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A framelet-based iterative maximum-likelihood reconstruction algorithm for spectral CT

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Abstract

Standard computed tomography (CT) cannot reproduce spectral information of an object. Hardware solutions include dual-energy CT which scans the object twice in different x-ray energy levels, and energy-discriminative detectors which can separate lower and higher energy levels from a single x-ray scan. In this paper, we propose a software solution and give an iterative algorithm that reconstructs an image with spectral information from just one scan with a standard energy-integrating detector. The spectral information obtained can be used to produce color CT images, spectral curves of the attenuation coefficient $\mu(r, E)$ at points inside the object, and photoelectric images, which are all valuable imaging tools in cancerous diagnosis. Our software solution requires no change on hardware of a CT machine. With the Shepp–Logan phantom, we have found that although the photoelectric and Compton components were not perfectly reconstructed, their composite effect was very accurately reconstructed as compared to the ground truth and the dual-energy CT counterpart. This means that our proposed method has an intrinsic benefit in beam hardening correction and metal artifact reduction. The algorithm is based on a nonlinear polychromatic acquisition model for x-ray CT. The key technique is a sparse representation of iterations in a framelet system. Convergence of the algorithm is studied. This is believed to be the first application of framelet imaging tools to a nonlinear inverse problem.

Keywords

spectral CT; beam-hardening artifacts; maximum likelihood; framelet; soft thresholding; iterative reconstruction

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1. Introduction

Since the introduction of x-ray computed tomography (CT) in 1972, the filtered back-projection (FBP) reconstruction algorithm is the most popular in the CT field. In the FBP algorithm, a linear attenuation coefficient, which is assumed energy-independent, is reconstructed. In reality, when x-rays penetrate the human body, photons with low energy will be mostly absorbed with high-energy photons left, making the average energy higher, which is called the hard-beam effect. In particular, the attenuation coefficients reconstructed by the FBP algorithm will have a poor contrast resolution of low-density soft tissues, which cannot meet the demand of diagnosis of challenging lesions. Actually, there have been many iterative reconstruction algorithms [1–5] since the first commercial CT scanner was developed in 1972, and a brief review of iterative reconstruction method is given in [5]. Similar to the FBP algorithm, typical iterative algorithms suffer from the beam-hardening effect, as the spectrum of x-rays is ignored in computing the underlying attenuation coefficients.

To resolve this issue, injecting iodine contrast material into the blood circulation is a popular method. This may cause complications to patients and will increase the cost of a CT scan. To reduce the iodine amount, we can take into account the continuous spectrum information of x-rays. It will provide more anatomical information.

There are several methods to realize spectral CT. In dual-energy CT [6–10], the machine scans an object with two energy spectra. One such CT equipment is GE Discovery CT750 HD [11]. This is only available in a few high-end CT machines. To avoid two spectra, another strategy is to use a single x-ray spectrum but with energy-discriminative detectors to capture responses of different energies. In this setting, some spectral CT iterative reconstruction algorithms were developed in [12–18]. Also, there are some efforts to improve hardware of CT machines for reducing beam-hardening artifacts, such as pre-filtering the x-ray beam [19, 20].

In addition to hardware improvement, we can realize spectral CT by developing new algorithms on the existing CT hardware platform. Early methods intend to reduce beam-hardening streak artifacts [21–26], amongst the idea of spectral CT emerged [21, 26]. In [26], an iterative maximum-likelihood polychromatic algorithm for CT (IMPACT) was developed. The basic idea is to find the attenuation coefficient distribution that maximizes the log-likelihood function. It was demonstrated in [26] that the IMPACT algorithm is good at eliminating beam-hardening artifacts, and it can be applied in reducing metal artifacts [27, 28].

As we all know, high x-ray radiation exposure is harmful to public health and it may increase the risk of cancer. Hence, we have to reduce the radiation dose. To this end, we

propose an iterative spectral CT reconstruction algorithm assuming only one standard x-ray scan, which invokes a nonlinear inverse problem. Our iterative algorithm uses the polychromatic acquisition model [21, 26] with the framelet sparsity [29, 30]. Each iteration consists of three sub-steps. In one sub-step, we apply a step of rescaled gradient descent method to decrease the ℓ_2 -norm error in fitting the polychromatic acquisition model. Another sub-step is a thresholding operation on framelet coefficients which is performed to make the image sparse in the framelet domain. Non-negativity is another sub-step where negative components are set to 0 to keep positivity of the attenuation coefficients. These three sub-steps can be performed in any order within an iterative cycle.

With the Shepp–Logan phantom, we have found that although the photoelectric and Compton components were not perfectly reconstructed, their composite effect was very accurately reconstructed as compared to the ground truth and the dual-energy CT counterpart. This means that our proposed method has an intrinsic benefit in beam-hardening correction and metal artifact reduction. However, since there are substantial errors in photoelectric and Compton scatter components obtained using our method, the performance in material decomposition with our method may not be as good as that with dual-energy CT. Additionally, the phantom is very mathematical, piecewise constant. Further work is needed to evaluate with more realistic phantoms.

The rest of the paper is organized as follows. In section 2, we introduce the data acquisition model, framelet-based image processing methods, and our proposed algorithm. In section 3, convergence of the algorithm is studied. In section 4, Experimental results are demonstrated. Conclusions are given in section 5.

2. Models and methods

2.1. Models for reconstruction and data acquisition

We will use a polychromatic acquisition model in our algorithm. Consider an x-ray tube with a spectral curve $I_0(E)$ as in figure 1. Here the smooth part is due to the effect of bremsstrahlung, while the spikes are from characteristic κ lines for certain atoms.

Let $\mu(r, E)$ be the x-ray attenuation coefficient at point $r \in \Omega$ for energy level E , where Ω is the image domain. It is known that the attenuation coefficient can be approximated by [21, 31, 32]

$$\mu(r, E) = \phi(r)E^{-3} + \theta(r)f_{\text{KN}}(E), \quad (1)$$

where the photoelectric component $\phi(r)$ and the Compton scatter component $\theta(r)$ are independent of energy E . Here $f_{\text{KN}}(E)$ is the Klein–Nishina function

$$f_{\text{KN}}(E) = \frac{1+\varepsilon}{\varepsilon^2} \left(\frac{2(1+\varepsilon)}{1+2\varepsilon} - \frac{1}{\varepsilon} \ln(1+2\varepsilon) \right) + \frac{1}{2\varepsilon} \ln(1+2\varepsilon) - \frac{1+3\varepsilon}{(1+2\varepsilon)^2}, \quad (2)$$

where $\varepsilon = E/511$ keV.

Let L be a straight line passing through the object Ω presenting an x-ray beam. Denote by $I_0(E)$ the initial intensity of beam L with photon energy E . Then the intensity measured by a detector after beam L passing through Ω is [33]

$$I = \int_0^\infty I_0(E) \exp \left(- \int_L \mu(r, E) dr \right) dE. \quad (3)$$

From (3) one can see that the reconstruction of $\mu(r, E)$ from I is a nonlinear inverse problem. In the standard CT theory, (3) is changed to a linear inverse problem by the mean value theorem of integrals

$$I = \exp \left(- \int_L \mu(r, \bar{E}) dr \right) \int_0^\infty I_0(E) dE, \quad (4)$$

where $\bar{E}$ is a so-called equivalent energy. The spectral information is lost, however, in $\mu(r, \bar{E})$.

The subject goal of this paper is to develop an iterative reconstruction algorithm solving (3) without linearizing it to (4). As described in [26], with the same notation in a discrete setting, we denote by E_k , $k = 1, \dots, K$, the photon energy levels E , by y_i , $i = 1, \dots, M$, the measured intensity data I , and by μ_{jk} , $j = 1, \dots, J$ and $k = 1, \dots, K$, the linear attenuation coefficients $\mu(r, E)$. Here i is the index for x-ray beams L , j is the index for points r , and k is the index for energy levels E .

Now (3) can be discretized to give the expected number of photons detected along projection line i as

$$\hat{y}_i = \sum_{k=1}^K b_{ik} \exp \left(- \sum_{j=1}^J l_{ij} \mu_{jk} \right). \quad (5)$$

Here $b_{ik} = I_{ik} S_k E_k$ is the total intensity detected by detector for projection line i in the absence of absorber for incident photons of energy E_k , where I_{ik} is the incident number of photons, and S_k is a detector sensitivity factor. The factor E_k is used here because detectors of CT (most CT machines use scintillation-type detectors) are energy-counting, while in nuclear medicine photon-counting detectors are used. The measurement data y_i approximately follow a Poisson distribution [34] with expectation equal to $\hat{y}_i$, where the probability density function is

$$p(y_i) = \frac{\hat{y}_i^{y_i}}{y_i!} \exp(-\hat{y}_i).$$

Since for any i , y_i is an independent Poisson random variable with expectation $\hat{y}_i$, the log-likelihood is given by

$$L = \sum_{i=1}^M (y_i \log(\hat{y}_i) - \hat{y}_i). \quad (6)$$

As in [21, 26, 31, 32] we decompose μ_{jk} according to (1):

$$\mu_{jk} = \phi_j \Phi(E_k) + \theta_j \Theta(E_k), \quad (7)$$

where $\Phi(E_k) = E_0^3 / E_k^3$ for a reference energy E_0 , say $E_0 = 70$ keV, $\Theta(E_k) = f_{\text{KN}}(E_k) / f_{\text{KN}}(E_0)$, with f_{KN} as in (2). Under this normalization, ϕ_j and θ_j in (7) are actually the photoelectric and Compton scatter terms at energy E_0 , respectively. Now the number of unknowns ϕ_j and θ_j is $2J$. To further simplify the problem and reduce the number of unknowns, De Man *et al* [26] applied the least squares fit to ϕ_j and θ_j , $j = 1, \dots, J$, and obtained an analytic expression of ϕ_j and θ_j with respect to μ_j , so they reduced the number of unknowns to J .

In our study, we do not reduce the number of unknowns ϕ_j and θ_j in order to retain full spectral information. Substituting (7) into (5) we get an acquisition model

$$\hat{y}_i = \sum_{k=1}^K b_{ik} \exp \left(- \sum_{j=1}^J l_{ij} (\phi_j \Phi(E_k) + \theta_j \Theta(E_k)) \right). \quad (8)$$

Substituting (8) into (6), we further get the log-likelihood $L(\phi, \theta)$ on ϕ and θ , which are column vectors of ϕ_j and θ_j for $j = 1, \dots, J$, respectively.

2.2. Tight frame and framelets

In this subsection, we give a brief introduction to tight frame and framelet. They will be used for sparse representation of the energy-dependent attenuation coefficients.

Let $\mathcal{H}$ be a Hilbert space. A countable subset $\mathcal{F} \subset \mathcal{H}$ is called a frame if there are positive constants $C_2 \leq C_1$ such that

$$C_2 \|f\|^2 \leq \sum_{g \in \mathcal{F}} |\langle f, g \rangle|^2 \leq C_1 \|f\|^2, \quad \forall f \in \mathcal{H}, \quad (9)$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the inner product and the norm for the Hilbert space, respectively. If $C_1 = C_2$, the system $\mathcal{F}$ is called a tight frame. If (9) holds for $C_1 = C_2 = 1$, the tight frame is called a normalized frame or a Parseval frame. Thus

$$\|f\|^2 = \sum_{g \in \mathcal{F}} |\langle f, g \rangle|^2, \quad \forall f \in \mathcal{H}, \quad (10)$$

which is equivalent to

$$f = \sum_{g \in \mathcal{F}} \langle f, g \rangle g, \quad \forall f \in \mathcal{H}. \quad (11)$$

An orthonormal basis is a normalized tight frame, because conditions (10) and (11) hold for any orthonormal bases in $\mathcal{H}$. A normalized tight frame is, however, not necessarily an orthonormal basis. The analysis operator (decomposition operator) $D: \mathcal{H} \mapsto \ell_2(\mathbb{Z})$ is defined by $Df = \{\langle f, g \rangle\}_{g \in \mathcal{F}}$ for any $f \in \mathcal{H}$, while the synthesis operator $R: \ell_2(\mathbb{Z}) \mapsto \mathcal{H}$ is defined by $Rw = \sum_{g \in \mathcal{F}} w_g g$ for any $w_g \in \ell_2(\mathbb{Z})$. It is clear that $R = D^*$, and (11) implies $D^*D = I$. A normalized tight frame system can be over complete and redundant. The advantage of tight frame system over orthonormal basis for image representation is discussed in [29].

There are many tight frame systems that can sparsely represent piecewise smooth functions, including framelets and curvelets. Following our previous experiences in tight frame based image processing [29, 30], we use framelets introduced in [35] to sparsely represent the photoelectric part and Compton scatter part of the attenuation coefficients. We omit the details and interested readers may consult [30, 35, 36].

Instead, we focus on the description of discrete framelet transforms, as numerical computations are eventually performed in the discrete form. Each framelet system is associated with a family of discrete sequences $\{\mathbf{h}_i\}_{i=0}^s$, called filters. Among them, $\mathbf{h}_0$ is called the low-pass filter, and the others high-pass filters. The one level non-downsampling discrete framelet transform is given by

$$D = \begin{bmatrix} \mathbf{h}_0^* \\ \vdots \\ \mathbf{h}_s^* \end{bmatrix}, \quad (12)$$

where $\mathbf{h}_i^*$, $i = 0, \dots, s$, is the matrix form of the discrete convolution with kernel $\mathbf{h}_i$. If we want to get a multi-level decomposition, we just need to apply the matrix D in (12) to $\mathbf{h}_0^*$ recursively. When the filters satisfies a unitary extension principle [36], the rows of D will form a normalized tight frame.

There is a family of framelet systems constructed in [35] from B-splines with different orders. In the univariate case, the piecewise constant B-spline framelet system is the well-known Haar wavelet system, whose filters are

$$h_0 = \frac{1}{2}[1, 1], \quad h_1 = \frac{1}{2}[1, -1].$$

Filters of the framelet from piecewise linear B-spline are

$$h_0 = \frac{1}{4}[1, 2, 1], \quad h_1 = \frac{\sqrt{2}}{4}[1, 0, -1], \quad h_2 = \frac{1}{4}[-1, 2, -1].$$

Filters for higher order B-spline framelets can be found in [35]. Our attenuation coefficients have two spatial directions. Therefore, bivariate framelet system will be used in our experiments. In this case, the filters are obtained by tensor products of the corresponding univariate filters. In particular, the filters in bivariate Haar wavelet system are

$$\begin{aligned} h_0 &= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad h_1 = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad h_2 = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \\ h_3 &= \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

This framelet system will be used in our experiments. Filters for higher-order bivariate framelet systems can be constructed in a similar way.

2.3. Method based on framelets

For applications in medical imaging we may assume x-ray attenuation coefficients for a patient to be piecewise constant, piecewise linear, or piecewise smooth to a certain degree. This allows us to use the framelet systems introduced above.

Note that ϕ_j and θ_j are required to be non-negative. Moreover, they should fit the polychromatic data acquisition model (5) with Poisson noise. Second, they should be non-negative. Thirdly they should be sparse in the framelet domain. Let $\phi^n = (\phi_j^n)$ and $\theta^n = (\theta_j^n)$ be the unknowns in the n th iterate. Our iterative algorithm forces ϕ^n and θ^n to satisfy these three conditions.

To fit the model (5) with Poisson noise, we need to minimize $-L(\phi, \theta)$ as given in (6) and (8). A well-known method to minimize $-L(\phi, \theta)$ is the scaled gradient descent (see [2, 37–41]). Motivated by this method, we set

$$\begin{cases} \phi_j^{n+\frac{1}{3}} = \phi_j^n + \delta_1 \Delta \phi_j^n, \\ \theta_j^{n+\frac{1}{3}} = \theta_j^n + \delta_2 \Delta \theta_j^n, \end{cases} \quad (13)$$

where

$$\begin{cases} \Delta \phi_j^n = -\frac{\partial L}{\partial \phi_j}(\phi^n, \theta^n) \left(\sum_{h=1}^J \left(\frac{\partial^2 L}{\partial \phi_j \partial \phi_h} + \frac{\partial^2 L}{\partial \phi_j \partial \theta_h} \right) (\phi^n, \theta^n) \right)^{-1}, \\ \Delta \theta_j^n = -\frac{\partial L}{\partial \theta_j}(\phi^n, \theta^n) \left(\sum_{h=1}^J \left(\frac{\partial^2 L}{\partial \theta_j \partial \phi_h} + \frac{\partial^2 L}{\partial \theta_j \partial \theta_h} \right) (\phi^n, \theta^n) \right)^{-1}. \end{cases} \quad (14)$$

Next, we project the $\phi_j^{n+\frac{1}{3}}$ and $\theta_j^{n+\frac{1}{3}}$ onto the non-negative cone to enforce the non-negativity

$$\phi_j^{n+\frac{2}{3}} = \max \left\{ \phi_j^{n+\frac{1}{3}}, 0 \right\}, \quad \theta_j^{n+\frac{2}{3}} = \max \left\{ \theta_j^{n+\frac{1}{3}}, 0 \right\}. \quad (15)$$

Finally, we promote the sparsity of $\phi^{n+\frac{2}{3}}$ and $\theta^{n+\frac{2}{3}}$ in the framelet domain as follows. We transform $\phi^{n+\frac{2}{3}}$ and $\theta^{n+\frac{2}{3}}$ to the framelet domain, set small coefficients to 0 by soft-thresholding, and transform the sparsified coefficients back to the (ϕ, θ) domain. More precisely,

$$\phi_j^{n+1} = \mathcal{W}^T T_{\lambda_1}(\mathcal{W} \phi^{n+\frac{2}{3}}), \quad \theta_j^{n+1} = \mathcal{W}^T T_{\lambda_2}(\mathcal{W} \theta^{n+\frac{2}{3}}), \quad (16)$$

where $T_\lambda(\cdot)$ is the soft-thresholding operator defined by $T_\lambda(x) = \frac{|x|-\lambda}{|x|}x$ if $|x| > \lambda$, and zero otherwise. Process (16) can also be viewed as a denoising procedure using framelet sparsity, and it was revealed in [29] that the solution minimizes an objective function involving a balanced sparsity of framelet coefficients.

Steps (13)–(16) are not necessarily performed in a specific order. Starting from initial values ϕ_j^0 and θ_j^0 , (13)–(16) as a group will be repeated until a termination criterion is met. The termination criterion used in our simulations and real data computation below is whether $\|\hat{y} - y\|_2 / \|y\|_2$ is 0.025.

Note that (14) follows from the strategy in [40, p 736]. In fact, we may rename ϕ_j and θ_j , $j = 1, \dots, J$, as x_p , $p = 1, \dots, 2J$, and write (8) as

$$\hat{y}_i = \sum_{k=1}^K b_{ik} \exp \left(- \sum_{j=1}^{2J} a_{ilk} x_l \right). \quad (17)$$

3. Convexity of the negative log-likelihood

Theorem 3.1

The negative log-likelihood $-L(\phi, \theta)$ as defined in (6) and (8) is convex in the region $y_i = \hat{y}_i$, $i = 1, \dots, M$.

Proof—Expressing $\hat{y}_i$ as in (17), the Hessian matrix of $-L(\phi, \theta)$ is $H = \left[-\frac{\partial^2 L}{\partial x_i \partial x_j} \right]_{i,j=1}^{2J}$. We need to show that H is positive semi-definite. That is, for any column vector $z = (z_i)_{i=1}^{2J}$, $z^T H z \geq 0$.

We have the following derivatives:

$$\begin{aligned} \frac{\partial \hat{y}_i}{\partial x_j} &= - \sum_k b_{ik} a_{ijk} \exp \left(- \sum_{l=1}^{2J} a_{ilk} x_l \right), \\ \frac{\partial^2 \hat{y}_i}{\partial x_h \partial x_j} &= \sum_k b_{ik} a_{ihk} a_{ijk} \exp \left(- \sum_{l=1}^{2J} a_{ilk} x_l \right), \\ \frac{\partial}{\partial x_j} \log(\hat{y}_i) &= \frac{1}{\hat{y}_i} \frac{\partial \hat{y}_i}{\partial x_j}, \\ \frac{\partial^2}{\partial x_h \partial x_j} \log(\hat{y}_i) &= - \frac{1}{\hat{y}_i^2} \frac{\partial \hat{y}_i}{\partial x_h} \frac{\partial \hat{y}_i}{\partial x_j} + \frac{1}{\hat{y}_i} \frac{\partial^2 \hat{y}_i}{\partial x_h \partial x_j}, \end{aligned}$$

and

$$\begin{aligned} - \frac{\partial L}{\partial x_j} &= \sum_{i=1}^M \left(1 - \frac{y_i}{\hat{y}_i} \right) \frac{\partial \hat{y}_i}{\partial x_j}, \\ - \frac{\partial^2 L}{\partial x_h \partial x_j} &= \sum_{i=1}^M \left(1 - \frac{y_i}{\hat{y}_i} \right) \frac{\partial^2 \hat{y}_i}{\partial x_h \partial x_j} + \frac{y_i}{\hat{y}_i^2} \frac{\partial \hat{y}_i}{\partial x_h} \frac{\partial \hat{y}_i}{\partial x_j}. \end{aligned} \quad (18)$$

Since y_i and $\hat{y}_i$ are all nonnegative

$$\frac{y_i}{\hat{y}_i^2} z^T \left[\frac{\partial \hat{y}_i}{\partial x_h} \frac{\partial \hat{y}_i}{\partial x_j} \right] z = \frac{y_i}{\hat{y}_i^2} \left(z_1 \frac{\partial \hat{y}_i}{\partial x_1} + \dots + z_{2J} \frac{\partial \hat{y}_i}{\partial x_{2J}} \right)^2 \geq 0. \quad (19)$$

On the other hand

$$\begin{aligned}
{}^t z \left[\frac{\partial^2 \hat{y}_i}{\partial x_h \partial x_j} \right] z &= {}^t z \left[\sum_k b_{ik} a_{ihk} a_{ijk} \exp \left(-\sum_l a_{ilk} x_l \right) \right] z \\
&= \sum_k b_{ik} (z_1 a_{i1k} + \cdots + z_{2J} a_{i,2J,k})^2 \exp \left(-\sum_l a_{ilk} x_l \right) \\
&\geq 0.
\end{aligned} \tag{20}$$

By (18), (19) and (20),

$${}^t z H z = \sum_{i=1}^M \left(1 - \frac{y_i}{\hat{y}_i} \right) {}^t z \left[\frac{\partial^2 \hat{y}_i}{\partial x_h \partial x_j} \right] z + \frac{y_i}{\hat{y}_i^2} {}^t z \left[\frac{\partial \hat{y}_i}{\partial x_h} \frac{\partial \hat{y}_i}{\partial x_j} \right] z \tag{21}$$

is ≥ 0 for all vectors z if $y_i \leq \hat{y}_i$, $i = 1, \dots, M$.

Initial values $\phi_j^0 = 0$ and $\theta_j^0 = 0$ will always fulfill the condition $y_i \leq \hat{y}_i$, $i = 1, \dots, M$, because zero ϕ_j and θ_j correspond to x-ray beams passing through the empty space. Consequently one may always start the iteration process from zero initial values. In the following theorem, we will prove that iterative $\hat{y}_i^n$ form a decreasing convergent sequence whose limit is y_i . This is a proof of the convergence of our algorithm without considering the effect of the nonnegativity and sparsification procedures.

Theorem 3.2

Suppose that the initial values of ϕ_j^0 and θ_j^0 , $j = 1, \dots, J$, are chosen so that $y_i \leq \hat{y}_i$, $i = 1, \dots, M$. Then the iterative algorithm as outlined in section 2.3 produces a decreasing sequence of $\hat{y}_i^n$ approaching the measured data y_i .

Proof—Note that

$$-\frac{\partial L}{\partial \phi_j} \leq 0, \quad -\frac{\partial L}{\partial \theta_j} \leq 0, \quad \sum_{h=1}^J \left(\frac{\partial^2 L}{\partial \phi_j \partial \phi_h} + \frac{\partial^2 L}{\partial \phi_j \partial \theta_h} \right) (\phi^n, \theta^n) < 0, \quad \sum_{h=1}^J \left(\frac{\partial^2 L}{\partial \theta_j \partial \phi_h} + \frac{\partial^2 L}{\partial \theta_j \partial \theta_h} \right) (\phi^n, \theta^n) < 0.$$

Then by (14) $\Delta \phi_j^n \geq 0$ and $\Delta \theta_j^n \geq 0$. Hence ϕ_j^n and θ_j^n are monotone increasing at every iteration. This implies that $\hat{y}_i^n$ is a monotone decreasing for each iteration. Since y_i is a lower bound for the decreasing sequence $\hat{y}_i^n$, our iterative algorithm converges, if we ignore the effect of (15) and (16).

4. Experimental results

In this section, we present our results on some synthesized data and real data scanned by a CT scanner.

4.1. Experiments with synthesized data

In this subsection, we show our results for synthesized data. We use the 2D Shepp–Logan phantom, a widely-used synthesized phantom for testing algorithms in CT. There are several ellipses in the phantom. In our simulations, we place different substances in the ellipses to generate our test phantom: the most outer ellipsoidal shell is filled by bone, and the majority part of the phantom by plexiglas; the two slant ellipses are air, and the remaining ellipses are made of aluminium respectively. We list in table 1 the intensities [26] of different substances placed in the Shepp–Logan phantom, which are used for generating the underlying photoelectric coefficients ϕ and Compton scatter coefficients θ . See figures 2(a) and (b) for the underlying ϕ and θ of our test phantom.

We use a fan-beam imaging geometry similar to the one used by Yu *et al* [42]. In this system, the circular scanning locus is of radius 57 cm, and the diameter of the FOV is 22.4 cm corresponding to 672 detector bins, each element of which is 0.033 cm. The resolution of reconstructed images is 256×256 , so that each pixel has a size of $0.078125 \text{ cm} \times 0.078125 \text{ cm}$. In our simulations, we use 180 views for a 360° scan for our test phantom. That is to say, the system matrix has 672×180 rows and 256×256 columns. This system matrix is energy-independent and determined by the CT system, whose element is the length of the intersection between one x-ray and one pixel of the scanned object (corresponding to the I_{ij} in (5)). We use 21 spectra of x-ray at 110 kV in figure 1(a) from 20 to 120 keV. The photon number is approximately 1.3×10^6 . All simulated measurements are contaminated by Poisson noise with the mean in (5). We use our proposed algorithm to reconstruct the photoelectric coefficients ϕ and Compton scatter coefficients θ . We choose step size $\delta_1 = 0.5$ for ϕ and $\delta_2 = 1$ for θ for our phantom. The threshold parameters λ_1 and λ_2 are chosen empirically. We stop the iteration after 3000 steps.

The reconstructed images by our algorithm are shown in figures 2 and 3. From figures 2(d)–(f), we can see that attenuation coefficients were reconstructed very well using our algorithm assuming only one scan. This shows that our algorithm can be applied to reconstruct piecewise constant structures. In particular, from the reconstruction result of aluminium, we can see that our algorithm can reduce metal artifact well. All reconstructed images of our phantom are kept in good shape.

Although the components ϕ and θ were not perfectly reconstructed, the composite effect in terms μ was very accurately reconstructed. To further illustrate the effectiveness of our method, profiles of the 64th row of the reconstructed ϕ and θ of our test phantom are plotted in figures 3(a) and (b). One can observe that all the profiles of the reconstructed ϕ and θ are very close to those of the corresponding phantom, which can further validate our conclusion made from the reconstructed images. In figures 3(a) and (b) for our test phantom, the profile is very steady and close to that of the real data, which shows that most of the artifacts are removed using our method.

Finally, to demonstrate the advantage of our algorithm over the dual-energy CT image reconstruction, we also perform simulation experiment for the image reconstruction of dual-energy CT using two spectral measurements. The reconstruction model is based on the x-ray linear attenuation composed by photoelectron absorption and Compton scattering,

$$\mu(r, E) = \phi(r)\Phi(E) + \theta(r)\Theta(E). \quad (22)$$

Here two energy spectra of the x-ray source operated at 110 and 80 kVp, and the photon numbers 1.3×10^6 and 1.7×10^6 respectively. The projection measurements of two spectra were simulated based on physical model (3). From projection measurements and (3), the corresponding projections of photoelectric image $\phi(r)$ and the Compton scatter image $\theta(r)$ can be computed using optimization method. Then, photoelectric image and the Compton scatter image were respectively reconstructed from their projections using simultaneous algebraic reconstruction technique (SART). Then, the corresponding attenuation coefficient $\mu(r, E)$ at each energy bin was computed from (22).

Dual-energy CT used the same projection views as those in our method. The results of dual-energy method are shown in figures 2(g)–(i). From figures 2(g)–(i), we can see there are still artifacts in the reconstructed images. In figures 3(a)–(c), the profiles of dual-energy CT are not very steady, especially in the Al regions. In contrast, the reconstructed μ with our method is much closer to the real data.

By our proposed algorithm, the relative errors of ϕ , θ and μ at monochromatic 70 keV are 0.4295, 0.1649 and 0.0218 respectively. By dual-energy method, the relative errors of ϕ , θ and μ at monochromatic 70 keV are 0.0814, 0.0313 and 0.0448. Although the relative errors of ϕ and θ by our method are much larger than those of dual-energy CT, the relative error of μ is smaller than that of dual-energy CT.

4.2. Experiments on real-world data

In this section, a real sheep head was scanned using a cone-beam imaging geometry. The sheep head was bought dead at a local meat market. The distances from the x-ray source to the rotation center and the detector are 109.76 cm and 139.83 cm respectively. We collected 1195 projections under a fast-continuous-rotation scanning mode. Then the effective projection sequence used for reconstruction was extracted with a structure similarity (SSIM) based method [43]. Among the final effective projection sequence, 200 equispaced projections were used for reconstruction. In this case, we just reconstructed the central slice with our algorithm upon the 200 projections views. After the rotation center was located with the method in [44, 45], two 1024×1024 components were reconstructed and used to obtain any spectral images at any energy according to (7).

In our experiments in this section, we use 21 spectra of x-ray at 110 kV in figure 1(a) from 20 to 120 keV. The photon number is approximately 1.3×10^6 . We take 300 iterations, and the step sizes of ϕ and θ are 0.5 and 1.5 respectively. The reconstructed photoelectric component ϕ and Compton scatter component θ are shown in figures 4(a) and (b) respectively, the reconstructed attenuation coefficient $\mu(r, E)$ at monochromatic 70 keV is shown in figure 4(c), and the $\mu(r, E)$ with color overlay by $\mu(r, E)$ at monochromatic 40, 70, 100 keV is in figure 4(d). This is a real color CT image using just one scan, which is the main difference between our work and the previous work. For example, regions A and B in figure 4(c) appear to have indistinguishable gray levels. In figure 4(d), however, they have

different colors. And the reconstructed ϕ in figure 4(a) appears more details than that in the reconstructed μ in figure 4(c) from gray levels, but more details can be found in figure 4(d). The color overlay is implemented as follows. Set the reconstructed at monochromatic energy levels 40, 70, 100 as red, green, and blue respectively, and then linearly merge them and empirically adjusted the weights.

In order to further demonstrate the merit of our spectral CT reconstruction algorithm, we draw spectral curves of the attenuation coefficient $\mu(x, E)$ at two points in regions A and B (figures 4(c) and (d)). These spectral curves (figure 5) clearly differentiate the spectral nature of the two regions. The spectral curves produced by our algorithm are a valuable tool in analyzing and diagnosing the nature of a tumor.

5. Conclusion

In this paper, we develop a framelet-based iterative algorithm for polychromatic CT, which can reconstruct two components using just one scan. Excellent results are obtained in simulations with synthesized and real data. It is also shown that our algorithm can eliminate both beam-hardening and metal streak artifacts effectively.

We have only showed, however, the convergence and stability of our algorithm without considering the effect of sparsification promotion in the framelet domain. We leave it as a question for further study.

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Figure 1.
X-ray tube spectrum distribution: (a) x-ray spectra at 110 kV; (b) x-ray spectra at 80 kV.

Figure 2.

Shepp–Logan phantom and reconstructed images (a)–(i) of our test phantom: (a)–(c) The simulated ϕ , θ and μ respectively; (d)–(f) The reconstructed ϕ , θ and μ using our proposed method respectively; (g)–(i) The reconstructed ϕ , θ and μ using the dual-energy CT at 80 and 110 kVp respectively.

Figure 3.

Profiles of our test phantom on the 64th row: (a) profile of reconstructed ϕ ; (b) profile of reconstructed θ ; (c) profile of reconstructed μ at reference energy E_0 .

Figure 4.

Reconstructed images (a)–(d) of real data: (a) reconstructed photoelectric component at monochromatic 70 keV; (b) reconstructed Compton scatter component at monochromatic 70 keV; (c) reconstructed attenuation coefficient $\mu(r, E)$ at monochromatic 70 keV; (d) image with color overlay by the attenuation coefficient at monochromatic 40, 70 and 100 keV.

Figure 5. Spectral curves of the attenuation coefficient $\mu(r, E)$ in regions A and B as marked in figures 4(c) and (d).

Table 1

Simulated ϕ , θ and $\mu_{70 \text{ keV}}$ for bone, air, plexiglas, and aluminium.

Substances	ϕ (1/cm)	θ (1/cm)	$\mu_{70 \text{ keV}}$ (1/cm)
<i>Bone</i>	0.1757	0.3109	0.4974
<i>Air</i>	1.7e-05	0.0002	0.0002
<i>Plexiglas</i>	0.0107	0.2072	0.2187
<i>Al</i>	0.2125	0.4274	0.6523