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A discrete convolution kernel for No-DC MRI

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Abstract

An analytical inversion formula for the exponential Radon transform with an imaginary attenuation coefficient was developed in 2007 (2007 *Inverse Problems* 23 1963–71). The inversion formula in that paper suggested that it is possible to obtain an exact MRI (magnetic resonance imaging) image without acquiring low-frequency data. However, this un-measured low-frequency region (ULFR) in the k-space (which is the two-dimensional Fourier transform space in MRI terminology) must be very small. This current paper derives a FBP (filtered back-projection) algorithm based on You's formula by suggesting a practical discrete convolution kernel. A point spread function is derived for this FBP algorithm. It is demonstrated that the derived FBP algorithm can have a larger ULFR than that in the 2007 paper. The significance of this paper is that we present a closed-form reconstruction algorithm for a special case of under-sampled MRI data. Usually, under-sampled MRI data requires iterative (instead of analytical) algorithms with L_1 -norm or total variation norm to reconstruct the image.

Keywords

tomography; image reconstruction; fourier transform; MRI

1. Introduction

For a two-dimensional (2D) object, the Radon transform is a set of one-dimensional (1D) line integrals of the object. Its analytical inversion was first investigated by Radon in 1917 [1]. Radon's inversion formula has wide applications, especially in x-ray CT (computed tomography) [2, 3].

If the Radon transform is weighted with an exponential function e^{at} in the integrand and a being a real number, the Radon transform becomes the exponential Radon transform. Analytical inversion formulas for the exponential Radon transform were developed in the early 1980s [4]. One important application of the exponential Radon transform is in SPECT (single photon emission computed tomography) [5, 6].

In 2007, the exponential weighting function e^{at} in the exponential Radon transform was extended to a complex exponential function $e^{(a+i\eta)t}$ (with a being the real part and η being the imaginary part) and an inversion formula was developed by You [7]. It was claimed that when the weighting function is $e^{i\eta t}$ (by letting $a = 0$), the inversion formula may find

applications in MRI (magnetic resonance imaging). MRI may be possible without measuring some low-frequency data at the center of the k -space, where the k -space is the two-dimensional Fourier transform space in MRI terminology. However, this un-measured low-frequency region (ULFR) in the k -space must be very small. The goals of this paper are to convert You's inversion formula into a practical reconstruction algorithm with a discrete convolution kernel, to investigate the size of the ULFR, and to derive a point spread function for this algorithm so that we can have a better insight. Computer simulations are presented.

2. Methods

2.1. Imaging geometry

Let $\vec{x} = (x, y)$ be a point in the 2D plane and $f(\vec{x})$ be a 2D object. The projection measurement $p(s, \theta)$ is the exponential Radon transform with an imaginary exponential function and is defined as

$$\begin{aligned}
 p(s, \theta) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\eta \vec{x} \cdot \vec{\theta}^\perp} f(\vec{x}) \delta(\vec{x} \cdot \vec{\theta} - s) d\vec{x} \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\eta \left(\hat{s} \vec{\theta} + t \vec{\theta}^\perp \right) \cdot \vec{\theta}} f\left(\hat{s} \vec{\theta} + t \vec{\theta}^\perp \right) \delta\left(\left(\hat{s} \vec{\theta} + t \vec{\theta}^\perp \right) \cdot \vec{\theta} - s \right) d\hat{s} dt \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\eta t} f\left(\hat{s} \vec{\theta} + t \vec{\theta}^\perp \right) \delta\left(\hat{s} + t \vec{\theta}^\perp \cdot \vec{\theta} - s \right) d\hat{s} dt \\
 &= \int_{-\infty}^{\infty} e^{2\pi i \left(\frac{\eta}{2\pi} \right) t} f\left(s \vec{\theta} + t \vec{\theta}^\perp \right) dt \quad (1)
 \end{aligned}$$

where $\vec{\theta} = (\cos \theta, \sin \theta)$, $\vec{\theta}^\perp = (-\sin \theta, \cos \theta)$, η is a real constant, and δ is the Dirac delta-function. The equations in (1) give a few different (yet equivalent) definitions of a weighted Radon transform with the weighting function $e^{i\eta t}$. When $\eta = 0$, (1) reduces to the conventional Radon transform. If $(i\eta)$ is replaced by a real number a , (1) is the exponential Radon transform. The t -axis is in the direction $\vec{\theta}^\perp$ along which the line integral of the object f is performed, and the s -direction is along a virtual detector (if one compares this situation with the x-ray CT). The orthogonal (s, t) -system is the (x, y) -system rotated by an angle θ , as indicated by $s\vec{\theta} + t\vec{\theta}^\perp$.

The last line of (1) resembles a 1D inverse Fourier transform with respect to the variable t . Taking the Fourier transform of (1) with respect to variable s yields

$$\begin{aligned}
P(\omega_s, \theta) &= \int_{-\infty}^{\infty} p(s, \theta) e^{-i2\pi s \omega_s} ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(s \vec{\theta} + t \vec{\theta}^{\perp}\right) e^{-i2\pi\left(s \omega_s - t \frac{\eta}{2\pi}\right)} dt ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(s \vec{\theta} + t \vec{\theta}^{\perp}\right) e^{i2\pi\left(s \vec{\theta} + t \vec{\theta}^{\perp}\right) \cdot \left(\omega_s \vec{\theta} - \frac{\eta}{2\pi} \vec{\theta}^{\perp}\right)} dt ds \\
&= F\left(\omega_s \vec{\theta} - \frac{\eta}{2\pi} \vec{\theta}^{\perp}\right)
\end{aligned} \tag{2}$$

where F is the 2D Fourier transform of f . In (2), $(\omega_s) \in (-\infty, \infty)$ is a (variable) frequency in the $\vec{\theta}$ direction and $-\eta/(2\pi)$ is a (fixed) frequency in the $\vec{\theta}^{\perp}$ direction. Thus (2) implies that P is a ‘slice’ of F . For a non-zero η , P does not pass through the origin; it has a distance $|\eta|/(2\pi)$ from the origin. Therefore P is not a ‘central slice’ in the 2D Fourier domain (i.e. the k -space). If the angle θ covers an angular range of π , then the k -space is measured except for the circular ULFR region at the center with a radius $|\eta|/(2\pi)$. The Fourier domain expression in (2) can be measured by one line of the k -space of MRI data, and the spatial domain expression in (1) can be obtained by taking the inverse Fourier transform of the MRI data.

2.2. Image reconstruction

The inversion formula associated with (1) is given by [7] as

$$f(\vec{x}) = \frac{1}{4\pi} \int_0^{2\pi} e^{-i\eta \vec{x} \cdot \vec{\theta}^{\perp}} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} \frac{e^{\eta(s-l)} + e^{-\eta(s-l)}}{2(s-l)} p(l, \theta) dl \Big|_{s=\vec{x} \cdot \vec{\theta}} d\theta. \tag{3}$$

We can rewrite this formula in a form that is more consistent with the filtered back-projection (FBP) algorithm:

$$f(\vec{x}) = \frac{1}{4\pi} \int_0^{2\pi} e^{-i\eta \vec{x} \cdot \vec{\theta}^{\perp}} \int_{-\infty}^{\infty} h\left(\vec{x} \cdot \vec{\theta} - l\right) p(l, \theta) dl d\theta, \tag{4}$$

where the convolution kernel h is obtained from (3) and is defined as

$$h(s) = \frac{d}{ds} \frac{e^{\eta s} + e^{-\eta s}}{2s} = \frac{\eta s (e^{\eta s} - e^{-\eta s}) - (e^{\eta s} + e^{-\eta s})}{2s^2} = \frac{\eta s \sinh(\eta s) - \cosh(\eta s)}{s^2}. \tag{5}$$

Discrete implementation of (5) requires considerations of the singularity at $s = 0$.

The kernel function $h(s)$ developed in [7] is unbounded and has a singularity at $s = 0$. The performance of a discrete algorithm depends on how this ill-behaved function $h(s)$ is discretized. In [7], the kernel $h(s)$ in (5) is implemented in such a way that the derivative d/ds is approximated by the finite difference. On the other hand, in our implementation, the derivative d/ds is first performed analytically and then the resultant continuous expression is discretized. To deal with the singularity of $h(s)$ at $s = 0$, our approach is different from that in [7]. In [7], a small positive value is added to the denominator so that division by zero is avoided. On the other hand, our method is similar to the derivation of the convolution kernel for the ramp filter. This is a standard method in many textbooks [8].

When $\eta = 0$, (5) reduces to the conventional ramp filter $h_{\text{ramp}}(s) = -1/s^2$ and its band-limited version can be obtained by applying a rectangular window function in the Fourier domain or, equivalently, by convolving a sinc function $\sin(\pi s)/(\pi s)$ in the spatial domain. The resultant

band-limited ramp filter has a convolution kernel $h_{\text{ramp}}(s)$ as $\frac{1}{2} \frac{\sin(\pi s)}{\pi s} - \frac{1}{4} \left(\frac{\sin(\pi s/2)}{\pi s/2} \right)^2$ [8]. Let the sampling interval be 1 and then the band-limited discrete convolution kernel can be obtained as [8]:

$$h_{\text{ramp}}(n) = \begin{cases} 1/4, & n=0 \\ 0, & n=\text{even} \\ -1/(n\pi)^2, & n=\text{odd} \end{cases} \quad (6)$$

In order to find a band-limited discrete version of (5), we first factor (5) into two factors

$$h(s) = [\eta s \sinh(\eta s) - \cosh(\eta s)] \frac{-1}{s^2} = [\eta s \sinh(\eta s) - \cosh(\eta s)] h_{\text{ramp}}(s). \quad (7)$$

In (7), the first factor $[\eta s \sinh(\eta s) - \cosh(\eta s)]$ does not contain any singularities and can be directly discretized as $[\eta n \sinh(\eta n) - \cosh(\eta n)]$ with $s = n$. The second factor in (7) $-1/s^2$ has a second-order singularity at $s = 0$ and it cannot be directly sampled. Fortunately, (6) is the band-limited discrete version of the ramp filter $h_{\text{ramp}}(s) = -1/s^2$. Combining $[\eta n \sinh(\eta n) - \cosh(\eta n)]$ and (6), an expression for a band-limited discrete version of (5) is obtained as

$$h(n) = \begin{cases} 1/4, & n=0 \\ 0, & n=\text{even} \\ [\eta n \sinh(\eta n) - \cosh(\eta n)]/(n\pi)^2, & n=\text{odd} \end{cases} \quad (8)$$

With this discrete convolver, an FBP reconstruction algorithm can be readily implemented in two steps:

Step 1: Perform discrete convolution $p(s_n, \theta_m) * h(n)$ with respect to the first variable of $p(s_n, \theta_m)$, where s_n and θ_m represent the discrete samples of s and θ , respectively. Let the filtered projection be $q(s_n, \theta_m)$.

Step 2: Perform weighted backprojection $f(\vec{x}) = \frac{1}{4\pi} \sum_m e^{-i\eta \vec{x} \cdot \vec{\theta}_m} q(\vec{x} \cdot \vec{\theta}_m, \theta_m)$. Numeric interpolation is required to approximate $q(\vec{x} \cdot \vec{\theta}_m, \theta_m)$ by $q(s_n, \theta_m)$ and $q(s_{n+1}, \theta_m)$ if $s_n < \vec{x} \cdot \vec{\theta}_m < s_{n+1}$.

2.3. Point spread functions (PSFs)

Combining (1) and (4) yields

$$\begin{aligned} f(\vec{x}) &= \frac{1}{4\pi} \int_0^{2\pi} e^{-i\eta \vec{x} \cdot \vec{\theta}^\perp} \left[h(s) * \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\eta \hat{\vec{x}} \cdot \vec{\theta}^\perp} f(\hat{\vec{x}}) \delta(\hat{\vec{x}} \cdot \vec{\theta} - s) d\hat{\vec{x}} \right] \Big|_{s=\vec{x} \cdot \vec{\theta}} d\theta \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\hat{\vec{x}}) \int_0^{2\pi} e^{i\eta (\hat{\vec{x}} - \vec{x}) \cdot \vec{\theta}^\perp} h\left(\hat{\vec{x}} \cdot \vec{\theta} - \vec{x} \cdot \vec{\theta}\right) d\theta d\hat{\vec{x}} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\hat{\vec{x}}) g\left(\hat{\vec{x}} - \vec{x}\right) d\hat{\vec{x}} \end{aligned} \quad (9)$$

where the point spread function (PSF) $g(\vec{x})$ is given as

$$g(\vec{x}) = \frac{1}{4\pi} \int_0^{2\pi} e^{i\eta \vec{x} \cdot \vec{\theta}^\perp} h(\vec{x} \cdot \vec{\theta}) d\theta. \quad (10)$$

We must point out that (9) and (11) do not hold on the whole plane, because the integrals in (9) contain a function $h(s)$, which is defined in (5) and is unbounded for $\eta \rightarrow 0$. If the object $f(\vec{x})$ does not have a finite support, (8) does not exist. The integrals in (9) have finite values only if the support of object $f(\vec{x})$ is finite. The integral values in (9) grow exponentially as the size of the support gets larger.

The point spread function $g(\vec{x}) = \frac{1}{4\pi} \int_0^{2\pi} e^{i\eta \vec{x} \cdot \vec{\theta}^\perp} h(\vec{x} \cdot \vec{\theta}) d\theta$ can be approximated by the delta function $\delta(\vec{x})$ in the region of $\vec{x}$ that is close to the origin. Outside this region $g(\vec{x})$ grows exponentially. For this reason, the inverse problem is theoretically non-invertible. One can only seek for approximate solutions. A similar case is investigated in [12]. As shown in

figure 2, the function g has a large positive spiking value at the origin and g is almost zero in a small region close to the origin. In this small region, the function g behaves like the delta function $\delta(\vec{x})$. However, outside this region, the function g deviates from zero dramatically.

Now we replace $h(s)$ in (10) by the δ -function $\delta(s)$, which is a point source, then $g(\vec{x})$ becomes the point spread function (PSF), $\rho(\vec{x})$, for the pure projection/backprojection pair without any tomographic filtering:

$$\rho(\vec{x}) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\eta \vec{x} \cdot \vec{\theta}^\perp} \delta\left(\vec{x} \cdot \vec{\theta}\right) d\theta. \quad (11)$$

In fact, we can go one step further. We will prove that the pure projection/backprojection pair without any tomographic filtering actually gives a shift-invariant PSF and we will derive a closed-form expression for $\rho(\vec{x})$ next.

Let a point source at an arbitrary location $\vec{x}_0$ be $\delta(\vec{x} - \vec{x}_0)$. The projection of $\delta(\vec{x} - \vec{x}_0)$ according to (1) is

$$p(s, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\eta \vec{x} \cdot \vec{\theta}^\perp} \delta(\vec{x} - \vec{x}_0) \delta(\vec{x} \cdot \vec{\theta} - s) d\vec{x} = e^{i\eta \vec{x}_0 \cdot \vec{\theta}^\perp} \delta(\vec{x}_0 \cdot \vec{\theta} - s) \quad (12)$$

and the backprojection of it yields

$$\begin{aligned} b(\vec{x}) &= \frac{1}{2} \int_0^{2\pi} e^{-i\eta \vec{x} \cdot \vec{\theta}^\perp} p(\vec{x} \cdot \vec{\theta}, \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} e^{-i\eta(-(x-x_0)\sin\theta + (y-y_0)\cos\theta)} \delta((x-x_0)\cos\theta \\ &\quad + (y-y_0)\sin\theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left[e^{-i\eta r \frac{\delta(\theta - \theta_1)}{r}} + e^{i\eta r \frac{\delta(\theta - \theta_2)}{r}} \right] d\theta = \frac{\cos(\eta r)}{r}. \end{aligned} \quad (13)$$

Let $\alpha(\theta) = (x - x_0)\cos\theta + (y - y_0)\sin\theta$. The equation $\alpha(\theta) = 0$ has two solutions: θ_1 and θ_2 , in $[0, 2\pi)$, satisfying

$$\begin{cases} \sin\theta_1 = -\frac{x-x_0}{r} \\ \cos\theta_2 = \frac{y-y_0}{r} \end{cases} \quad \text{and} \quad \begin{cases} \sin\theta_2 = \frac{x-x_0}{r} \\ \cos\theta_1 = -\frac{y-y_0}{r} \end{cases}$$

respectively, with $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. In the last line of (13), we used the following well-known property of the δ -function [9]:

$$\delta(c(\theta)) = \frac{\delta(\theta - \theta_1)}{|c'(\theta_1)|} + \frac{\delta(\theta - \theta_2)}{|c'(\theta_2)|}.$$

Here, $|c'(\theta_1)| = |c'(\theta_2)| = r$.

The closed-form expression for $\rho(\vec{\mathbf{x}})$ is thus

$$\rho(\vec{\mathbf{x}}) = \frac{\cos(\eta \|\vec{\mathbf{x}}\|)}{\|\vec{\mathbf{x}}\|}. \quad (14)$$

When $\eta = 0$, $\rho(\vec{\mathbf{x}})$ reduces to the well-known PSF $1/\|\vec{\mathbf{x}}\|$ for the conventional non-weighted projector/backprojector pair [8]. The availability of the PSF $\rho(\vec{\mathbf{x}})$ makes it possible to reconstruct the image by the ‘backprojection first, then filter’ algorithm [8, 13, 14]. In the ‘backprojection first, then filter’ algorithm, one first backprojects the measured projections into the image domain, obtaining an intermediate image. Then one de-convolves this intermediate image with the kernel $\rho(\vec{\mathbf{x}})$ by, for example, the iterative Richardson–Lucy algorithm [10, 11].

2.4. Computer simulation setup

The detector had $N = 128$ discrete samples. The projection data were analytically calculated. Both noiseless data and noisy data (with Gaussian k-space noise) were considered. The computer generated phantom consisted of multiple uniform discs of different radii, see figure 1. The number of views over 360° was 200. The image array size was $N \times N$. The two-step FBP algorithm presented at the end of section 2.2 was implemented in MATLAB and the convolution kernel was given in (8).

2.5. The meaning of η in terms of η

If an object's span has N samples, and the discrete Fourier transform of an N -sample data set has N samples in the Fourier domain. The kernel in the discrete Fourier transform is $e^{-i2\pi nk/N}$, where n is the spatial domain sample index and k is the Fourier domain (i.e., k-space) sample index. Comparing this kernel $e^{-i2\pi nk/N}$ with the projection kernel $e^{i\eta t}$ in (1),

we have $t = n$ and $\eta = \frac{2\pi k}{N\Delta t}$, maintaining $\eta t = 2\pi nk/N$. Without loss of generality, let $t = 1$ (i.e., using the sampling interval as the unit). This situation has the sampling intervals in

the spatial and Fourier domains to be $\Delta t = 1$ and $\Delta\eta = \frac{2\pi}{N}$, respectively. This η is the sampling interval in the k-space in an MRI acquisition.

3. Results

Figure 1 shows the reconstruction results with different values of η . It is observed that when $\eta > 2.0$, the reconstruction contains severe artifacts. The central line profiles indicate that the reconstruction is more accurate for a smaller value of $|\eta|$. The result with $\eta = 0$ is used as the gold standard for other results to compare with. When $|\eta| > 2.0$, a mask was used to zero out the image values outside a circular region so that the image can be better visualized.

According to the definition of $h(s)$ in (5), the growth rate of the exponential functions depends on the product ηs . When the product $|\eta s|$ is large, the algorithm is unstable and severe artifacts appear in regions away from the origin in terms of the scaled-distance ηr , where r is the distance to the origin. Therefore, the stability of the proposed reconstruction algorithm depends both on the radius η of the un-measured low-frequency region (ULFR) and on the distance r away from the image origin. When $|\eta| < 2.0$, no visible artifacts are found in image support. As the $|\eta|$ gets larger, the algorithm becomes more unstable and artifacts propagate towards the image support. When the artifact values are greater than the image values and the image is displayed with a gray-scale that is determined by minimum and maximum values of the artifact-affected image, the visibility of the image will be degraded. If the bright artifacts are outside the image support, masking out the image values outside the support region helps image visualization. If the bright artifacts propagate into the support region, not even the masking method can help.

The mean-square-error (MSE) is calculated for each image with respect to the true image within the image mask. The image mask is circular and larger than the image support.

Figure 2 shows the radial profiles of the PSF $g(\vec{x})$ and gives more insight into the inverse problem. It is observed that when $\|\vec{x}\|$ is smaller, $g(\vec{x})$ is a better approximation of $\delta(\vec{x})$. When $\|\vec{x}\|$ is larger, some oscillation appears. The amplitude of the oscillation becomes larger as $\|\vec{x}\|$ increases. The ‘good region’ shrinks as the value of $|\eta|$ increases, where the ‘good region’ is referred to the artifact-free region in the image in the spatial domain (instead of the k-space). Figure 2 shows the radial profile of the point spread function $g(\vec{x})$ calculated by (10).

4. Discussion and conclusions

In this paper, the inverse problem for the exponential Radon transform with an imaginary attenuation coefficient $i\eta$ is considered, and a discrete convolution kernel is suggested, so that an FBP image reconstruction algorithm can be readily implemented. The combined projection/backprojection operation is shown to be shift-invariant. A closed-form point spread function (PSF) for the projector/backprojector pair is obtained.

It is demonstrated that the FBP algorithm (4) for $\eta = 0$ with the convolver (8) is not a true inversion of projection transform (1), in the sense that the PSF $g(\vec{x}) \approx \delta(\vec{x})$ only valid in a central subregion of the 2D plane. This small ‘good’ region shrinks as the value of $|\eta|$ increases. If this ‘good’ region is large enough to contain a practical object, the algorithm may find some applications. The size of this ‘good’ region is strongly dependent on the value of η . Equivalently, we can assume that the object has a span of N samples and the k-

space sampling interval is η . The maximum value of $|\eta|$ that can give a reasonable reconstruction is approximately $2/\eta$. Our other computers simulations (by varying N from 64 to 4096, not presented) show that this value of $2/\eta$ is independent of the value of N . We represent the k-space with grids and the line spacing is η , where η is the sampling interval as shown in figure 3. The un-measured low-frequency region (ULFR) is a disc of a radius $2/\eta$. The k-space samples inside the ULFR are not required. There are about 13 samples that are not needed.

There is a possibility that the discrete convolution kernel $h(n)$ is not unique for $\eta = 0$, and a more stable $h(n)$ than (8) may exist and may offer a larger ‘good’ region in which $g(\hat{\mathbf{x}}) \approx \delta(\hat{\mathbf{x}})$ for a given value of η .

It is interesting to notice that according to the MSE results, the algorithm with $\eta = 0$ is less stable than the case with a small positive η . This is consistent with our observations for the inversion of the attenuated Radon transform [15, 16], where the system matrix is better conditioned when there is a small attenuation coefficient in the attenuated Radon transform. The system matrix is better conditioned than that without attenuation. However, as the attenuation coefficient increases, the system matrix becomes extremely ill-conditioned. A little bit of attenuation brings in more tomographic information because the opposite views (180° apart) measure different projections. When attenuation coefficients get larger, the exponential function suppresses the useful projection measurements under the noise level.

The MRI situation is similar to that in the attenuated Radon transform. When η is small, two parallel lines in the k-space measure more information than one line (with the same orientation) passing through the origin in the k-space. However, when the k-space missing region is large, it is extremely ill-condition to recover the missing data.

Usually, under-sampled MRI data requires iterative algorithms with L_1 -norm or total variation norm to reconstruct the image. The significance of this paper (at least in theory) is that we presented a closed-form reconstruction algorithm for under-sampled MRI data.

To conclude the paper, we offer an explanation for the unusual phenomenon that an image can be exactly reconstructed while the low frequency components as well as the DC component are not available. Our first hypothesis was that the k-space data might be analytic in the complex plane. An analytic function is highly redundant and any un-measured data can be mathematically obtained by using analytic continuation. Then we realized that this hypothesis is wrong, because the k-space signal does not satisfy the Cauchy–Riemann equations. The two-dimensional k-space signal is not analytic on the complex plane. Fortunately, the object has a finite support, and this implies that any one-dimensional line passing through the origin of the k-space is analytic. Therefore, analytic continuation is valid on any line passing through the origin of the k-space. We must point out that the usual analytic continuation method using power expansions is ill-posed and impractical. If the k-space is well sampled except for a low-frequency region, this unmeasured region in theory can be uniquely determined by the measured data. Our reconstruction algorithm is an indirect method to recover the unmeasured data. The significance of this paper is that we

present a closed-form method which can be thought of as an equivalent ‘analytic continuation’ method without requiring any power expansions.

When $|\eta|$ is large, the unmeasured region is large and the closed-form method does not perform well, because for larger regions of blank low-frequency k-space, recovery of this missing data becomes extremely ill-conditioned.

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Figure 1. FBP image reconstruction results using the two-step algorithm presented at the end of section 2.2. The ideal line profiles are plotted in dashed lines.

Figure 2. Radial line profiles for the point spread function of the two-step algorithm presented at the end of section 2.2 ($N = 128$).

Figure 3. The k-space is represented with grids and the line spacing is η . The ULFR contains 13 points.