a stopping rule that a pre-specified image contrast of 0.55 is reached.

	Method 1 (mathematical)	Method 2 (ignores electronic noise)	Method 3 (1000 realizations)
Recon.
ROI2 stdev	0.0358366, (Rank =6)	0.0345710, (Rank = 4)	0.0404598, (Rank =10)
	Method 4 (1 realization)	Method 5 (with a power $\gamma = 0.2$)	Method 6 (popular)
Recon.
ROI2 stdev	0.0393090, (Rank =9)	0.0291881, (Rank = 3)	0.0413375, (Rank = 11)
	Method 7 (exponential, 1 realization)	Method 8 (constant)	Method 9 (totally wrong, 1 realization)
Recon.
ROI2 stdev	0.0374451, (Rank = 7)	0.0378492, (Rank = 8)	0.2773169, (Rank = 12)
	Method 10 (exponential, true)	Method 11 (exponential, power $\gamma = 0.5$, true)	Method 12 (exponential, power $\gamma = 0.4$, 1 realization, 5-point average)
Recon.
ROI2 stdev	0.0348636, (Rank =5)	0.0254592, (Rank =2)	0.0252129, (Rank = 1)

Fig. 5.

Reconstructed images with Methods 1-12, using a stopping rule that a pre-specified image contrast of 0.60 is reached.

Fig. 6. Contrast vs. Iteration number for (Upper) Method #1 and (Lower) Method 7, respectively. The contrast is defined as $(ROI1-ROI2)/ROI2$.

Fig. 7. Normalized standard deviation in ROI 2 vs. Iteration number for (Upper) Method #1 and (Lower) Method 7, respectively.

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TABLE I

Parameters of the phantom (in MATLAB phantom format)

x_0 (center)	y_0 (center)	A (semi-axis)	B (semi-axis)	ϕ (rotation)	HU
0.0	-11.2	59.2	31.2	0	1050
0.0	-11.2	58.4	30.4	0	-100
0.0	-12.0	56.0	28.0	0	50
40.0	-12.0	8.0	8.0	0	600
-40.0	-12.0	8.0	8.0	0	600
0.0	-24.0	8.0	8.0	0	600
24.0	-20.0	6.0	6.0	15	-30
-24.0	-20.0	6.0	6.0	-15	-30

TABLE II

Ranking of the methods in terms of ROI 2 normalized standard deviation

Rank	Method
1 (best)	Method 12 (Exponential, 1 noise realization, 5-point running average)
2	Method 11 (Exponential, with a power γ , noiseless weight)
3	Method 5 (Mathematical with a power γ)
4	Method 2 (Mathematical, ignoring electronic noise)
5	Method 1 (Mathematical) [According to Fig. 4] Method 10 (Exponential, noiseless weight) [According to Fig. 5]
6	Method 10 (Exponential, noiseless weight) [According to Fig. 4] Method 1 (Mathematical) [According to Fig. 5]
7	Method 7 (Exponential, 1 noise realization)
8	Method 3 (1000 noise realization) [According to Fig. 4] Method 8 (Constant weight) [According to Fig. 5]
9	Method 8 (Constant weight) [According to Fig. 4] Method 4 (Mathematical, 1 noise realization) [According to Fig. 5]
10	Method 6 (Popular) [According to Fig. 4] Method 3 (1000 noise realization) [According to Fig. 5]
11	Method 4 (Mathematical, 1 noise realization) [According to Fig. 4] Method 6 (Popular) [According to Fig. 5]
12 (worst)	Method 9 (Totally wrong)