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Theory of Linear and Nonlinear Gain in a Gyroamplifier using a Confocal Waveguide

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Abstract

The linear and nonlinear theory of a gyroamplifier using a confocal waveguide is presented. A quasi-optical approach to describing the modes of a confocal waveguide is derived. Both the equations of motion and the mode excitation equation are derived in detail. The confocal waveguide circuit has the advantage of reducing mode competition but the lack of azimuthal symmetry presents challenges in calculating the gain. In the linear regime, the gain calculated using the exact form factor for the confocal waveguide agrees with an azimuthally averaged form factor. A beamlet code including velocity spread effects has been written to calculate the linear and nonlinear (saturated) gain. It has been successfully benchmarked against the MAGY code for azimuthally symmetric cases. For the confocal waveguide, the beamlet code shows that the saturated gain is reduced when compared with results obtained using an azimuthally averaged form factor. The beamlet code derived here extends the capabilities of nonlinear gyroamplifier theory to configurations that lack azimuthal symmetry.

I. Introduction

A. Gyrotron Physics

The physics of the interaction between an electromagnetic wave and an electron beam for application to gyrotrons has been studied extensively [1]. One of the important challenges in gyrotron research is the development of powerful gyrotron amplifiers, especially at high frequencies [2–10]. As the gyrotron frequency increases, it is advantageous for the gyrotron amplifier to operate in a higher order mode of the interaction circuit to minimize space charge effects and ohmic loss. One possible method of reducing the mode competition in overmoded waveguides is the use of a confocal waveguide structure [6,11]. However, this configuration results in a coupling between the electron beam and the electromagnetic wave that is nonuniform in the azimuthal direction. Such nonuniformity is also found in other gyrotron configurations, such as split resonators [12,13] and quasioptical gyrotrons [14–16]. The purpose of this paper is to present a detailed linear and nonlinear gyrotron interaction theory that accounts for the nonuniform azimuthal variation of the electromagnetic mode in confocal waveguides.

The generalized linear and nonlinear theory of gyrotron traveling wave amplifiers has been previously derived [1,2]. The nonlinear equations take into account the geometry of the interaction, namely the coupling factor that describes the overlap between the

electromagnetic mode and the electron beam. Gyrotron amplifier theory, as developed in [2], has focused on analytical solutions for TE modes of a circular pipe. In this paper, we derive the quasi-optical approximation for the open modes of a confocal resonator, and use this to develop the gyrotron equations in parallel to [2]. The open geometry of the confocal resonator is inherently lossy, and we show the analytical solution for these losses. We subsequently derive both the equations of motion and field excitation equations, presented in the appendix, in the context of beamlets.

B. DNP/NMR at MIT

The Francis Bitter Magnet Lab at MIT currently has gyrotron oscillators for dynamic-nuclear-polarization-enhanced nuclear magnetic resonance (DNP/NMR) research at 140, 250, 330, and 460 GHz [17–23]. Previously, pulsed DNP has been achieved using an IMPATT diode driver [24, 25]. The pulse length of 50 ns at 35 mW was able to excite 1% of the sample's linewidth. In order to capture the entire linewidth, a shorter and more powerful pulse is needed, on the order of 100 W to 1 kW at 1 to 10 ns. Gyro-amplifiers are a good candidate for generation of the pulses needed for pulsed DNP/NMR. Additionally, the frequency scaling of gyro-amplifiers is a useful feature for accessing various frequencies. To date, amplification of short pulses has been demonstrated at 140 GHz [26], and at 250 GHz [10].

Contemporary gyro-amplifiers take advantage of a variety of design approaches. Lossy-wall gyro-amplifiers have been designed and operated at 35 GHz [5, 27] and at 95 GHz [28]. An alternative design feature is a helically-corrugated interaction circuit [8,29]. We present a confocal interaction circuit as an alternative to lossy-wall designs. Gyrotrons with confocal circuits continue to be studied intensively [30–34].

II. Quasi-optical Approximation of a Confocal Resonator

A. Membrane function

We begin with a description of the confocal geometry. As seen in Fig. 1, the confocal geometry consists of two mirrors positioned such that their radius of curvature R_c is equal to that of their separation distance, $L_\perp$ ($R_c = L_\perp$). The aperture, or total width of each mirror, is $2a$. As this geometry is not closed, the width is adjusted in order to either increase or decrease the diffractive loss of the supported HE_{mn} modes. These supported spatial modes have m variations along $\hat{x}$ and n variations along $\hat{y}$. Fig. 1 shows the HE_{06} mode, in which there are six variations along the $\hat{y}$ direction.

For a closed waveguide, supported modes are described by the membrane function Ψ , which obeys the wave equation:

$$\nabla_\perp^2 \Psi + k_\perp^2 \Psi = 0 \quad (1)$$

in which the transverse wavenumber $k_\perp$ is real ($k_\perp = k_{\perp r}$). In consideration of the fact that the confocal waveguide is not a closed geometry, we introduce a quasi-optical approach to finding an approximate membrane function [35,36]. The formulation for a closed geometry

[1] relies on the exact solution to the wave equation in Eq. 1. Translational symmetry in the $\hat{z}$ component reduces the problem to two dimensions. Additionally, we assume a solution that is based on a modified plane wave in the $\hat{y}$ direction:

$$U(x, y) = B(x, y)e^{-ik_{\perp}y} \quad (2)$$

As indicated in Eq. 2, we assume a wave propagating in the $\hat{y}$ direction. Variation in x is absorbed into the function $B(x, y)$, an appropriate step if assuming the paraxial

approximation of $k_y = \sqrt{k_{\perp}^2 - k_x^2} \simeq k_{\perp} - \frac{k_x^2}{2k_{\perp}}$. Now, the wave equation Eq. 1 becomes

$$\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2} - 2ik_{\perp} \frac{\partial B}{\partial y} = 0 \quad (3)$$

Since the field distribution is a modified plane wave propagating in the $\hat{y}$ direction, we can neglect $\frac{\partial^2 B}{\partial y^2}$ as small compared to the term $2ik_{\perp} \frac{\partial B}{\partial y}$. With this simplification, Eq. 3 becomes

$$\left[\frac{\partial^2}{\partial x^2} - 2ik_{\perp} \frac{\partial}{\partial y} \right] B(x, y) = 0 \quad (4)$$

Using the form of Eq. 2 for U , we provide a general solution to the membrane function Ψ as two counter-propagating waves:

$$\Psi = -iU - iU^* \quad (5)$$

The confocal geometry, when examined with a quasi-optical approximation, lends itself readily to a Gaussian beam solution for $B(x, y)$. Thus, U may be expressed as

$$U(x, y) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{w_0}{w(y)}} \exp\left(-\frac{x^2}{w^2(y)}\right) \times \exp\left(-ik_{\perp}y - i\frac{k_{\perp}x^2}{2R(y)} + \frac{i}{2}\arctan\frac{y}{b}\right) \quad (6)$$

where $b = k_{\perp}w_0^2/2$, following the derivations in [35]. Furthermore, w and R are

$$w^2(y) = w_0^2 \left[1 + \left(\frac{2y}{k_{\perp}w_0^2}\right)^2 \right] \quad (7)$$

$$\frac{1}{R(y)} = \frac{y}{y^2 + (k_{\perp r} w_0^2 / 2)^2} \quad (8)$$

The perpendicular wavevector component, $k_{\perp r}$, may be found by considering the boundary conditions of Eq. 6 under the confocal geometry. In particular, at a coordinate $(x, y) = (0, L_{\perp}/2)$ we know that the phase front curvature $R(y)$ needs to match the confocal mirror radius, R_c . Consequently, $k_{\perp r}$ is found to be

$$k_{\perp r} = \frac{\pi}{L_{\perp}} \left(n + \frac{2m+1}{\pi} \arcsin \sqrt{\frac{L_{\perp}}{2R_c}} \right) \quad (9)$$

Since $L_{\perp} = R_c$ for the confocal geometry,

$$k_{\perp r} = \frac{\pi}{L_{\perp}} \left(n + \frac{m}{2} + \frac{1}{4} \right) \quad (10)$$

These results are in agreement with those derived by Nakahara [37], following Goubau [38]. In this section we have derived the quasi-optical approximation of the membrane function that describes the field distribution of a confocal resonator. Because the confocal geometry is open, RF power may leak out of the sides of the waveguide. Thus, a mode HE_{mn} will, whilst propagating axially, lose power through transverse diffraction. This leads to a loss per distance that is useful for suppressing undesired (lower-order) modes that do not support the microwave field intended for amplification.

B. Diffractive losses in a confocal resonator

We follow the derivation in [11, 36] in our discussion of losses incurred by waveguide modes HE_{mn} in the confocal geometry. The wave vector $k_{\perp}$ is decomposed into both a real and an imaginary component, $k_{\perp r} + ik_{\perp i}$. Eq. 10 is the real component; a more general expression is needed:

$$k_{\perp} = \frac{\pi}{L_{\perp}} \left(n + \frac{m}{2} + \frac{1}{4} + \frac{\delta}{\pi} \right) - i \frac{\Lambda}{2L_{\perp}} \quad (11)$$

In Eq. 11 we have introduced a small phase shift δ as well as an imaginary component for $ik_{\perp i}$. Λ is directly related to the diffraction losses. As discussed in Section A, the quasi-optical solution to the modes in a confocal geometry is found by superimposing propagating Gaussian waves to form a standing wave between the curved mirrors. Diffraction occurs because the transverse width of the confocal mirrors is insufficient to capture the transverse extent of the waveguide modes HE_{mn} . With each subsequent reflection, the Gaussian wave loses a fraction of its power. Schematically, lower-order modes (with fewer variations in the

y-direction) have a broader transverse “footprint”, so more readily diffract for a given confocal aperture width. This effect is shown in Fig. 2.

Λ is written in terms of the Fresnel diffraction parameter C_F and the radial spheroidal wavefunction expressed in prolate spheroidal coordinates [36,39] $R_{0,m}^{(1)}(\xi_1, \xi_2)$:

$$\Lambda = 2 \ln \left[\sqrt{\frac{\pi}{2C_F}} \frac{1}{R_{0,m}^{(1)}(C_F, 1)} \right] \quad (12)$$

The Fresnel diffraction parameter defined as $C_F = k_{\perp} a^2 / L_{\perp}$, where a is half the width of the confocal mirrors. In Eq. 12, $R_{0,m}^{(1)}(C_F, 1)$ is a function of C_F as well as m , which specifies the transverse mode content of the HE_{mn} waveguide modes. For the first three values of m (0, 1, 2) the values of $R_{0,m}^{(1)}(C_F, 1)$ are shown in Fig. 10 and $\log(\Lambda)$ are shown in Fig 4, both as a function of C_F .

In the wave vector component k_z , losses incurred are due to the imaginary component, k_{zi} . If we consider a *loss rate* in terms of decades per axial distance, then the loss rate is expressed as

$$\text{Loss Rate} = \frac{20}{\ln(10)} k_{zi} \quad (13)$$

in which

$$k_{zi} = \text{Im} \sqrt{\left(\frac{\omega}{c}\right)^2 - k_{\perp}^2} = -\frac{k_{\perp r} k_{\perp i}}{k_{zr}} \quad (14)$$

where $k_{zr} = \sqrt{\frac{\omega^2}{c^2} - k_{\perp}^2}$ is assumed to be not close to 0 (i.e. the mode is not close to cutoff; $k_{2r}^2 \gg 2k_{\perp r} k_{\perp i}$). We calculate the attenuation in decades per unit length as a function of C_F (or aperture a). Figure 5 shows a comparison between the theoretical loss rate and numerical simulation results from the commercial numerical software program CST Microwave Studio. For the theoretical loss rate, Eq. 13 was used. These results are for a frequency of 140 GHz in a confocal geometry with $R_c = 6.83$ mm.

C. Field equations and RF Lorentz force

With the general solution of Eq. 5, we may write the vector fields of $\vec{E}$ and $\vec{H}$ as

$$E_x = \frac{k}{k_{\perp r}^2} \frac{\partial \Psi}{\partial y} = \frac{k}{k_{\perp r}} (U^* - U) + \frac{k}{2k_{\perp r}^3} \frac{\partial^2}{\partial x^2} (U^* - U) \quad (15)$$

$$E_y = -\frac{k}{k_{\perp r}^2} \frac{\partial \Psi}{\partial x} = i \frac{k}{k_{\perp r}^2} \frac{\partial}{\partial x} (U + U^*) \quad (16)$$

$$H_x = \frac{k_{zr}}{k_{\perp r}^2} \frac{\partial \Psi}{\partial x} = -i \frac{k_{zr}}{k_{\perp r}^2} \frac{\partial}{\partial x} (U + U^*) \quad (17)$$

$$H_y = \frac{k_{zr}}{k_{\perp r}^2} \frac{\partial \Psi}{\partial y} = \frac{k_{zr}}{k_{\perp r}} (U^* - U) + \frac{k_{zr}}{2k_{\perp r}^3} \frac{\partial^2}{\partial x^2} (U^* - U) \quad (18)$$

along with the general solution that $H_z = U + U^*$ and, because of the transverse electric nature of this waveguide mode, $E_z = 0$, with $k = \omega/c$.

Equations 15–18 describe the transverse components under the quasi-optical approximation. Using these components, the RF Lorentz ($\vec{G}$) force may be found. The components of this force are necessary in order to analyze the interaction in this waveguide mode. A thin, annular electron beam of radius R_g (the “guiding center”) is injected into the confocal cavity, as seen in Fig. 5. At each position around the guiding center exist populations of electrons with gyroradius r_c . The coordinate transform between the guiding center (X, Y) and the beam center (x, y) is given by

$$x = X + r_c \cos \theta \quad (19)$$

$$y = Y + r_c \sin \theta \quad (20)$$

It is particularly useful to find the radial and azimuthal components of the Lorentz force (G_r & G_θ) at every guiding center due to the azimuthally-symmetrical electron beam [1].

We start with the radial component of the RF Lorentz force.

$$G_r = (E_x - \beta_z H_y) \cos \theta + (E_y + \beta_z H_x) \sin \theta + \beta_\perp H_z \quad (21)$$

where β_z and $\beta_\perp$ are the axial and perpendicular electron velocities, v_z and $v_\perp$, normalized to the speed of light c , respectfully. We expand Eq. 21 in azimuthal harmonics:

$$G_r = \sum_l G_{lr} \exp(-il\theta) \quad (22)$$

As we are concerned with the fundamental harmonic, the first coefficient of the expansion ($l = 1$) should be found. With a coordinate transformation to the guiding center (X, Y) , we may use Eqs. 15–18, 21 to express G_{1r} in terms of the function U .

$$G_{1r} = -\frac{1}{2} \frac{k - \beta_z k_{zr}}{k_{\perp r}} (U - U^*) - \frac{1}{2} \frac{k - \beta_z k_{zr}}{2k_{\perp r}^3} \left(\frac{\partial^2 U}{\partial X^2} - \frac{\partial^2 U^*}{\partial X^2} \right) - \frac{1}{2} \frac{k - \beta_z k_{zr}}{k_{\perp r}^2} \left(\frac{\partial U}{\partial X} + \frac{\partial U^*}{\partial X} \right) + \beta_\perp k_{\perp r} r c \frac{1}{2} (U - U^*) \quad (23)$$

In the limit of $k_{\perp r} c \rightarrow 0$, and using Eqs. 5 and 18 we can further reduce Eq. 23 with reference to Ψ as

$$G_{1r} = -i \frac{k - \beta_z k_{zr}}{2k_{\perp r}^2} \left(\frac{\partial \Psi}{\partial X} + i \frac{\partial \Psi}{\partial Y} \right) \quad (24)$$

Equation 24 relates the fundamental harmonic's radial RF Lorentz force to the membrane function, Ψ , the latter of which we have found via the quasi-optical approximation of the confocal field distribution. An analogous calculation may be computed for the azimuthal component of the Lorentz force, G_θ , which relates to the Cartesian field components as

$$G_\theta = (E_y + \beta_z H_x) \cos \theta - (E_x - \beta_z H_y) \sin \theta \quad (25)$$

Using a similar expansion $G_\theta = \sum_l G_{l\theta} e^{-il\theta}$, and restricting ourselves to the fundamental harmonic ($l = 1$), a similar treatment leads to an expression for $G_{1\theta}$ in terms of Ψ :

$$G_{1\theta} = -\frac{k - \beta_z k_{zr}}{2k_{\perp r}^2} \left(\frac{\partial \Psi}{\partial X} + i \frac{\partial \Psi}{\partial Y} \right) \quad (26)$$

Equations 24 and 26 comprise force terms that will be used in deriving the self-consistent set of gyro-TWT equations.

We have shown that within the quasioptical approximation of the membrane function Ψ (Ex. 5) we reduce the theory of the open waveguide gyro-TWT to the theory of the closed

waveguide gyro-TWT [1]. The derivations in the Appendices help to apply the theory [1] to this particular geometry under the simplifications introduced by the Gaussian beam approximation.

III. Beamlets

The self-consistent, nonlinear equations that describe the interaction physics in a gyro-amplifier were developed by Yulpatov [40] and later generalized by Nusinovich and Li [2]. The derivation of the gain equations is summarized in Appendix A. We introduce the concept of *beamlets* as a tool to be used for numerically solving the gain equations for a gyro-amplifier that is not cylindrically symmetric.

The spatial geometry of the interaction between the electron beam and the electromagnetic wave is embodied by the form factor, L_s (for the fundamental harmonic, $s = 1$), of the waveguide mode. The form factor is described fully in Appendix A. In the case of the confocal cavity, the form factor takes on different values at different guiding centers. This is seen in Fig. 6, which schematically features the annular electron beam discretized, overlaid on to the HE_{06} spatial mode of the confocal waveguide.

The field pattern is clearly different at individual points around the guiding radius of the electron beam, and this must be taken into account for the simulation of the gain from this interaction. In fact, the local coupling factor L_1 can be significantly different at various locations about the annular electron beam's interaction with the confocal mode. In Appendix A, we numerically find this variation in L_1 , shown in Fig. 7. The value of $R_g = 1.6$ mm was chosen to optimize coupling to the second and fifth peaks along the y-direction. A smaller beam diameter to couple to the third and fourth peaks was not considered practical due to the required cathode size and space charge effects.

IV. Linear Gain

In this section we show that the asymmetry of a mode and electron beam is irrelevant in the computation of linear gain. The linear gain derived assumes that the magnetic field is uniform and constant over the interaction length. It suffices to simply take the average of the form factor, $\langle |L_1|^2 \rangle$ in the computation of linear gain.

We begin with the differential equations describing the evolution of electron phase, electron energy, and field amplitude, denoted respectively by [1] as θ , w , and C . The equations are derived in Appendix A. That derivation presents a stationary theory for the waveguide excitation due to an electron beam. In this approach, the field equation is of the first order with respect to axial distance, z , and is appropriate to describe the excitation of a waveguide mode at a single frequency. When operating close to cutoff, a different approach based on a second order field equation is taken, as discussed in [41].

A fundamental approximation taken in this derivation is that open modes of the confocal waveguide are treated with the excitation theory assuming closed waveguides. This assumption is appropriate when the confocal modes are considered to be confined, and therefore excitation of leaky waves by the electron beam is not included.

For N guiding centers, these differential equations become

$$\frac{d\theta_n}{dz} = \frac{1}{1-bw_n} \left\{ \mu w_n - \Delta + \frac{1}{2\sqrt{1-w_n}} \text{Im}[CL_{1n}e^{-i\theta_n}] \right\} \quad (27)$$

$$\frac{dw_n}{dz} = -\frac{\sqrt{1-w_n}}{1-bw_n} \text{Re}[CL_{1n}e^{-i\theta_n}] \quad (28)$$

$$\frac{dC}{dz} = -\frac{1}{N} I_o \sum_{n=1}^N \frac{1}{4\pi} \int_0^{2\pi} \frac{\sqrt{1-w_n}}{1-bw_n} L_{1n} * e^{-i\theta_n} d\theta_{0n} \quad (29)$$

where the subscript n denotes the n th guiding center of an annular electron beam. Effectively, the net field amplitude is the sum of contributions of electrons at all guiding centers, which in a numerical code will be discretized. The parameters b , μ , and Δ are related to the initial conditions of the electron beam-RF interaction at the beginning of the circuit ($z = 0$) and are defined in Appendix A. Additionally, the values of θ_{0n} , w_{0n} and C at $z = 0$ are, respectively, a uniform distribution of the electrons over the entire Larmor radius in phase (0 to 2π), $w_{n/z=0} = 0$, and $C(z=0)$ numerically determined by the power in the waveguide mode. Furthermore, the integral in Eq. 29 is taken over the initial phase θ_{0n} at $z = 0$. In general, the terms L_{1n} are different for different guiding centers. Using the substitution

$$C = \bar{C} e^{-i\Delta z} \quad (30)$$

as well as a first order expansion

$$\theta_n = \theta_{0n} - \Delta z + \theta_n^{(1)} \quad (31)$$

$$w_n = w_n^{(1)} \quad (32)$$

we can rewrite Eqs. 27–29 in the following form:

$$\frac{d\theta_n^{(1)}}{dz} = (\mu - \Delta b) w_n^{(1)} + \frac{1}{2} \text{Im}[\bar{C} L_{1n} e^{-i\theta_{0n}}] \quad (33)$$

$$\frac{dw_n^{(1)}}{dz} = -\text{Re}[\bar{C}L_{1n}e^{-i\theta_{0n}}] \quad (34)$$

$$\frac{d\bar{C}}{dz} - i\Delta\bar{C} = -\frac{1}{N}I_0 \sum_{n=1}^N \frac{1}{2\pi} \int_0^{2\pi} \left[\left(b - \frac{1}{2}\right) w_n^{(1)} + i\theta_n^{(1)} \right] \frac{1}{2} L_{1n}^* e^{i\theta_{0n}} d\theta_{0n} \quad (35)$$

The superscript (1) denotes the first-order terms for w_n and θ_n . To handle the integral over θ_{0n} in Eq. 35, we introduce a change of variables:

$$\bar{w}_n^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} w_n^{(1)} L_{1n}^* e^{i\theta_{0n}} d\theta_{0n} \quad (36)$$

$$\bar{\theta}_n^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} \theta_n^{(1)} L_{1n}^* e^{i\theta_{0n}} d\theta_{0n} \quad (37)$$

With this change of variables, Eqs. 33–35 become

$$\frac{d\bar{\theta}_n^{(1)}}{dz} = (\mu - \Delta b) \bar{w}_n^{(1)} - \frac{i}{4} \frac{1}{2\pi} \int_0^{2\pi} \bar{C} L_{1n} L_{1n}^* d\theta_{0n} \quad (38)$$

$$\frac{d\bar{w}_n^{(1)}}{dz} = -\frac{1}{2} \bar{C} L_{1n} L_{1n}^* \quad (39)$$

$$\frac{d\bar{C}}{dz} - i\Delta\bar{C} = -\frac{1}{N} I_0 \frac{1}{2} \sum_{n=1}^N \left[\left(b - \frac{1}{2}\right) \bar{w}_n^{(1)} + i\bar{\theta}_n^{(1)} \right] \quad (40)$$

As a final step, we note that we can sum over the N guiding centers index, n , by taking the express average over the guiding centers:

$$\bar{\bar{\theta}}^{(1)} = \frac{1}{N} \sum_{n=1}^N \bar{\theta}_n^{(1)} \quad (41)$$

$$\overline{\overline{w}}^{(1)} = \frac{1}{N} \sum_{n=1}^N \overline{w}_n^{(1)} \quad (42)$$

Finally, we can simplify Eqs. 38–40 to:

$$\frac{d\overline{\overline{\theta}}^{(1)}}{dz} = (\mu - \Delta b) \overline{\overline{w}}^{(1)} - \frac{i}{4} \overline{C} \frac{1}{N} \sum_{n=1}^N |L_{1n}|^2 \quad (43)$$

$$\frac{d\overline{\overline{w}}^{(1)}}{dz} = -\frac{1}{2} \overline{C} \frac{1}{N} \sum_{n=1}^N |L_{1n}|^2 \quad (44)$$

$$\frac{d\overline{C}}{dz} - i\Delta \overline{C} = -\frac{1}{2} I_0 \left(b - \frac{1}{2} \right) \overline{\overline{w}}^{(1)} - \frac{i}{2} I_0 \overline{\overline{\theta}}^{(1)} \quad (45)$$

In considering linear gain, we assume that $\frac{d}{dz} = i\Gamma$. Consequently, Eqs. 43–45 can be reduced to

$$(\mu - \Delta b) \overline{\overline{w}}^{(1)} - i\Gamma \overline{\overline{\theta}}^{(1)} - \frac{i}{4} \langle |L_1|^2 \rangle \overline{C} = 0 \quad (46)$$

$$i\Gamma \overline{\overline{w}}^{(1)} + \frac{1}{2} \langle |L_1|^2 \rangle \overline{C} = 0 \quad (47)$$

$$\frac{1}{2} I_0 \left(b - \frac{1}{2} \right) \overline{\overline{w}}^{(1)} + \frac{i}{2} I_0 \overline{\overline{\theta}}^{(1)} + (i\Gamma - i\Delta) \overline{C} = 0 \quad (48)$$

in which $\langle |L_1|^2 \rangle \equiv \frac{1}{N} \sum_{n=1}^N |L_{1n}|^2$ is the averaged form factor over the guiding centers. We note that Eqs. 46–48 show that the reduced first-order expansion of considering multiple beamlets has simplified to the standard expression for linear growth as derived in [2], provided that the form factor in question is computed from an average of all guiding radii.

Indeed, with the variable $I'_0 = \frac{1}{4} I_0 \langle |L_1|^2 \rangle$, we can find that the polynomial dictating the gain term Γ is

$$\Gamma^2(\Gamma - \Delta) + \Gamma(b - 1)I'_0 + (\mu - \Delta b)I'_0 = 0 \quad (49)$$

Thus, the linear gain when considering form factors that depend on the guiding center is identical to that of an approach that averages over all of the guiding centers. This fact reflects the superposition principle in the linear regime. This is useful computationally,

because it shows that if the form factor L_1 is calculated from $\sqrt{\langle |L_1|^2 \rangle}$, the resulting value can be used in Eq. 49 to compute the linear gain directly, but not the nonlinear gain.

V. Nonlinear Code Results

The nonlinear beamlet code was developed for application to interaction circuits which lack azimuthal symmetry (e.g. a confocal waveguide operating in the HE_{06} mode). For the beamlet method, the annular beam is divided into discrete beamlets, each with a coupling factor L_{1n} calculated according to their relative location in the field distribution (c.f. Fig. 6 and 8). A second method using a single beamlet and beam averaged coupling coefficient $\langle L_1 \rangle$ is presented for comparison. For the beamlet method in all instances, 100 total beamlets are utilized, which well-discretizes the spatial variance of the fields. As a means of verifying the beamlet code, the gain for a circular waveguide operating in the azimuthally symmetric TE_{03} mode was calculated and compared with the predictions of MAGY [42]. Figure 9 shows that the gain predicted by MAGY and the beamlet code are almost identical.

The gain of a confocal waveguide operating in the HE_{06} mode is shown in Fig. 9 for both the beamlet method and the beam averaged coupling coefficient approach. These simulations were performed for 140 GHz with 50 mW input power, 5.085 T, 37 kV, 3 A beam current, confocal rail spacing $R_c = 6.83$ mm, a pitch factor α of 0.9, and a loss per unit length of 3 dB/cm, which represent typical operating values.

A difference of ~ 1.7 dB in peak gain is computed between the code using an averaged L_1 and that of the beamlet simulation as seen in Fig. 9. This 1.7 dB difference in peak gain is observed to be invariable over a range of input power, as seen in Fig. 11 for the same operating conditions. Although the saturated gain difference of 1.7 dB appears small, it is important when considering power. For the case shown in Fig. 9, the estimated saturated output power is 500 W for the averaged case, but is only 360 W when accurately calculated by the beamlet theory.

Electron beam velocity spread is an important factor that can impact the predicted gain of a gyrotron amplifier and has been studied analytically as well as numerically [43,44]. We take the approach used in [44] to model the effect of a non-cold electron beam (a distribution of electron velocities is introduced). The velocity spread model, which conserves total energy, assumes a Gaussian distribution in the perpendicular velocity component:

$$f(\beta_{\perp}) = \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{(\beta_{\perp} - \beta_{\perp 0})^2}{\sigma^2}\right) \quad (50)$$

where $\beta_{\perp 0}$ is the mean value of $\beta_{\perp}$ and σ is the width.

The developed code was used to study the effect of velocity spread on the saturated gain in the confocal mode. The calculated gain in Fig. 12 is shown for an operating point of 5.085 T, 37 kV, 3 A beam current, a pitch factor of 0.8, confocal radius 6.83 mm, 3 dB/cm waveguide loss, and 1 W input power at 140 GHz.

The difference between saturated gain as calculated with an averaged coupling factor and the beamlet approach is shown in Fig. 12.

The results in Fig. 12 emphasize the importance of properly accounting for the effect of velocity spread on circuit gain. Again, as in Fig. 12, the use of a beam averaged coupling coefficient is found to overpredict the saturated circuit gain.

VI. Discussion

The equations of motion for electrons and the mode excitation equations for the microwave fields have been derived. Using the quasi-optical approximation for the spatial modes of a confocal waveguide, we see that there is a difference in the peak gain as calculated for the case of an averaged coupling factor, L_1 , or using a beamlet code to sample the interaction at different guiding centers. Due to nulls in the confocal modes, portions of the annular electron beam are not interacting strongly with the mode. Consequently, electrons at these guiding centers may not fully bunch, and the energy that they carry will remain in the transverse velocity component. Qualitatively, this phenomenon of imperfect bunching due to local nulls in the electromagnetic spatial modes of an interaction circuit will occur in waveguide designs that support such spatial modes, and may be accounted for in numerical simulations by using beamlets. In the case of the confocal waveguide modes, it is seen that the interaction between electrons in a given guiding center with the spatial mode depends on the location of the guiding center relative to the confocal geometry. Therefore, calculating an averaged interaction value may over-sample the net coupling between the electron beam and the waveguide mode. This will result not only in a higher predicted gain but also in a diminished effect from increased waveguide loss, as sections of the electron beam that in reality are not interacting with the waveguide mode are included via the averaging calculation. In the beamlet case, which accurately accounts for the asymmetry of the waveguide mode, certain regions of the electron beam do not factor prominently into the gyrotron interaction. Consequently, those beamlets that are interacting with a strong field region amount to the entirety of the gyrotron interaction. Increased waveguide loss lowers the field strength, which adversely affects the bunching mechanism, leading to a greater difference in the predicted gain when comparing the averaged coupling against the beamlet code. We note that the numerical results presented do not distinguish between diffractive and ohmic loss, and therefore the conclusions apply to both physical situations. Finally, the inclusion of velocity spread was demonstrated. At moderate velocity spread values, the difference in calculated gain has only a minor dependence on the absolute velocity distribution. This is consistent with the argument that it is the spatial distribution of asymmetric modes, and the poor coupling between beam and mode at certain spatial locations, that contributes to a difference in calculated saturated gain.

The work presented has up to this point not considered the effect of guiding center drift. The $\vec{E} \times \vec{B}$ force may introduce an azimuthal rotation of the guiding center positions. For a purely transverse electric mode, such as the HE_{06} mode of the confocal waveguide, the axial component of the electric field is zero. Thus, the possible sources for this azimuthal guiding center drift are due to the radial electric components from the DC space charge depression and the microwave field. For the operating point presented in Fig. 8 (5.085 T, 37 kV, $\alpha = 0.9$, and a pipe radius of 3.54 mm) the DC space charge depression is about 0.5 kV. This results in a total accumulated azimuthal drift over a 20 cm interaction length of about 7 degrees, which is small enough to be negligible in the analysis. The transverse microwave E-field also can contribute to the azimuthal drift. For 500 W of power, this E-field is an order of magnitude smaller than the DC E-Field and it also is only present in the last centimeter of the gain section, so that it can be ignored.

This article has focused solely on an annular beam configuration. A sheet beam, as demonstrated in [45], might represent an alternate approach to enhance the coupling if the sheet beam is aligned to interact with the middle peak of an HE_{0m} mode of a confocal waveguide with m odd. Additionally, a large orbit gyrotron could also be used [46].

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Appendices

A. Derivation of governing equations for a gyro-twt

A. Motion equations

The motion equations that govern the dynamics of an electron in a waveguide field begin with the fundamentals [1]:

$$\frac{dp_{\perp}}{dz} = -\frac{e}{v_z} \operatorname{Re} \left(CG_{\theta} e^{i(\omega t - k_z r, z)} \right) \quad (\text{A.1})$$

$$p_{\perp} \frac{d\psi}{dz} = \frac{e}{v_z} \text{Re} \left(C G_r e^{i(\omega t - k_{zr} z)} \right) \quad (\text{A.2})$$

where C is the waveguide mode amplitude, $p_{\perp}$ is the perpendicular electron momentum, G_{θ} and G_r are the RF Lorentz force components, and ψ is the electron phase in its gyro-orbit. In order to keep track of the phase advance due to electrons gyrating in the magnetic field, using the longitudinal momentum p_z we introduce the variable $h_H = eB_0/cp_z$ (which conveniently allows us to write the Larmor radius as $r_c = \frac{p_{\perp}}{h_H p_z}$) and B_0 is the magnetic field.

The longitudinal momentum is $p_z = \gamma mc\beta_z$. Then, the gyrophase can be defined as

$$\theta = h_H z + \psi \quad (\text{A.3})$$

Equation A.3 is useful to quantify the phase of the waveguide mode with respect to the gyrophase:

$$\omega t - k_{zr} z - \theta = -\vartheta \quad (\text{A.4})$$

Furthermore, as follows from Eqs. 24, 26

$$\frac{dp_{\perp}}{dz} = \frac{e}{v_z} \text{Re} \left(C \frac{k - \beta_z k_{zr}}{2k_{\perp r}} L_1(X, Y) e^{-i\vartheta} \right) \quad (\text{A.5})$$

where the coupling factor L_1 is

$$L_1(X, Y) = \frac{1}{k_{\perp r}} \left(\frac{\partial \Psi}{\partial X} + i \frac{\partial \Psi}{\partial Y} \right) \quad (\text{A.6})$$

We can then take Eqs. A.3 and A.4 and find that

$$\frac{d\vartheta}{dz} = \frac{d\psi}{dz} + \frac{\Omega + k_{zr} v_z - \omega}{\beta_z c} \quad (\text{A.7})$$

where $\Omega = eB_0/\gamma mc$ is the cyclotron frequency. Now, we rewrite Eq. A.2 as

$$p_{\perp} \frac{d\vartheta}{dz} + \frac{\omega - k_{zr} v_z - \Omega}{v_z} p_{\perp} = \frac{e}{v_z} \text{Re} \left(C(-i) \frac{k - \beta_z k_{zr}}{2k_{\perp r}} L_1(X, Y) e^{-i\vartheta} \right) \quad (\text{A.8})$$

For $\gamma_0, \beta_{z0}, \beta_{\perp 0}$, and Ω_0 defined at the beginning of the interaction ($z = 0$), we use normalized electron energy, w , and the parameter b :

$$w = 2 \frac{1 - \frac{k_{zr}}{k} \beta_{z0}}{\beta_{\perp 0}^2} \frac{\gamma_0 - \gamma}{\gamma_0} \quad (\text{A.9})$$

$$b = \frac{k_{zr}}{k} \beta_{\perp 0}^2 \frac{1}{2\beta_{z0} (1 - \frac{k_{zr}}{k} \beta_{z0})} \quad (\text{A.10})$$

(It can be seen that the product $bw = \frac{k_{zr}}{k} \frac{1}{2\beta_{z0}} \frac{\gamma_0 - \gamma}{\gamma_0}$). The components of momentum may be rewritten to incorporate normalized electron energy as follows:

$$p_z = p_{z0} (1 - bw) \quad (\text{A.11})$$

$$p_{\perp} = p_{\perp 0} \sqrt{1 - w} \quad (\text{A.12})$$

We introduce the parameter μ ,

$$\mu = \frac{\beta_{\perp 0}^2 \left(1 - \frac{k_{zr}^2}{k^2} \right)}{2 \left(1 - \frac{k_{zr}}{k} \beta_{z0} \right)} \quad (\text{A.13})$$

The term $\frac{\omega - k_{zr} v_z - \Omega}{\beta_z c}$ in Eq. A.7 may be expressed with b and μ as

$$\frac{\omega - k_{zr} v_z - \Omega}{\beta_z c} = \frac{\omega}{c\beta_{z0} (1 - bw)} \left(\frac{\omega - k_{zr} v_{z0} - \Omega_0}{\omega} - \mu w \right) \quad (\text{A.14})$$

Therefore, Eq. A.7 is conveniently expressed as

$$\frac{d\vartheta}{dz} = -\frac{\omega}{c\beta_{z0}(1-bw)} \left(\frac{\omega - k_{zr}v_{z0} - \Omega_0}{\omega} - \mu w \right) + \frac{em\gamma_0}{p_{z0}(1-bw)} \frac{1}{p_{\perp 0} \sqrt{1-w}} \times \text{Re} \left(-iC \frac{1-\beta_{z0} \frac{k_{zr}}{k}}{2k_{\perp r}} k L_1(X, Y) e^{-i\vartheta} \right)$$

(A.15)

For further convenience, we may define the detuning parameter :

$$\Delta = \frac{\omega - k_{zr}v_{z0} - \Omega_0}{\omega} \quad (\text{A.16})$$

and normalize length as $z' = kz$. Then, the relative gyrophase ϑ , in normalized coordinates z' , varies as

$$\frac{d\vartheta}{dz'} = \frac{1}{\beta_{z0}(1-bw)} \left(\mu w - \Delta + \frac{1 - \frac{k_{zr}}{k} \beta_{z0}}{\gamma_0 \beta_{\perp 0} \sqrt{1-w}} \text{Im} \left[\frac{eC}{2mc^2 k_{\perp r}} L_1 e^{-i\vartheta} \right] \right) \quad (\text{A.17})$$

To find the differential of the normalized electron energy, w , we may take the derivative of Eq. A.12:

$$\frac{dw}{dz'} = -2 \frac{\sqrt{1-w}}{1-bw} \left(1 - \beta_{z0} \frac{k_{zr}}{k} \right) \frac{1}{\gamma_0 \beta_{z0} \beta_{\perp 0}} \times \text{Re} \left(\frac{eC}{2mc^2 k_{\perp r}^2} L_1 e^{-i\vartheta} \right) \quad (\text{A.18})$$

We denote additional normalized variables with a prime:

$$\mu' = \frac{\mu}{\beta_{z0}} \quad (\text{A.19})$$

$$\Delta' = \frac{\Delta}{\beta_{z0}} \quad (\text{A.20})$$

$$C' = \frac{eC}{mc^2 \gamma_0 \beta_{z0} \beta_{\perp 0}} \left(1 - \beta_{z0} \frac{k_{zr}}{k} \right) \frac{1}{k_{\perp r}} \quad (\text{A.21})$$

The above simplifications allow us to summarize the variation of both the relative gyrophase, ϑ , and normalized electron energy, w , with respect to normalized coordinates z' :

$$\frac{d\vartheta}{dz'} = \frac{1}{1-bw} \left(\mu' w - \Delta' + \frac{1}{\sqrt{1-w}} \text{Im} \left[C' \frac{1}{2} L_1 e^{-i\vartheta} \right] \right) \quad (\text{A.22})$$

$$\frac{dw}{dz'} = -2 \frac{\sqrt{1-w}}{1-bw} \text{Re} \left(C' \frac{1}{2} L_1 e^{-i\vartheta} \right) \quad (\text{A.23})$$

Equations A.22 and A.23 fully characterize the dynamical properties of an electron beam interacting with a waveguide mode at any guiding center (X, Y) .

B. Mode excitation equations

Just as an electron is affected by the presence of a waveguide mode's EM field, the waveguide mode itself is also changed by the electron beam. Following the derivation in [1], the mode amplitude C_s (of the s -th mode of the waveguide) interacts with the electron beam via the mode's electric field as integrated over the waveguide cross section $S_\perp$:

$$\frac{dC_s}{dz} = -\frac{1}{N_s} \int_{S_\perp} \vec{j}_\omega \cdot \vec{E}_s^* ds_\perp \quad (\text{A.24})$$

in which $\vec{j}_\omega$ is the current density component at the angular frequency ω . The normalization factor N_s of the s -th mode is

$$N_s = \frac{c}{4\pi} \int_{S_\perp} (\vec{E}_s \times \vec{H}_s^* + \vec{E}_s^* \times \vec{H}_s) \cdot \vec{z}_0 ds_\perp = \frac{c}{2\pi} \text{Re} \int_{S_\perp} (\vec{E}_s \times \vec{H}_s^*) \cdot \vec{z}_0 ds_\perp = 4P_s \quad (\text{A.25})$$

In Eq. A.25, P_s is the Poynting vector. The normalization factor N_s is explicitly calculated in Appendix Section B. Since the electric field is purely transverse, the integrand in Eq. A.24 is $\vec{j}_\omega \cdot \vec{E}_s^* = \vec{j}_{\omega\perp} \cdot \vec{E}_s^*$. Additionally, we know that by charge conservation in the electron beam, at any cross sectional slice the charge entering and exiting is conserved: $j_z dt = j_{z0} dt_0$. So, in considering the transverse current, we can relate it to the axial component by

$\vec{j}_\perp d(\omega t) = \vec{j}_\perp \frac{j_{z0}}{j_z} d(\omega t_0)$. Transforming to frequency space [1]:

$$\vec{j}_{\omega\perp} \cdot \vec{E}_s^* = \frac{1}{\pi} \int_0^{2\pi} \vec{j}_\perp \cdot \vec{E}_s^* e^{-i\omega t} d(\omega t) \quad (\text{A.26})$$

Using the formulation to relate $d(\omega t)$ to $d(\omega t_0)$, Eq. A.26 may be rewritten as

$$\frac{1}{\pi} \int_0^{2\pi} \frac{j_{\perp} \vec{j}_{z0}}{j_z} e^{-i\omega t} \vec{E}_s^* d(\omega t_0) = \frac{1}{\pi} j_{z0} \int_0^{2\pi} \frac{\vec{p}_{\perp}}{p_z} \vec{E}_s^* e^{-i\omega t} d(\omega t_0) \quad (\text{A.27})$$

Using the relations for momentum in Eqs. A.11 and A.12 as well as recognizing that the area integral in Eq. A.24 reduces to $(j_{z0} S_{\perp})$, the differential equation for the growth of the amplitude C (dropping the mode number s) becomes

$$\frac{dC}{dz} = \frac{1}{N} \frac{1}{\pi} (j_{z0} S_{\perp}) \int_0^{2\pi} \frac{p_{\perp 0} \sqrt{1-w}}{p_{z0}(1-bw)} E_{1\vartheta}^*(X, Y) e^{i\vartheta} d\vartheta_0 \quad (\text{A.28})$$

in which $E_{1\vartheta}$ is the azimuthal component of the electric field (which interacts with the gyrating current component). The interaction is at the fundamental harmonic, so in the usual expansion $E_{\vartheta} = \sum E_{l\vartheta} e^{-il\vartheta}$ we are interested in $l = 1$.

Using the field Eqs. 15 and 16 and the fact that $E_{\vartheta} = E_y \cos \vartheta - E_x \sin \vartheta$, the fundamental harmonic $E_{1\vartheta}$ is

$$E_{1\vartheta} = -\frac{k}{2k_{\perp r}^2} \left(\frac{\partial \Psi}{\partial X} + i \frac{\partial \Psi}{\partial Y} \right) = -\frac{k}{2k_{\perp r}} L_1 \quad (\text{A.29})$$

This is a result that follows the derivation for G_{1r} in Eq. 21. Thus, the amplitude differential dC/dz is rewritten as

$$\frac{dC}{dz} = -\frac{1}{N} \frac{1}{\pi} I_b \int_0^{2\pi} \frac{p_{\perp 0} \sqrt{1-w}}{p_{z0}(1-bw)} \frac{k}{2k_{\perp r}} L_1^* e^{i\vartheta} d\vartheta_0 \quad (\text{A.30})$$

where I_b is the beam current. It is convenient to rewrite Eq. A.30 in normalized form:

$$\frac{dC'}{dz'} = -\frac{1}{N} \frac{1}{\pi} I_b \frac{e}{mc^2 \gamma_0 \beta_{\perp 0} \beta_{z0} k_{\perp r}} \left(1 - \frac{k_{z0}}{k} \beta_{z0} \right) \times \frac{\beta_{\perp 0}}{\beta_{z0} k_{\perp r}} \int_0^{2\pi} \frac{\sqrt{1-w}}{1-bw} \frac{L_1^*}{2} e^{i\vartheta} d\vartheta_0 \quad (\text{A.31})$$

We make one final simplification and introduce the normalized current, I_0 , defined as

$$I_0 = 2 \left(\frac{e I_b}{mc^3} \right) \frac{1 - \frac{k_{z0}}{k} \beta_{z0}}{\gamma_0 \beta_{z0}^2 \left(\frac{k_{\perp r}}{k} \right)^2} \frac{c^3}{\omega^2 N} \quad (\text{A.32})$$

where

$$\frac{eI_b}{mc^3} = \frac{I_b(A)}{17000(A)} \quad (\text{A.33})$$

In Eq. A.33 the fraction $\frac{mc^3}{e}$ reduces to 17000 amps in SI units. Finally, the differential of the normalized amplitude C' in normalized units z' is expressed as

$$\frac{dC'}{dz'} = -I_0 \frac{1}{2\pi} \int_0^{2\pi} \frac{\sqrt{1-w}}{1-bw} \frac{L_1^*}{2} e^{i\vartheta} d\vartheta_0 \quad (\text{A.34})$$

Equations A.22, A.23, and A.34 comprise the (normalized) self-consistent set of equations that describe gyro-TWT operation. In Eq. A.34, a loss term may be introduced in the system if we consider an imaginary component to the axial wavevector, k_{zi} . Properly normalized,

loss in the system is represented by $\frac{dC'}{dz'} \sim -k'_{zi} C'$ where $k'_{zi} = k_{zi}/k$.

In the absence of loss, it is convenient to derive an energy conservation relation. Since energy is transferred between the electrons and the field (power $\sim |C|^2$), we may use Eqs. A. 34 and A.23 to find that

$$|C'|^2 - |C'_{z=0}|^2 = I_0 \frac{1}{2\pi} \int_0^{2\pi} w d\vartheta_0 \quad (\text{A.35})$$

The form of Eq. A.35 is an expression of the conservation of energy in the system and relates the change of microwave energy to that of the electrons.

B. Normalization factor

The normalization factor first introduced in Eq. A.25 has an analytical expression dependent on the field distribution of the operating mode. In our case, the quasi-optical approximation of the confocal resonator is used to find an analytical solution for the normalization factor. To begin, we may relate the normalization factor N_s to the field distribution U by taking the explicit cross product in Eq. A.25. Expanding the cross product, we find that

$$\vec{E}_s \times \vec{H}_s^* \cdot \vec{z}_0 = E_x H_y^* - E_y H_x^* \quad (\text{B.1})$$

Referring to Eqs. 15 – 18, Eq. B.1 is equivalently

$$E_x H_y^* - E_y H_x^* = \frac{k k_{zr}}{k_{\perp r}^2} (U - U^*)^2 + \frac{k k_{zr}}{k_{\perp r}^4} \left(\frac{\partial U}{\partial X} + \frac{\partial U^*}{\partial X} \right)^2 \quad (\text{B.2})$$

Equation B.2 is a general result; we may consider the quasi-optical approximation of the field distribution in a confocal resonator by using the expression for U given in Eq. 6 and write the integral in Eq. A.25:

$$\begin{aligned} N_s = & \frac{c}{2\pi} \frac{k k_{zr}}{k_{\perp r}^2} \iint \left[-4 \sqrt{\frac{2}{\pi}} \frac{w_0}{w} \exp \left(-\frac{2x^2}{w^2} \right) \right. \\ & \sin^2 \left(k_{\perp r} y + \frac{k_{\perp r} x^2}{2R} - \frac{1}{2} \arctan \frac{y}{b} \right) + \\ & \frac{1}{k_{\perp r}^2} 4 \sqrt{\frac{2}{\pi}} \left(\frac{2\pi}{w^2} \sqrt{\frac{w_0}{w}} \exp \left(-\frac{x^2}{w^2} \right) \cos \left(k_{\perp r} y + \frac{k_{\perp r} x^2}{2R} \right. \right. \\ & \left. \left. - \frac{1}{2} \arctan \frac{y}{b} + \frac{k_{\perp r} x}{R} \sqrt{\frac{w_0}{w}} \exp \left(-\frac{x^2}{w^2} \right) \sin (k_{\perp r} y \right. \right. \\ & \left. \left. + \frac{k_{\perp r} x^2}{2R} - \frac{1}{2} \arctan \frac{y}{b} \right) \right)^2 \left. \right] dxdy \quad (\text{B.3}) \end{aligned}$$

Equation B.3 is an exact solution; we may take two approximations that will result in an analytical expression for the integrals. First, the argument of the trigonometric functions,

$k_{\perp r} y + \frac{k_{\perp r} x^2}{2R} - \frac{1}{2} \arctan \frac{y}{b}$, describes a phase front. Under a paraxial approximation, the large term $k_{\perp r} y$ dominates, and therefore the argument of the trigonometric functions reduces to that single term. Second, we notice that in Eq. B.3 the first term comes from the cross product component $E_x H_y^*$ whereas the remaining two are from $E_y H_x^*$. As has been seen in the field distribution of the confocal resonator modes, the transverse term $E_x \gg E_y$. It follows that the first term of Eq. B.3 dominates, and we may drop the remaining two. With these approximations, the normalization reduces to

$$N_s \simeq \frac{c}{2\pi} \frac{k k_{zr}}{k_{\perp r}^2} \iint \left[-4 \sqrt{\frac{2}{\pi}} \frac{w_0}{w(y)} \times \exp \left(-\frac{2x^2}{w^2(y)} \right) \sin^2(k_{\perp r} y) \right] dxdy \quad (\text{B.4})$$

The limits of integration are $-\infty < x < \infty$ and $-L_{\perp}/2 < y < L_{\perp}/2$. We find that the normalization may be approximated as

$$\begin{aligned} N_s & \simeq -\frac{c}{\pi} \frac{k k_{zr}}{k_{\perp}^2} L_{\perp} w_0 \\ \frac{c^3}{\omega^2 N_s} & = -\frac{c^3 \pi k_{\perp}^2}{\omega^2 c k k_{zr} L_{\perp} w_0} = -\frac{\pi}{k^3 k_{zr}} \frac{k_{\perp}^2}{L_{\perp} \sqrt{L_{\perp}}} \sqrt{k_{\perp}} \quad (\text{B.5}) \end{aligned}$$

in which the final step is found from evaluating the radius $R(y=\frac{L_{\perp}}{2})=L_{\perp}$ in Eq. 8 and therefore

$$w_0 = \sqrt{\frac{L_{\perp}}{k_{\perp r}}} \quad (\text{B.6})$$

Fig. 1. The confocal geometry including the field distribution of the HE_{06} mode. The dashed line represents the annular electron beam.

Fig. 2.

The confocal geometry including the field distribution of the HE_{06} mode at 140 GHz (a) and HE_{04} mode at 94 GHz (b). This field distribution is found for an aperture of $2a = 4$ [mm] and $R_c = 6.83$ [mm].

Fig. 3.

The exact values of $R_{0,m}^{(1)}(C_F, 1)$ plotted against C_F for various values of m ; the lines connect calculated points.

Fig. 4. The exact values of $\log(\Lambda)$ plotted against C_F for various values of m ; the lines connect calculated points.

Fig. 5. Comparison of loss rate due to diffraction using Eq. 13 and numerical simulation software CST for an HE_{06} mode. Here, $R_c = 6.83$ mm and the frequency is 140 GHz.

Fig. 6.

An exploded view of the geometry at every guiding center. (X, Y) are the coordinates of the location of the guiding center.

Fig. 7.

Schematic of a HE_{06} mode supported by a confocal geometry. The annular electron beam is shown as a dashed circle, and is discretized into distinct beamlets, shown as red dots. The coupling factor at different beamlets depends on the (asymmetric) field distribution of the HE_{06} mode.

Fig. 8.

Coupling factor L_1 around an electron beam interacting with the HE_{06} mode of a confocal geometry with mirror radius $R_c = 6.83$ mm and beam radius $R_g = 1.6$ mm. Numerically, the coupling factor varies significantly with guiding center location.

Fig. 9.

The gain for a TE_{03} mode as predicted by the beamlet and MAGY codes. The operating condition simulated is for a 141 GHz signal at 5.085 T, 37 kV, a pitch factor of 0.9, 3 A beam current, input power 50 mW, and a pipe radius of 3.54 mm.

Fig. 10. Confocal circuit gain versus interaction distance calculated using a single beam averaged coupling factor $\langle L_1 \rangle$, and 100 beamlets.

Fig. 11.

The peak confocal circuit gain versus input power as calculated using the beam averaged coupling factor method and the beamlet method.

Fig. 12.

Confocal circuit gain versus interaction distance for various specified RMS velocity spread values. Solid curves are calculated using a beam averaged coupling factor $\langle L_1 \rangle$ and dashed curves are calculated using the beamlet method.