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Motion of a Rigid Cylinder Between Parallel Plates in Stokes Flow: Part 1: Motion in A Quiescent Fluid and Sedimentation

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Abstract

A general numerical scheme for solution of two-dimensional Stokes equations in a multiconnected domain of arbitrary shape [1,2] is applied to the motion of a rigid circular cylinder between plane parallel boundaries. Numerically generated boundary-conforming coordinates are used to transform the flow domain into a domain with rectilinear boundaries. The transformed Stokes equations in vorticity-stream function form are then solved on a uniform grid using an iterative algorithm. In Part I coefficients of the resistance matrix representing the forces and torque on the cylinder due to its translational motion parallel or perpendicular to the boundaries or due to rotation about its axis are calculated. The solutions are obtained for a wide range of particle radii and positions across the channel. It is found that the force on a particle translating parallel to the boundaries without rotation exhibits a minimum at a position between the channel centerline and the wall and a local maximum on the centerline.

The resistance matrix is utilized to calculate translational and angular velocities of a free particle settling under gravity in a vertical channel. It is found that the translational velocity has a maximum at some lateral position and a minimum on the centerline; the particle angular velocity is opposite in sign to that of a particle rolling along the nearer channel wall except when the gap between the particle and the wall is very small. These results are compared with existing analytical solutions for a small cylindrical particle situated on the channel centerline, and with solutions of related 3-D problems for a spherical particle in a circular tube and in a plane channel. It is shown that the behavior of cylindrical and spherical particles in a channel in many cases is qualitatively different. This is attributed to different flow patterns in these two cases. The motion of a spherical particle in a circular tube has qualitative and quantitative features similar to those for a cylindrical particle in a plane channel.

1. Introduction

The problems of particle motion in channels of arbitrary shape are of considerable interest for such applications as hemorheology and chemical technology. For example, it is important to be able to predict trajectory of a particle in a bifurcating channel for particles of regular and irregular shapes. With this goal in mind, we have developed a numerical scheme capable of solving the 2-D equations for creeping flow in a multiconnected domain of arbitrary shape. In the present work we applied this general scheme to a problem of creeping flow around a circular cylinder situated between two plane parallel boundaries. Two

parameters characterizing the geometry of the problem, the ratio of the diameter of the cylinder to channel width, and the distance of the cylinder axis from the center of the channel, will be varied within a wide range. Despite its significance this fundamental problem has received relatively little attention in the past. The lack of a natural coordinate system for the problem makes it difficult to obtain an exact analytical solution to the problem. Howland and Knight [3] solved the governing biharmonic equation in the form of infinite series for the problem of a circular cylinder slowly rotating midway between the infinite plane parallel boundaries. The method of solution was limited to a symmetric location of the cylinder. The coefficients were reported for three values of the ratio of particle radius to channel width 0.15, 0.2 and 0.25; the torque on the cylinder for these values was compared with the torque acting on the cylinder rotating in unbounded fluid.

Other solutions of the flow around a cylindrical particle symmetrically situated in a channel were obtained by the method of reflections for small radius-to-width ratios. Harrison [4], Faxen [5] and Takaisi [6] calculated the flow field around a circular cylinder placed in a Poiseuille flow midway between two parallel planes. Faxen [5] and later Takaisi [7] considered flow around a cylinder moving perpendicular to its axis midway between two parallel fixed planes. The cylinder was translating parallel to the channel walls; the fluid far from the cylinder was assumed to be at rest. The values for the drag on the cylinder computed in these two studies disagree for radius-to-width ratios greater than 0.05. The drag on a cylinder translating in a quiescent fluid in the direction perpendicular to plane channel walls was determined by Westberg [8], for the case when the particle center is situated on the channel centerline.

The only solution for the asymmetric case was obtained by Harper and Chang [9]. The authors utilized the method of matched asymptotic expansions to obtain solution for radius-to-width ratios smaller than 0.01. The results are in agreement to the order given with those of Faxen [5] and Takaisi [6] for the symmetric case. It may be noted that Faxen obtained a higher order approximation than that of Takaisi and Harper and Chang and hence his solution has a wider range of validity.

The methods used in the previous studies yield accurate solutions only in limiting cases when the ratio of particle size to channel width is small and the particle is far from the channel walls; the series expansions become divergent when these conditions are violated.

In the present study the numerical procedure [10] utilizing a numerically generated boundary-conforming coordinate system is applied to solve a variety of flow problems for the geometry of a circular cylinder between infinite plane parallel boundaries. In Part I the results are presented for the force and torque on a particle translating parallel or perpendicular to the walls or rotating about its axis in otherwise quiescent fluid. These results are then used to calculate the translational and angular velocities of the particle settling under gravity in a vertical channel. In Part II [1] the motion of a circular particle in Poiseuille and Couette flow between parallel plates is considered, and the force and torque on a stationary particle, as well as the translational and angular velocities for a free particle, are calculated.

2. Mathematical Statement of the Problem

We consider the creeping flow around a cylindrical particle of radius R' in a channel of width H , Fig. 1. We render all variables dimensionless using H as the characteristic length scale and U as the characteristic velocity; U will be defined in the particular examples considered below. The momentum and continuity equations in the dimensionless variables are:

$$-\nabla p + \nabla^2 \mathbf{V} = 0 \quad (2.1a)$$

$$\nabla \cdot \mathbf{V} = 0 \quad (2.1b)$$

where $p = (p' H) / (\mu U)$ is the dimensionless modified pressure (i.e. it includes the gravitational term) and

$\mathbf{V} = \mathbf{V}' / U$, is the dimensionless velocity.

Introducing the dimensionless vorticity, $\omega = \nabla \times \mathbf{V}$, $\omega = (\omega' H) / U$, and the stream function ψ given by:

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}, \quad (2.2)$$

where $\psi = \psi' / UH$, eqn (2.2) can be presented in the vorticity-stream function form:

$$\nabla^2 \omega = 0 \quad (2.3)$$

$$\nabla^2 \psi = -\omega \quad (2.4)$$

The system of eqns (2.3) and (2.4) is solved subject to the appropriate boundary conditions. The force and torque on the particle and pressure drop in the channel are obtained from the calculated vorticity-stream function fields.

The no-slip and impermeability boundary conditions are imposed on the particle surface and channel walls. At the inlet and outlet sections of the channel “infinity” boundary conditions are imposed. In the numerical solution of eqns (2.3) and (2.4) the infinity boundary conditions are imposed at a finite distance from the particle. Typically, 1.5–3 channel widths on each side of the particle are sufficient for the calculated characteristics to be independent of channel length.

Once the solutions to eqns (2.4) and (2.5) are obtained, the force and torque on the particle are computed by integrating over the particle surface; it can be shown (e.g. Ref. [10]) that the components of the force and torque on the particle can be expressed as:

$$F_1 = \int (p dy - \omega dx) \quad (2.5)$$

$$F_2 = - \int (p dx - \omega dy) \quad (2.6)$$

$$T = - \int (px + 2u + \omega y) dx - \int (py - 2v + \omega x) dy \quad (2.7)$$

where F_1 is the x component, and F_2 is the y component of the force, $F = F'/\mu U$ and torque, $T = T'H(\mu U)$.

From the linearity of the governing equations and the boundary conditions it follows that the force and torque on the particle can be presented as a sum of the following terms: (a) a translational part due to the particle motion with velocity (U_c, V_c) with the fluid velocity being zero at all other boundaries; (b) a rotational part due to the particle rotation with angular velocity Ω with the fluid velocity being zero at all other boundaries, and (c) a part due to the motion of fluid at the boundaries of the region while the particle is at rest. Thus the force and torque exerted by the fluid on the particle can be expressed as a sum

$$\begin{pmatrix} F_1 \\ F_2 \\ T \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} U_c \\ V_c \\ \Omega \end{pmatrix} + \begin{pmatrix} F_{10} \\ F_{20} \\ T_0 \end{pmatrix} \quad (2.8)$$

where the elements of the resistance matrix A_{ij} and the elements of the last column are functions of the geometry of the problem. It has been shown [11,12] that the certain conditions, which are satisfied in the present case, the matrix A_{ij} is symmetric. The variables F_{10} , F_{20} , T_0 are the components of the force and torque acting on a quiescent particle ($U_c = V_c = \Omega = 0$).

In order to determine the coefficients in eqn (2.8) the forces and torque on a quiescent particle in a moving fluid are first calculated, yielding the last column in eqn (2.8). To determine the rest of the coefficients, we calculate the forces and torque for three different sets of velocities $(U_{c1}, V_{c1}, \Omega_1)$, $(U_{c2}, V_{c2}, \Omega_2)$, and $(U_{c3}, V_{c3}, \Omega_3)$ and solve the set of eqns (2.8) for the nine coefficients A_{ij} ($i, j = 1, 2, 3$). In the present calculations the following sets were considered: $(U_{c1}, 0, 0)$, $(0, V_{c2}, 0)$, and $(0, 0, \Omega_3)$. It should be noted that, theoretically, there are only six independent coefficients in the matrix A_{ij} due to its symmetry. Therefore, it appears that only six equations generated by two sets of velocities are sufficient for their

determination. In reality, however, the forces and torques are calculated with certain numerical error. As a result, the computed matrix is not perfectly symmetric and it is necessary to determine all nine coefficients of the matrix. Once the coefficients in eqn (2.8) are known, one can determine the translational and angular velocities of a free particle ($F_x = F_z = 0$, $T = 0$) by resolving eqn (2.9) with respect to U_c , V_c and Ω .

3. Solution Procedure

Below we briefly describe the solution algorithm used in this work. For detailed description of the method Ref. [10] should be consulted. To be able to accurately solve problems with general curvilinear geometry we have utilized a coordinate transformation of the governing eqns (2.3) and (2.4). The coordinate transformation has been determined by solution of two elliptic partial differential equations (e.g. Ref. [2]). The transformed governing equations were discretized using central differences and solved in the new coordinate plane by the successive-overrelaxation method (e.g. Ref. [13]). Despite the linearity of the equations their solution has to be accomplished in an iterative manner because of the lack of the boundary conditions for the vorticity eqn (2.3.). The boundary conditions for eqn (2.3) were obtained from eqn (2.4) with an additional constraint of no-slip. Because this boundary condition contains unknown values of the stream function it has to be updated at each iteration until the final convergence.

In addition, the value of the stream function in the multiconnected domain is known at the interior surfaces up to an unknown constant. The value of this constant can be determined by requiring the pressure to be a single-valued function of position. To this end we solved the governing equations twice: once for each of two sets of arbitrarily specified constants at each surface. The resulting solution then can be superposed to yield the solution which has a single-valued pressure field [10].

Once the vorticity and stream function fields are available other parameters of interest, such as forces and torques on the bodies, can be calculated by integrating around the respective bodies. It can be shown [2] that integration of the force components can be accomplished around any contour containing the body. This fact was utilized in the present work by calculating the integrals away from the bodies to reduce the errors due to both mesh distortion and high gradients of the variables in the near field.

4. Particle Motion in a Quiescent Fluid

In this section we will study the flow of fluid due to the motion of the particle assuming that at the boundaries of the channel, including the inlet and outlet, the fluid is quiescent. Thus, in this case the last column in eqn (2.8) is zero, and the force and torque on the particle can be expressed in terms of the nine coefficients, $A_{ij}(y, R)$, as functions of position of the center of the particle with respect to the lower channel wall, y , and the particle radius, R . From symmetry of the problem it can be concluded that no horizontal force on the particle is generated due to particle translation perpendicular to the walls with velocity V_c ; thus, $A_{12} = 0$ and, due to symmetry of the matrix [11], $A_{21} = 0$. In addition, no torque on the particle is generated due to the particle translation perpendicular to the walls; thus, $A_{32} = 0$ and, due to

the symmetry of the matrix, $A_{23} = 0$. Therefore, the force and torque on a cylindrical particle in otherwise quiescent fluid can be expressed in terms of only four coefficients: A_{11} , A_{22} , A_{33} , and A_{13} . In most cases, the computations are done for $0 \leq y \leq 0.999(1-R)$ and $0.15 \leq R \leq 0.45$, i.e. the smallest gap between the particle and the wall is 0.1 % of the channel width; in certain cases the computations are done for gaps as small as 0.001% of the channel width, and also for smaller particles, $R=0.1$.

Translational motion parallel to the channel walls

First, we consider a problem of horizontal translation of a circular particle between two plane parallel walls with velocity U_c ; this quantity is considered a characteristic velocity for non-dimensionalization. Faxen [5] and Takaisi [7] obtained approximate solutions as expansions in R for the symmetric geometry. The expression for the force given by Faxen has a singularity at $R \cong 0.313$ and that given by Takaisi at $R \cong 0.200$. It is clear, however, that the force is finite for $R < 0.5$ and increases asymptotically to infinity as R tends to 0.5. Thus, the upper limits of applicability of the expansions are smaller than the aforementioned values.

Figure 2 compares the present calculations of the force for $0.15 \leq R \leq 0.45$ with the results of Faxen and Takaisi. [It can be noted, that this force is equal to the coefficient A_{11} in eqn (2.8).] The numerical solution is in agreement with Faxen's solution (a more accurate of the two expansions) up to $R = 0.25$.

Figure 3 shows the force acting on a particle translating with unit velocity parallel to the channel walls, $A_{11}(y, R)$, as a function of the position of the center of particle. The results are presented for the upper half of the channel due to the symmetry of the problem with respect to the centerline of the channel. The force has a maximum at the centerline ($y = 0.5$), reaches a local minimum at a position between the centerline and the wall, and increases indefinitely as the particle approaches the wall, i.e. as $y \rightarrow 1-R$. The positions of the minima are: $y_{\min} = 0.75$ for $R = 0.15$; 0.725 for $R = 0.2$; 0.70 for $R = 0.25$; 0.67 for $R = 0.3$; 0.63 for $R = 0.35$; 0.595 for $R = 0.4$; and 0.542 for $R = 0.45$.

A qualitatively similar behavior of the force was reported by Harper and Chang [9] for $R \leq 0.01$. In order to gain more insight in this phenomenon, we consider a problem with a simpler geometry. Consider a long slab of thickness $2d$ translating with unit velocity in a channel parallel to the plane boundaries. The solution to this problem, if end effects are neglected, can be easily obtained in closed form. The horizontal force on the slab per unit length is:

$$F_1 = \frac{(1+2d)(4d^2+2d+1) - 6d(1-2y)^2}{(1-2d)(y-d)(1-y-d)(1-3y+3y^2+d^2+d)} \quad (4.1)$$

where y is the coordinate of the middle (centerline) of the slab, and d is dimensionless half-thickness. The behaviour of the force is qualitatively similar to that found for cylindrical

particle; i.e. the force has a minimum inside the interval $0.5 - y \leq 1 - d$, and goes to infinity as y tends to $1 - d$. If $d = 0$ (infinitely thin slab) the minimum of the force is located at:

$$y_{\min} = (3 + \sqrt{3})/4 \cong 0.79.$$

When an eccentrically situated particle moves with unit velocity parallel to the boundaries, a torque on the particle is generated, $T = A_{31}(y, R)$. Figure 4 shows the coefficient A_{31} for different particle positions and radii. An interesting feature of these curves is a plateau at intermediate values of y , where the torque remains nearly constant before a steep increase near the wall. When a particle rotates with unit angular velocity about its axis a horizontal force is generated, $F_1 = A_{13}(y, R)$. As it was shown by Happel and Brenner [11] this force is equal to the torque exerted on the particle translating with unit velocity parallel to the boundaries, $T = A_{31}(y, R)$. Although numerically $A_{31} \neq A_{13}$ the difference is too small to be distinguishable on the graph.

Translational motion perpendicular to the channel walls

Westberg [8] obtained an approximate solution to the problem of flow around a cylinder translating perpendicular to the walls of a plane channel; the cylinder is situated midway between the walls. Westberg's solution for the drag force was expressed as an expansion to $R^{1/2}$. The force has a maximum at $R \cong 0.313$ and then decreases and becomes negative. Clearly, in reality there should not be a singularity and the force should increase monotonically to infinity as the particle radius approaches 0.5. Thus, the above value gives an upper estimate for the range of validity of Westberg's results. In the present calculations the particle velocity V_c is taken as the characteristic velocity. The present numerical solution, Fig. 5(a), is in agreement with Westberg's for $R \leq 0.2$ radius. Figure 5(b) depicts the coefficient $A_{22}(y, R)$ as a function of position of the center of the particle for particles of different sizes. Unlike the horizontal force, A_{11} , the vertical force is a monotonic function of position.

Rotation of the cylinder around its axis

An analytical solution to the problem of a cylinder rotating with constant angular velocity midway between two plane parallel boundaries was obtained by Howland and Knight [3] in the form of an infinite series. The series coefficients were calculated only for three particle sizes, $R = 0.15, 0.2$ and 0.25 . Figure 6(a) shows their results for the torque on the particle together with the present numerical solution; the relative error for the three values of radius does not exceed 2.7%. The characteristic velocity used in non-dimensionalization for this problem is $\Omega' H$, where Ω' is the angular velocity of the particle. Figure 6(b) shows the torque on a particle rotating with unit angular velocity, $A_{33}(y, R)$, as a function of position of the center of the particle and particle radius. The torque increases monotonically from a minimum at the centerline of the channel to infinity when the particle approaches the wall.

The flow induced in a closed channel by rotation of the particle represents an example of Stokes' flow with closed streamlines. Such flows are of considerable theoretical interest and have been studied by a number of investigators (e.g. Ref. [14]). It is interesting to compare

our numerical results obtained for a symmetrically-situated, rotating particle of $R = 0.2$ with the numerical solution by Pan and Acrivos [14] for a rectangular cavity where the fluid motion is generated by the uniform translation of the top wall. It is known [15] that far from the moving wall (i.e. asymptotically) the flow should consist of an infinite set of eddies having length-to-width ratios equal to 1.39. The numerical results of Pan and Acrivos indicate that in the case $L = 5$ this ratio is approached closely in the third eddy from the moving wall. The present results show that for $L = 5$ the first eddy closest to the particle has a width-to-length ratio approximately equal to 1.37, while for $L = 7$ this ratio is equal to 1.39 in the first eddy.

5. Particle Settling Under Gravity in a Vertical Channel

In this section, solutions are presented for the sedimentation of a cylindrical homogeneous particle under gravity in a quiescent fluid in a vertical channel with plane parallel boundaries. Thus, the x -axis is directed downward, and y is the distance between the particle center and the left wall of the channel. The characteristic velocity for this problem is defined by:

$$U = \pi H^2 \Delta \rho g / \mu \quad (5.1)$$

where $\rho = \rho_p - \rho$ is the difference between the particle and fluid densities and g is the gravitational acceleration. The dimensionless buoyancy force on the particle is:

$$F_g = R^2 \quad (5.2)$$

The values of particle translational and angular velocities, $U_c(y, R)$ and $\Omega(y, R)$, respectively, can be determined from the relationship (2.8):

$$\begin{pmatrix} R^2 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix} \begin{pmatrix} U_c \\ \Omega \end{pmatrix}, \quad (5.3)$$

where the coefficients A_{11} , $A_{13} = A_{31}$ and A_{33} were presented in the preceding section. In the geometry considered the velocity component perpendicular to the boundaries equals zero, thus the dimension of the resistance matrix is reduced to two. The net torque on the particle is assumed to vanish if the particle is homogeneous and the motion is sufficiently slow (quasi-steady motion). However, if the geometrical center and the center of gravity of a particle are different, a net torque on the particle would result.

Figure 7(a) shows the velocity of a cylindrical particle settling along the centerline of a vertical channel; the velocity is presented as a function of the particle radius. The computations were done for radii $0.15 \leq R \leq 0.45$. These results are shown by the solid line

while the dotted line is an extrapolation for smaller and larger particles. When the particle radius goes to 0 or 0.5 the settling velocity goes to zero; it reaches the maximum at an intermediate radius value between 0.15 and 0.2.

The distribution of settling velocity across the channel for particles of different sizes is shown in Figure 7(b). The velocity has a maximum at some eccentric position of the particle, whereas at the centerline it has a local minimum. In order to better understand this behavior we can again consider a long slab of thickness $2d$ with sides parallel to the channel walls, settling under gravity. The settling velocity of the slab can be easily found from eqns (5.3) and (2.8). The characteristic velocity used in the non-dimensionalization is $U = \frac{2H}{\rho g l \mu}$, where ρ is the difference between densities of the slab and the liquid. For an infinitely thin slab the maximum velocity is reached at $y = (3 + \sqrt{3})/6 \cong 0.79$. Qualitatively, the velocity profile is similar to the one for cylindrical particle suggesting that the reason for the eccentric location of the maximum of settling velocity is the asymmetric backflow on either side of the slab or the particle.

Figure 8 shows the angular velocity of a cylindrical particle settling in a vertical channel. The angular velocity is zero midway between the walls, as expected from symmetry, and increases as the particle is displaced towards the wall. The direction of rotation is opposite to that for rolling along the nearer channel wall. At some lateral position the particle reaches its maximum angular velocity. After this point the angular velocity decreases rapidly as the particle approaches the wall. At some point near the wall the direction of rotation changes to that for rolling along the nearer wall and finally the angular velocity becomes zero as the particle touches the wall. An explanation for this behavior is that the backflow created by the particle moving forward goes along the more distant wall and causes steeper velocity gradients on that side of the particle.

This reversal of direction of rotation occurs very close to the wall and so we could not detect it in our routine computations which were done for particle-wall separation of up to 0.001 of channel width. Because of the computational expense involved we performed computations up to 0.0001 gap only for one particle size $R = 0.25$. The results of this analysis are shown on the inset of Fig. 8, on a magnified scale. The angular velocity at the last point computed ($y = 0.7499$) is nearly zero and cannot be distinguished from zero on the plot.

In the absence of the particle, the pressure distribution in the channel is hydrostatic: when a settling particle is present, an additional pressure difference, Δp , is generated due to the particle motion. Figure 9 shows the pressure difference created by a settling particle. The pressure in the fluid is highest on the forward side of the particle. The pressure difference is a function of both particle size and lateral position. For a given particle size the pressure drop is maximal when the particle settles along the channel centerline.

6. DISCUSSION

The results presented in this paper constitute a complete study of the motion of a cylindrical particle in a channel with plane parallel boundaries when the fluid far from the particle is quiescent. Indeed, the complete description of fluid-particle interaction is characterized by

the coefficients of the resistance matrix (2.8), $A_{11}(y, R)$, $A_{22}(y, R)$, $A_{33}(y, R)$ and $A_{13}(y, R)$ which have been calculated in the present work. Several characteristics of the motion exhibit qualitatively interesting behavior and deserve further discussion.

We have found that the drag force on a particle translating parallel to the walls exhibits a local maximum at the centerline and a minimum at a position between the centerline and the wall (Fig. 3). A similar behavior was reported by Harper and Chang [9] in a study of the same problem by the method of matched asymptotic expansions; their solution is valid for $R < 0.01$ and the minimum of drag was found approximately at $y_{\min} \cong 0.775$. We have also shown that the force on a slab between two parallel boundaries exhibits qualitatively similar behavior.

We shall now compare these results to the solutions of related 3-D problems: translational motion of a sphere in a circular tube [11] and in a plane channel [16]. Happel and Brenner [11] utilized the method of reflections to obtain the solutions which is valid for small particle radius and small eccentricity values. Their solution for the drag force exhibits non-monotonic behavior and reaches its minimal value at an eccentricity-to-tube radius ratio of approximately 0.4.

The drag on a sphere translating in a plane channel exhibits a different behavior [16]. It has a minimum at the channel centerline and monotonically increases to infinity as the particle approaches the wall. Thus, the behavior of the drag on a spherical particle translating in a plane channel differs from both the drag on the sphere translating in a tube and the drag on an infinite cylinder translating in a plane channel.

The source of this phenomenon appears to be in the strong backflow that is generated in the latter two cases due to the condition of zero net flow through any cross-section of the conduit. Indeed, both a cylinder in a channel and a sphere in a tube leave a finite cross-section of the conduit open for fluid flow, while a sphere in a channel leaves an infinite cross-section of the channel open. Although Ganatos *et al.* [16] found a weak backflow in the larger gap between the sphere and the wall it is apparently not strong enough to change the behaviour of the force.

The behavior of the torque on a translating cylindrical particle, Fig. 4, also differs from the behavior of the torque on a sphere translating between two parallel walls. In the 2D case the torque increases monotonically from the centerline to the channel to the wall although a plateau exists at certain locations, whereas in the 3-D case [16] the torque becomes negative near the centerline, crosses zero and goes to infinity as the sphere approaches the wall.

The behavior of the torque on a rotating particle in the 2-D case (Fig. 6b) and in 3-D case [16] are qualitatively similar. The force on a particle translating perpendicular to the walls (Fig. 5(b), present work and Ref. [16]) also exhibits similar trends in both cases.

Finally, the results of the present work for the case of a particle settling under gravity are compared with the related 3-D cases. We have found that in the case of a cylindrical particle settling in a vertical channel the settling velocity has a maximum at a position between the wall and the centerline (Fig. 7a). We have also shown that a slab settling in a vertical channel

exhibits a similar behavior. Bungay and Brenner [12] reported a similar result for a sphere settling in a vertical tube; they showed that for very small particles the maximum lies at a dimensionless radius of 0.41 as measured from the axis of the tube. Bungay and Brenner [17] also showed that for a closely-fitting sphere settling in a tube the maximum of velocity is reached at an eccentricity parameter $e = b/(R_0 - R) \cong 0.98$, where b is the distance between the tube axis and the center of the sphere. R_0 is the radius of the tube and R is the radius of the sphere. This maximum settling velocity is 2.1 times greater than settling velocity along the axis of the tube. For the present 2-D case the analog of e is a parameter $Z = (y - 0.5)/(0.5 - R)$; our calculations give for the position of maximum velocity $Z = 0.85$ for $R = 0.3$, $Z = 0.93$ for $R = 0.35$, and $Z = 0.95$ for $R = 0.4$. For $R = 0.4$ the maximum of the settling velocity is 2.2×10^{-4} , whereas if the particle settles along the channel centerline its velocity is 1.0×10^{-4} , hence, the ratio is 2.2—in close agreement with the asymptotic result of Bungay and Brenner [17].

We have found that the angular velocity of a cylindrical torque-free particle settling under gravity exhibits a peculiar behavior: the sign of the velocity is opposite to that of a particle rolling along the nearer wall except for in a very narrow region when the particle almost touches the wall (Fig. 8). This behavior is attributed to the strong backflow in the larger gap between the particle and the wall. The asymptotic solution of Bungay and Brenner [17] for a closely fitting sphere in a tube also predicts a rotation in the direction opposite to that for rolling along the nearer side of the wall; the direction of rotation changes at some lateral position, e , close to unity. If we now turn to our Fig. 8 where the profile of Ω is shown in detail for $R = 0.25$, we can see that the direction of rotation changes at $y \cong 0.7485$ corresponding to $Z = 0.994$, i.e. very close to unity.

Ganatos *et al.* [16] studied the problem of a sphere settling in an inclined channel. They also found that the angular velocity changes sign, but the change of sign occurs almost immediately at the channel centerline for $R/H = 0.4$; at $e \cong 0.64$, for $R/H = 0.25$; and at $e \cong 0.78$ for $R/H = 0.05$. After the sign change the angular velocity sharply increases and its maximum (which is positive) is several times greater than the magnitude of the minimum (which is negative). Clearly, this behavior of the angular velocity for a sphere between plane boundaries is distinctly different from the 2-D case and from the case of a sphere settling in a tube.

The profile of the pressure drop created by a cylinder settling between two plane boundaries is shown in Fig. 9. The pressure drop changes slowly from the maximum at the centerline until it almost reaches the wall and then steeply drops to zero when the particle touches the wall. Bungay and Brenner [12,17] came to similar conclusions for both small and large particles.

The resistance matrix calculated in the present paper will be utilized in the following Part 2 in a detailed study of the motion of a cylindrical particle in 2-D Poiseuille and Couette flows.

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Fig. 1.
Geometry of the problem.

Fig. 2. Horizontal force on a particle as a function of particle radius. The particle is situated in the middle of the channel and translating with unit velocity parallel to the boundaries. (1) Present solution; (2) Faxen [5]; (3) Takaisi [7].

Fig. 3. Horizontal force on a particle translating with unit velocity parallel to the boundaries as a function of the position of its center and particle radius.

Fig. 4.
Torque on a particle translating with unit velocity parallel to the boundaries.

Fig. 5.

Vertical force on a particle situated at the center of the channel and translating perpendicular to the channel walls with unit velocity. (1) Present solution; (2) Westberg [8]. (b) Vertical force on a particle translating with unit velocity perpendicular to the boundaries as a function of position of the particle center and particle radius.

Fig. 6.

(a) Torque on a particle situated at the center of the channel and rotating with unit angular velocity. — Present solution; ---Extrapolation; ● Howland and Knight [3]. (b) Torque on a particle rotating with unit velocity as a function of position of the particle center and particle radius.

Fig. 7.

(a) Sedimentation velocity for a particle settling along the channel centerline as a function of particle radius, (b) Sedimentation velocity for a torque-free particle as a function of position of the particle center and particle radius.

Fig. 8.
Distribution of angular velocity for a torque-free particle settling in a vertical channel.

Fig. 9.
Distribution of pressure difference due to a cylindrical particle settling in a vertical channel.