Published in final edited form as:

IEEE Signal Process Lett. 2017 July ; 24(7): 1078–1082. doi:10.1109/LSP.2017.2702004.

Noise Correlation Effect on Detection: Signals in Equicorrelated or Autoregressive(1) Gaussian

Hadi Kasasbeh, Ramanarayanan Viswanathan, and Lei Cao

The authors are with the Department of Electrical Engineering, University of Mississippi, University, MS 38677

Abstract

In this letter, we consider the effect of noise correlation on the error performance of binary hypothesis signal detection, when one of two deterministic signals is received in correlated Gaussian noise. For the likelihood ratio detection scheme, analytical performance results are derived for equicorrelated and autoregressive order one models. Although it is known previously that the best signal lies in the direction of eigenvector corresponding to the minimum eigenvalue of the noise covariance matrix, our investigation of the variation of mean signal-to-noise power ratio as a function of correlation parameter (i) shows how correlation leads to increased probability of error up to a point, beyond which monotonic decrease in error probability with increasing correlation is possible and (ii) provides a max-min signal design solution for the unknown correlation parameter case. Numerical results are also included for some specific signals.

Keywords

Signal detection; Gaussian noise; multiple sensors; correlation; autoregressive

I. Introduction

A collection of sensors are employed in a variety of situations in order to enhance information gathering and processing operations. Depending upon the capability of sensor networks, either condensed (or pre-processed) information or the raw data from the sensor sites may be combined at a fusion center (FC). The former case is usually referred to as decentralized processing architecture, whereas the latter case is termed as the centralized processing architecture. When signal processing is carried out in order to ascertain whether a target of interest is present in the received signals at the sensor sites or not, the problem becomes a target detection problem.

In this work, we consider the detection of the presence of one of two possible deterministic signals in correlated Gaussian noise. A frequent question that arises is how the statistical correlation among the sensor observations affects the overall detection performance of a detection (centralized as well as decentralized) scheme? Only partial answers to this question seem to be available in the literature. Our study here for a restricted Gaussian observation provides new results for the centralized detection scheme. The literature on the effect of correlation on signal detection is sparse. One reason is that the type of correlation, i.e., the structure of covariance matrix, could be different depending on the models for

different scenarios. Second, analytical results are generally possible only for some structured covariance matrices and under asymptotic analysis involving a very large number of sensors [1] [2]. An analysis of decentralized detection of correlated Gaussian observations involving two sensors, each with one bit quantization, yielded some interesting and surprising results [3] [4].

In general, it can be expected that the positive correlation among the sensor observations may lead to poor performance when compared to the uncorrelated case, unless the signal vectors under the two hypotheses can be designed appropriately, as can be seen from the textbook example for the two sensor case in [5] [6] and more recently, for the general case of arbitrary number of sensors [7]. Optimum signal design requires complete knowledge of noise covariance matrix. In some situations, this information may be unavailable or partially known, and we would like to examine, for a given signal set, the variation of probability of detection error (P_e) at the FC, as a function of a correlation parameter. This approach provides a max-min solution to signal design that maximizes the minimum mean signal-to-noise power ratio, as the correlation parameter varies over its admissible interval.

II. Problem Formulation

Consider a set of M sensors monitoring a region of interest to ascertain the presence of one of two signals (hypothesis H_1 or H_0) of interest. The signals, which are assumed to be deterministic, are received in additive Gaussian noise, as shown in

$$H_1: Z = \chi_1 + V \quad (1)$$

$$H_0: Z = \chi_0 + V,$$

where $Z^T = [Z_1 \dots Z_M]$, $V^T = [V_1 \dots V_M]$, $\chi_1^T = [\chi_{11} \dots \chi_{1M}]$, $\chi_0^T = [\chi_{01} \dots \chi_{0M}]$. The noise at different sensors are correlated and represented by vector V , which is Gaussian distributed, with zero mean and a positive definite covariance matrix $C = [C_{ij}]$, $\{i, j\} \in \{1, \dots, M\}$. It is assumed that the signals χ_1 , χ_0 are completely known at the individual sensors and the fusion center. In the centralized test, all sensors send their observations (i.e Z_j , $j = 1, \dots, M$) to the FC through orthogonal and error-free reporting channels. Based on the available information at the FC, the FC makes a final decision on which hypothesis is true using a likelihood ratio test (LRT).

From standard results on LRT for the detection of deterministic signals in correlated Gaussian noise, we decide H_1 if

$$T \geq \gamma \equiv \frac{2 \log \lambda + \chi_1^T C^{-1} \chi_1 - \chi_0^T C^{-1} \chi_0}{2} \quad (2)$$

and H_0 otherwise, where $T = Z^T C^{-1}(\chi_1 - \chi_0)$ and λ equals the ratio of prior probabilities, namely $P(H_0)/P(H_1)$, for Bayes criterion, and equals the value that provides a desired false alarm probability, for Neyman-Pearson criterion. Let $W = C^{-1}(\chi_1 - \chi_0)$. Probabilities of

false alarm and miss detection are given as $P_F = Q\left(\frac{\gamma}{\sqrt{W^T C W}}\right)$ and

$P_M = Q\left(\frac{-\gamma + (\chi_1 - \chi_0)^T W}{\sqrt{W^T C W}}\right)$ respectively. $Q(\cdot)$ is the upper tail function of the standard Gaussian distribution.

III. Analysis of Centralized Detection Define

Define $K = \frac{(E(T|H_1) - E(T|H_0))^2}{\text{Var}(T|H_0)}$ as the (mean) signal-to-noise ratio (SNR), also referred to as the deflection coefficient. K can be further simplified as $K = W^T C W$. Without loss of generality, we may assume that under H_0 , $\chi_0^T = [0, \dots, 0]_{1 \times M}$ and under H_1 , χ_1 is replaced with $\chi_1 - \chi_0$. For simplicity, let $X = \chi_1 - \chi_0$ in the sequel. Then, the detection error probability is given as

$$P_e = P(H_1)Q\left(\frac{-\gamma + X^T W}{\sqrt{W^T C W}}\right) + P(H_0)Q\left(\frac{\gamma}{\sqrt{W^T C W}}\right) = P(H_1)Q\left(\frac{-2 \log(\lambda) + K}{2 \sqrt{K}}\right) + P(H_0)Q\left(\frac{2 \log(\lambda) + K}{2 \sqrt{K}}\right) \quad (3)$$

It can be shown that $\frac{dP_e}{dK} < 0$. Similarly, for the NP criterion, for a fixed value of P_F , $\frac{dP_M}{dK} < 0$. Hence, P_e is a monotonic decreasing function with the increase of SNR K , and $P_e \rightarrow 0$ as $K \rightarrow \infty$.

A. Equicorrelated Gaussian

In this case, the covariance matrix is specified by $C_{ii} = \sigma^2$, $C_{ij} = \sigma^2 \rho$, $i \neq j$. For convenience, σ^2 is taken to be unity in the sequel, as any SNR can be achieved by scaling appropriately the signal level. For this covariance matrix, its inverse, eigenvalues, and eigenvectors are all available in closed-form for finite M [8]. In fact, the $(i, j)^{\text{th}}$ element of the inverse matrix $D = C^{-1}$ is given by

$$D_{ij} = \begin{cases} \frac{1 + (M-2)\rho}{(1-\rho)(1+(M-1)\rho)} & i=j \\ \frac{-\rho}{(1-\rho)(1+(M-1)\rho)}, & i \neq j \end{cases}$$

Our goal is to study the effect of ρ on the detection error probability. By knowing C^{-1} , SNR K can be found as

$$K = b\alpha + (a - b)\beta \quad (4)$$

where

$$\alpha = \left(\sum_{j=1}^M X_j \right)^2, \beta = \sum_{j=1}^M X_j^2,$$

$$a = \frac{1 + (M - 2)\rho}{(1 - \rho)(1 + (M - 1)\rho)} \text{ and } b = \frac{-\rho}{(1 - \rho)(1 + (M - 1)\rho)}.$$

Lemma 1—For any real $X^T = [X_1, \dots, X_M]$, $0 \leq \frac{\alpha}{\beta} \leq M$.

Proof: The result can be obtained by using Cauchy-Schwartz inequality $|\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \cdot \langle \mathbf{v}, \mathbf{v} \rangle$ and setting $\mathbf{u} = X$ and $\mathbf{v} = \mathbf{1}^T$.

Because of a monotonic decreasing relationship between P_e and K , $\frac{dK}{d\rho} \geq 0$ is equivalent to $\frac{dP_e}{d\rho} \leq 0$, and vice versa. Hence from (4), K increases with ρ if

$$\frac{dK}{d\rho} = \frac{db}{d\rho}\alpha + \left(\frac{da}{d\rho} - \frac{db}{d\rho} \right) \beta > 0. \quad (5)$$

Consequently, we can arrive at the following equivalent condition,

$$F(\rho) \triangleq \frac{(1 + (M - 1)\rho)^2}{1 + (M - 1)\rho^2} > \frac{\alpha}{\beta}. \quad (6)$$

It can be seen that $F(\rho)$ is a monotonic increasing function of ρ and its minimum value of 0 is attained at the minimum value of $\rho = -\frac{1}{M-1}$, whereas its maximum value of M is attained when ρ takes its maximum value of 1. In addition, $F(0) = 1$. Using lemma 1 and (6), we have the following result.

Corrolary 1—The ρ that minimizes SNR K is given as the solution to:

$$F(\rho) \triangleq \frac{(1 + (M - 1)\rho)^2}{1 + (M - 1)\rho^2} = \frac{\alpha}{\beta}. \quad (7)$$

The above result shows that the correlation point ρ at which the SNR reaches its minimum value is identical for all signals X that have the same $\frac{\alpha}{\beta}$ value.

Proposition 1—For a good detection performance, it is desirable that $\frac{dP_e}{d\rho} < 0$ when $\rho > 0$, and $\frac{dP_e}{d\rho} > 0$ when $\rho < 0$.

The proposition simply indicates that the correlation introduced at the sensor sites can be exploited advantageously by a ‘good’ detection system, thereby providing improved performance beyond that achieved with zero ρ . It is known that, given the knowledge of covariance matrix, the signal X can be chosen as the vector proportional to the eigenvector corresponding to the minimum eigenvalue of the covariance matrix C so that K is maximized [5]– [7]. Using the eigenanalysis of the covariance matrix [8], we have the following lemma.

Lemma 2—For the equicorrelated case, the minimum eigenvalues, corresponding eigenvectors and SNRs satisfy the following conditions:

For $\rho > 0$, the minimum eigenvalue is $\lambda_{min} = 1 - \rho$ and the corresponding eigenvector ($V_{min} = [V_1, \dots, V_M]$) satisfies the condition $\sum_{j=1}^M V_j = 0$. Notice that the best signal X is proportional to V_{min} . Hence, this requires that $\sum_{j=1}^M X_j = 0$.

Setting $\alpha = 0$ in (4) we have $K = \frac{\beta}{1 - \rho}$ which is the maximum achievable SNR, for a given positive ρ . As $\rho \rightarrow 1$, we have $K \rightarrow \infty$ and $P_e \rightarrow 0$.

For $\rho < 0$, the minimum eigenvalue is $\lambda_{min} = (1 + (M - 1)\rho)$ and the corresponding eigenvector V_{min} satisfies the condition $V_1 = V_2 = \dots = V_M$. Hence, $X = X_1[1 \dots 1]^T$.

Putting the corresponding signal in (4), we have $K = \frac{\beta}{1 + (M - 1)\rho}$ which is the maximum achievable SNR, for a given negative ρ . Hence, as $\rho \rightarrow -\frac{1}{M - 1}$, $K \rightarrow \infty$ and $P_e \rightarrow 0$. We also see that when $\rho \rightarrow 0$, $K \rightarrow \beta = MX_1^2$.

A few remarks follow.

- (5) and (6) determine how the SNR (hence the detection error) varies with the variation of correlation for any given signal X . These results are new and have not been seen in the literature.
- What happens if the signal set is chosen as $X = X_1[1 \dots 1]^T$ while the correlation happens to be positive? In this case, (7) is not satisfied, i.e., $F(\rho) < \frac{\alpha}{\beta} = M$, for any $0 < \rho < 1$, and hence P_e will increase monotonically with respect to ρ .

Ultimately, the error rate will be equivalent to that of a single sensor case when $\rho = 1$ (perfect correlation). This result has been known for finite sample and asymptotic cases (e.g. [1] [7]) although the derivation of (5) and (6) that leads to this result is new.

- Next, consider the case of a general signal such that $0 < \frac{\alpha}{\beta} < M$. In this case, P_e increases with ρ until the point ρ^* at which $F(\rho^*) = \frac{\alpha}{\beta}$. When ρ is further increased above ρ^* , P_e decreases and reaches zero when $\rho = 1$ (perfect correlation). Hence, except for the identical signals case, a strong positive correlation is always beneficial. By analogy, we can also conclude that for negative correlation, except for the signal that aligns with the eigenvector corresponding to positive correlation condition, the probability of error will approach zero as ρ approaches $-\frac{1}{M-1}$.
- **Max-Min Signal Design**
Without knowing ρ , by utilizing (7), we can set up the following max-min problem to find the best signal X^* , among all possible signal sets, that maximizes the minimum SNR achieved, as ρ takes values over the interval $(\frac{-1}{M-1}, 1)$:

$$X^* = \arg \max_X (\arg \min_{\rho} (K)). \quad (8)$$

Solution: Solution to (7) provides $\rho^+ = \arg \min_{\rho} (K)$, with the corresponding K^+ obtained from (4) as follows:

$$K^+ = \frac{\beta}{1 + (M-1)(\rho^+)^2}. \quad (9)$$

If we scale each X_i by a constant $\sqrt{g}$, $g > 0$, we end up scaling both K and β by g . Hence, without any loss of generality, we can constrain the power of the signal X as unity, so that $\beta = \|X\|^2 = 1$. The value of ρ^+ that is to be used in (9) is the solution to (7) and hence, it is a function of α , (β fixed at 1). Hence, if there exists an $X^* = X$, which is captured by a particular α , ($\beta = 1$), for which $\rho^+ = 0$ is the solution to (7), then K^+ in (9) will be maximized. That is, the max-min solution to X in (8) is the solution to X in (7) with $\rho = 0$. Therefore, the max-min solution X^* satisfies $\frac{\alpha}{\beta} = 1$, which can be explicitly written as follows:

$$\left(\sum_{j=1}^M X_j^* \right)^2 = \sum_{j=1}^M X_j^{*2} \quad (10)$$

- If ρ is fixed at 0, the noise being Gaussian, the noise components at sensors are all independent and hence the optimal signal X could be any signal, with the SNR solely determined by β . However, for the max-min design, although the max-min point corresponds to $\rho = 0$, the optimal max-min signal X^* has to satisfy (10).

B. Autoregressive (AR(1)) Model

In this case, the covariance matrix is given by $C_{ii} = 1$, $C_{ij} = \rho^{|i-j|}$, i, j . This matrix was considered previously in asymptotic analysis involving a large number of sensors [2], [9], [10]. For this Toeplitz matrix, the inverse is given as follows:

$$C^{-1} = \frac{1}{(1-\rho^2)} \begin{bmatrix} 1 & -\rho & 0 & \dots & 0 \\ -\rho & 1+\rho^2 & -\rho & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\rho & 1+\rho^2 & -\rho \\ 0 & \dots & 0 & -\rho & 1 \end{bmatrix}.$$

It is easy to show that the SNR (K) in this case is given as follows:

$$K = \frac{-\rho^2}{(1-\rho^2)}\gamma + \frac{-2\rho}{(1-\rho^2)}\eta + \frac{(1+\rho^2)}{(1-\rho^2)}\delta, \quad (11)$$

where $\gamma = X_1^2 + X_M^2$, $\eta = \sum_{j=1}^{M-1} X_j X_{j+1}$ and $\delta = \sum_{j=1}^M X_j^2$. Hence,

$$\frac{dK}{d\rho} = \frac{-2\rho}{(1-\rho^2)^2}\gamma + \frac{-2(1+\rho^2)}{(1-\rho^2)^2}\eta + \frac{4\rho}{(1-\rho^2)^2}\delta. \quad (12)$$

Therefore, $\frac{dP_e}{d\rho} < 0$ if

$$G(\rho) \triangleq \frac{\rho}{1+\rho^2} > \frac{\eta}{(2\delta - \gamma)}. \quad (13)$$

It can be seen that $G(\rho)$ is a monotonic increasing function of $-1 < \rho < 1$ and that its minimum value of $\frac{-1}{2}$ is approached as ρ approaches -1 , whereas its maximum value of $\frac{1}{2}$ is approached when ρ approaches 1. Also, $G(0) = 0$.

Lemma 3— $\frac{\eta}{(2\delta - \gamma)}$ is bounded over the interval $(\frac{-1}{2}, \frac{1}{2})$.

Proof: Using Cauch-Schwartz inequality: $\left(\sum_{j=1}^{M-1} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{M-1} a_j^2\right) \left(\sum_{j=1}^{M-1} b_j^2\right)$ to the sequences $a_j = X_j$, $b_j = X_{j+1}$, it can be seen that $\eta^2 \leq (\delta - X_M^2)(\delta - X_1^2)$. The bound follows by taking square root on both sides and then diving by $2\delta - \gamma$

Hence, for any given ρ , there exists a signal set that provides the best performance. For this AR(1) model, in general, no closed-form results for the minimum eigenvalue and the corresponding eigenvector are available. For given values of M and ρ , we rely on numerical analysis to design a best signal set and find the corresponding error performance.

Remarks:

- For $M = 3$, explicit solution for eigenvalues and eigenvectors can be obtained easily. One eigenvalue is $1 - \rho^2$. The minimum eigenvalue is given by

$$\lambda_{\min} = 1 - \frac{-\rho^2 |\rho| \sqrt{(\rho^2 + 8)}}{2} \text{ and a corresponding eigenvector satisfies } X^T = [x, x_2, x], \text{ where } x_2 = -x \operatorname{sign} \rho \left(\frac{(|\rho| + \sqrt{\rho^2 + 8})}{2} \right).$$

- Max-Min Signal Design

Proceeding along lines similar to the previous subsection, the max-min solution for the AR(1) model is given by $\rho = 0$ and $\eta = 0$. The corresponding signal X^*

$$\text{satisfies } \sum_{j=1}^{M-1} X_j^* X_{j+1}^* = 0.$$

IV. Numerical Results

The following numerical results are presented for the equicorrelated and AR(1) model cases. In all cases, $\sigma^2 = 1$, $P(H_1) = 0.5$ and two values of $M = 3$, $M = 6$ are considered. Also, for hypothesis H_0 , the signal was assumed to be zero. Then $X = \chi_1$. We programmed in *MATLAB*® to obtain the numerical results.

Scenario 1: Identical signals at all sensors

Signals under H_1 for the two values of M are assumed to be $\chi_1 = [1 \ 1 \ 1]$ and $\chi_1 = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$, respectively. Fig. 1(a) and 1(b) plot P_e against ρ for this scenario. According to corollary 1 for the equicorrelated case, notice that the signal set assumed is not optimal for ρ positive, but is optimal for negative correlation. In fact, for positive correlation, as ρ tends to 1, the error probability tends to the value achieved with a single sensor.

Scenario 2: Signals at sensors are not identical

We assume $\chi_1 = [1 \ 5 \ 3]$ and $\chi_1 = [1 \ 5 \ 3 \ 1 \ 5 \ 3]$. Fig. 2(a) and 2(b) plot P_e vs. ρ for this scenario. For the equicorrelated case, for ρ greater than a certain positive value, P_e decreases with ρ , thereby providing perfect detection as correlation coefficient approaches 1. A similar behavior is seen for the autoregressive case. As shown analytically, the peak value of P_e is obtained when (7) is satisfied for equicorrelated case (greater relation in (13) becomes equality for AR(1) model).

Scenario 3: Best signal sets

This scenario studies the behavior of P_e with respect to ρ when we have the ability to design the received signals at the sensors using the knowledge of the covariance matrix. Here, we constrain signal power as $\|\chi_1\|^2 = M$, with χ_1 as an eigenvector corresponding to the minimum eigenvalue of the covariance matrix. Fig. 3(a) and 3(b) plot P_e against ρ for the two correlation models.

In scenarios 1 and 3, the signal power $\|\chi_1\|^2$ equals M , whereas for scenario 2, it is a much higher value (equals 35 for $M=3$ and 70 for $M=6$). Hence, the probability of error is significantly lower in scenario 2. For scenario 3, Table I presents some best signal designs in the case of 3 sensors, for some values of ρ .

From Table I, for equicorrelation, we see that $\sum_{j=1}^M X_j$ is M for negative ρ and is 0 for positive ρ , thus agreeing with lemma 2. For positive correlation, $(M-1)$ eigenvalues are repeated at minimum values, thus the eigenvector corresponding to the minimum eigenvalue

lies in the spanned $(M-1)$ dimensional space. For negative correlation (recall $\rho > \frac{-1}{M-1}$), there is a single, non-repeated minimum eigenvalue. For the autoregressive model, results in Table I agree with the first remark under lemma 3.

V. Conclusion

In this letter, for the centralized detection of one of two deterministic signals in Gaussian noise, with either equicor-related covariance or AR(1) model covariance, we show analytically how the correlation between the sensors affects the detection performance for any signal set. We also derived a max-min solution for the signal set that achieves the maximum of minimum SNR, as ρ takes values over the admissible range. Numerical results are presented to show the behavior of P_e as a function of ρ , for representative signal sets.

Acknowledgments

This work is supported by NASA EPSCoR program under grants NNX13AB31A and NNX14AN38A.

References

1. Chamberland JF, Veeravalli VV. Decentralized detection in sensor networks. *IEEE Transactions on Signal Processing*. Feb; 2003 51(2):407–416.
2. Chamberland JF, Veeravalli VV. How dense should a sensor network be for detection with correlated observations? *IEEE Transactions on Information Theory*. Nov; 2006 52(11):5099–5106.
3. Willett P, Swaszek P, Blum R. The good, bad and ugly: distributed detection of a known signal in dependent gaussian noise. *IEEE Transactions on Signal Processing*. Dec; 2000 48(12):3266–3279.
4. Chen P-N, Papamarcou A. Likelihood ratio partitions for distributed signal detection in correlated gaussian noise. *Proceedings of 1995 IEEE International Symposium on Information Theory*. Sep. 1995 :118–.
5. Poor, HV. An introduction to signal detection and estimation. Springer Science & Business Media; 2013.
6. Kay, SM. Fundamentals of Statistical Signal Processing Volume II Detection Theory. Upper Saddle River, New Jersey 07458: Prentice Hall PTR; 1998.

7. Chepuri SP, Leus G. Sparse sensing for distributed detection. *IEEE Transactions on Signal Processing*. Mar; 2016 64(6):1446–1460.
8. Graybill, F. *Matrices with Applications in Statistics*. Wadsworth International Group; Belmont, CA: 1983.
9. Li W, Dai H. Distributed detection in wireless sensor networks using a multiple access channel. *IEEE Transactions on Signal Processing*. Mar; 2007 55(3):822–833.
10. Chamberland JF, Veeravalli VV. Wireless sensors in distributed detection applications. *IEEE Signal Processing Magazine*. May; 2007 24(3):16–25.

(a) Equicorrelation.

(b) Autoregressive.

Fig. 1.
 P_e vs. ρ with identical signals at sensors.

(a) Equicorrelation.

(b) Autoregressive.

Fig. 2.
 P_e vs. ρ for not all identical signals.

(a) Equicorrelation.

(b) Autoregressive.

Fig. 3.
 P_e vs. ρ with best signal design.

TABLE I

Best signals (χ_1^T) designed for $M=3$ in scenario 3.

ρ	Equicorrelation	Autoregressive
-0.999	-	[0.7071 1.4142 0.7071]
-0.7	-	[0.7549 1.3639 0.7549]
-0.5	[1 1 1]	[0.7871 1.3271 0.7871]
-0.3	[1 1 1]	[0.8191 1.2877 0.8191]
0	[1.7321 0 0]	[-0.8660 1.2248 - 0.8660]
0.3	[0.7071 0.7071 - 1.4142]	[-0.8191 1.2877 - 0.8191]
0.5	[0.7071 0.7071 - 1.4142]	[-0.7870 1.3271 - 0.7870]
0.7	[-1.2633 0.0812 1.1821]	[-0.7548 1.3640 - 0.7548]