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Asymptotics for a Class of Dynamic Recurrent Event Models

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Abstract

Asymptotic properties, both consistency and weak convergence, of estimators arising in a general class of dynamic recurrent event models are presented. The class of models take into account the impact of interventions after each event occurrence, the impact of accumulating event occurrences, the induced informative and dependent right-censoring mechanism due to the data-accrual scheme, and the effect of covariate processes on the recurrent event occurrences. The class of models subsumes as special cases many of the recurrent event models that have been considered in biostatistics, reliability, and in the social sciences. The asymptotic properties presented have the potential of being useful in developing goodness-of-fit and model validation procedures, confidence intervals and confidence bands constructions, and hypothesis testing procedures for the finite- and infinite-dimensional parameters of a general class of dynamic recurrent event models, albeit the models without frailties.

Keywords and Phrases

consistency; compensators; counting processes; full models; marginal models; martingales; repair models; sum-quota accrual; weak convergence

1 Introduction and Background

Recurrent events pervade many disciplines such as the biomedical and public health sciences, engineering sciences, social and political sciences, economic sciences, and even sporting events. Examples of such events are non-fatal heart attacks, hospitalization of a patient with a chronic disease, migraines, breakdown of an electronic or mechanical system, discovery of a bug in a software, disagreement in a marriage, change of a job, Dow Jones Industrial Average (DJIA) decreasing by at least 200 points during a trading day, a perfect baseball game in the Major Leagues, a goal scored in a World Cup soccer game, and many others. The mathematical modeling of recurrent events, together with the development of statistical inference procedures for the models, are of paramount importance.

There are two approaches to the specification of mathematical models for recurrent events. The first is a *full* specification of the probability measure on the measurable space induced by the monitoring of the recurrent event. This is done by specifying the joint distributions of

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the calendar times of event occurrences, or equivalently the joint distributions of the inter-event times. Alternatively, the probability measure can be specified as a measure on the space of paths of the stochastic process arising from the monitoring of the recurrent event. The simplest and perhaps most common full parametric model is when the counting process associated with the event accrual is assumed to follow a homogeneous Poisson process (HPP), in which case the inter-event times are independent and identically distributed (IID) with common negative exponential distribution. One may also specify a nonparametric model by simply assuming that the inter-event time distribution is some unknown continuous distribution, resulting in the IID renewal model. The general dynamic model of interest in this article is of the full model variety.

The second modeling approach is referred to as *marginal* modeling. In its basic form, the event position within a unit is utilized as a stratifying variable, and a (marginal) probability measure is specified for each of the resulting strata. This approach was pioneered in the papers [16, 19]. It should be observed that the class of full models subsumes the class of marginal models. However, proponents of the marginal modeling approach espouse this marginal approach since it generally leads to an easier interpretation of model parameters though, at the same time, it may be difficult to justify a full model which is consistent with the specified marginal models. In fact, there could be several full models that are consistent with the marginal models.

An IID distributional specification for the inter-event times is clearly an oversimplification since it will often be the case that after an event occurrence some type of intervention, such as a corrective measure or a repair, will be performed, thereby altering the distribution of the time to the next event occurrence. Furthermore, time-dependent concomitant variables could also impact the distributions of the inter-event times, and within a unit the interevent times may be correlated owing to unobserved latent variables. The number of event occurrences could also impact these distributions, such as when event occurrences weakens the unit, thereby stochastically shortening the time to the next event occurrence. Due to practical and unavoidable constraints, the monitoring of the event could also only be performed over a finite, possibly random, observation window, and thus a *sum-quota accrual scheme* ensues wherein the number of observed event occurrences is a random variable which is informative about the event occurrence mechanism. This finite monitoring constraint also produces a right-censored observation, which could not be ignored in performing inference because of selection bias issues. The class of dynamic recurrent event models proposed in [12] incorporates the above considerations. This class of models is a specific member of the class of models of interest in this article. The major goals of this article are to obtain the asymptotic properties of semi-parametric estimators of the model parameters for the general class of dynamic recurrent event time models of the type in [12]. Note that algorithmic issues of the semi-parametric estimators for the model in [12] were dealt with in [14].

This article focuses on the large-sample properties of semiparametric estimators for the parameters of the class of dynamic models described in section 2. These semiparametric estimators are described in section 3. Consistency properties of the estimators will be established in section 5, while weak convergence properties will be developed in section 6.

2 Class of Dynamic Models

In this section we describe the general class of dynamic models of interest. In the sequel, $(\Omega, \mathcal{F}, \mathbf{P})$ is the basic probability space on which all random entities are defined. Consider a study in which units are to be monitored over the calendar time $[0, s^*]$, where $s^* \in (0, \infty)$ is a fixed calendar time. For a unit in this study we suppose that there is a $1 \times q$ vector of possibly time-varying bounded covariates $\mathbf{X} = \{\mathbf{X}(s) : s \in [0, s^*]\}$. We shall denote by $\mathbf{N}^\dagger = \{N^\dagger(s) : s \in [0, s^*]\}$ the counting process such that $N^\dagger(s)$ is the number of observed event occurrences over the period $[0, s]$. There is also a random time τ such that the unit is only under observation over the period $[0, \tau]$, so its at-risk process is $\mathbf{Y}^\dagger = \{Y^\dagger(s) = I\{\tau \geq s\} : s \in [0, s^*]\}$, where $I\{\cdot\}$ is the indicator function. With the definition of $\mathbf{N}^\dagger$ and $\mathbf{Y}^\dagger$, note that when $\tau < s^*$, no more events are observed over $(\tau, s^*]$ for the unit, so that $N^\dagger(s) - N^\dagger(\tau) = 0$ for $s \in (\tau, s^*]$, and this is reflected in the probabilistic statement in equation (1) below. We shall denote by $\mathcal{F} = \{\mathcal{F}_s : s \in [0, s^*]\}$ a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{N}^\dagger$, $\mathbf{Y}^\dagger$, and $\mathbf{X}$ are $\mathcal{F}$ -adapted and, in addition, $\mathbf{Y}^\dagger$ and $\mathbf{X}$ are also $\mathcal{F}$ -predictable.

The class of dynamic models of interest postulates that for $k \in \{1, 2, \dots\}$ and with $dN^\dagger(s) \equiv N^\dagger((s+ds)-) - N^\dagger(s-)$, as $ds \downarrow 0$ and for $s \in [0, s^*]$,

$$\mathbf{P}\{dN^\dagger(s) \geq k | \mathcal{F}_{s-}\} = Y^\dagger(s) [\lambda(s|\mathbf{X}(s))I\{k=1\} + o_p(1)I\{k \geq 1\}] ds, \text{ a. e. } -[\mathbf{P}], \quad (1)$$

where

$$\lambda(s|\mathbf{X}(s)) = \lambda_0[\mathcal{E}(s)]\rho[s, N^\dagger(s-); \alpha]\psi[\mathbf{X}(s)\beta]. \quad (2)$$

In (2), $\mathcal{E} = \{\mathcal{E}(s) : s \in [0, s^*]\}$ is an $\mathcal{F}$ -predictable process with paths that are piecewise left-continuous, nonnegative, $\mathcal{E}(s) \leq s$, and piecewise differentiable with derivative satisfying $\mathcal{E}'(s) \geq 0$; $\lambda_0(\cdot)$ is an unknown baseline hazard rate function with cumulative hazard function $\Lambda_0(\cdot) = \int_0^\cdot \lambda_0(s) ds$ is a known nonnegative bounded function over $\mathcal{R}_+ \times \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \dots\}$ and with $\rho(s, 0; \alpha) = 1$ and $\alpha \in \mathcal{R}^q$ is an unknown q -dimensional parameter; and $\psi(\cdot)$ is a known nonnegative link function on $\mathcal{R}$ and with $\beta \in \mathcal{R}^p$ an unknown p -dimensional regression parameter. The process $\mathcal{E}$ is called the *effective age process*. We shall assume that $\tau \sim G(\cdot)$, where $G(\cdot)$ is some distribution function which does not involve $(\lambda_0(\cdot), \alpha, \beta)$, hence it is considered a nuisance parameter. The regressor $\mathbf{X}$ is a vector-valued bounded and predictable process whose probabilistic structure may also contain some unknown nuisance parameters. A technical condition that we will assume (see the paper [11]) is that the counting process $\mathbf{N}^\dagger$ is non-explosive over $[0, s^*]$, that is, $\mathbf{P}\{N^\dagger(s^*) < \infty\} = 1$. This condition necessarily imposes a constraint on the form of the function $\rho(\cdot, \cdot; \cdot)$ and the model parameters.

The model parameter of main interest is

$$\theta = (\Lambda_0(\cdot), \alpha, \beta) \in \Theta \equiv \mathcal{C} \times \mathbb{R}^q \times \mathbb{R}^p \quad (3)$$

where $\mathcal{C}$ is some class of cumulative hazard functions on $\mathbb{R}_+$, which will typically be a nonparametric class. Thus, θ will be a semiparametric parameter. Defining the process $\mathbf{M}^\dagger = \{M^\dagger(s; \theta) : s \in [0, s^*]\}$ with $M^\dagger(s; \theta) = N^\dagger(s) - A^\dagger(s; \theta)$ and where

$$A^\dagger(s; \theta) = \int_0^s Y^\dagger(v) \lambda_0[\mathcal{C}(v)] \rho[v, N^\dagger(v-); \alpha] \psi[\mathbf{X}(v); \beta] dv, \quad (4)$$

the model is tantamount to the condition that $\mathbf{M}^\dagger$ is a zero-mean square-integrable $\mathfrak{F}$ -martingale. The model specified in (1) and (2) is a slightly more general version of those in [12] and [14] since we allow the ρ -function to also directly depend on s aside from $N^\dagger(s-)$. For more background about this class of models and many specific models subsumed by this class of models, see [12, 14]. This general class of models includes as special cases models that have been considered in the biostatistics and reliability settings. To mention two specific models, if $\mathfrak{E}(s) = s - S_{N^\dagger(s-)}$ with $0 = S_0 < S_1 < S_2 < \dots$ being the times of successive event occurrences, so $\mathfrak{E}(\cdot)$ represents the backward recurrence time function, the model coincides with resetting the age of the unit to zero after each event occurrence, which is referred to in the reliability literature as a perfect repair; while if we have $\mathfrak{E}(s) = s$, then we say that a minimal repair is performed after each event occurrence. If the latter specification is further coupled with $\rho(v, k; \alpha) = 1$, then we recover the Andersen-Gill multiplicative intensity model [4]; also the Cox proportional hazards (PH) model [7] when $\psi(v) = \exp(v)$.

We suppose there are n units in the study, so we will observe n independent and identically distributed (IID) copies $\mathfrak{D}_n \equiv \mathfrak{D} = (\mathbf{D}_1, \mathbf{D}_2, \dots, \mathbf{D}_n)$ of the basic observable $\mathbf{D} = (N^\dagger, Y^\dagger, \mathfrak{E}, \mathbf{X})$. We denote by $\mathcal{D}$ the sample space of $\mathbf{D}$, so that the sample space for $\mathfrak{D}$ is $\mathcal{D}^n$. A larger filtration on $(\Omega, \mathcal{F}, \mathbf{P})$ is formed from the n unit filtrations according to

$$\mathfrak{F} = \bigvee_{i=1}^n \mathfrak{F}_i = \sigma \left(\bigcup_{i=1}^n \mathfrak{F}_i \right).$$

Inference on the model parameter $\theta = (\Lambda_0(\cdot), \alpha, \beta)$, or relevant functionals of θ , are to be based on the realization of $\mathfrak{D}_n$. Properties of the inferential procedures are to be examined when $n \rightarrow \infty$.

We shall use functional notation in the sequel. Thus, for a possibly vector-valued function g defined on $\mathcal{D}$, $\mathbf{P}g$ will represent the theoretical expectation of $g(\mathbf{D})$, while $\mathbb{P}g \equiv \mathbb{P}_n g$ will represent the empirical expectation of g given $\mathfrak{D}_n$. That is, $\mathbf{P}g = \int g(\mathbf{d}) \mathbf{P}(d\mathbf{d})$ and

$\mathbb{P}g = \frac{1}{n} \sum_{i=1}^n g(\mathbf{D}_i)$. The theoretical and empirical covariances of g are defined, respectively, via $\mathbf{V}g = \mathbf{P}(g - \mathbf{P}g)^{\otimes 2}$ and $\mathbb{V}g = \mathbb{P}(g - \mathbb{P}g)^{\otimes 2}$, where, for a column vector a , we write $a^{\otimes 0} = 1$, $a^{\otimes 1} = a$, and $a^{\otimes 2} = aa^T$.

3 Semiparametric Estimators

3.1 Doubly-Indexed Processes

The intensity model in (4) has the distinctive feature that the baseline hazard rate $\lambda_0(\cdot)$ is evaluated at time s at the effective age $\mathcal{E}(s)$. Since of interest is to infer about $\lambda_0(\cdot)$ or $\Lambda_0(\cdot)$, we need to de-couple $\lambda_0(\cdot)$ from $\mathcal{E}(\cdot)$. As demonstrated in [17, 13, 14] such de-coupling is facilitated through the use of doubly-indexed processes.

Let $t^* \in (0, \infty)$ be fixed, and define $\mathcal{S} = [0, s^*]$ and $\mathcal{T} = [0, t^*]$. Form $\mathcal{Q} = \mathcal{S} \times \mathcal{T}$. For our purpose we define the following $\mathcal{Q}$ -indexed processes associated with the $(N^\dagger, Y^\dagger, \mathcal{E}, \mathbf{X})$ processes for one unit: $\mathbf{Z} = \{Z(s, t) : (s, t) \in \mathcal{Q}\}$, $\mathbf{N} = \{N(s, t) : (s, t) \in \mathcal{Q}\}$, $\mathbf{A} = \{A(s, t; \theta) : (s, t) \in \mathcal{Q}\}$, and $\mathbf{M} = \{M(s, t; \theta) : (s, t) \in \mathcal{Q}\}$, where

$$\begin{aligned} Z(s, t) &= I\{\mathcal{E}(s) \leq t\}; \\ N(s, t) &= \int_0^s Z(v, t) N^\dagger(dv); \\ A(s, t; \theta) &= \int_0^s Z(v, t) A^\dagger(dv; \theta); \\ M(s, t; \theta) &= N(s, t) - A(s, t; \theta) = \int_0^s Z(v, t) M^\dagger(dv; \theta). \end{aligned}$$

As an interpretation, note that $N(s, t)$ is the number of occurrences of the recurrent event over the period $[0, s]$ and for which the effective ages on these occurrences are at most t . We introduce the following notation: For a finite subset $\mathbf{T} \subset \mathcal{T}$, $N(\cdot, \mathbf{T}) \equiv (N(\cdot, t) : t \in \mathbf{T})$ and similarly for the other processes.

Proposition 1—*Let $\mathbf{T} \subset \mathcal{T}$ be a finite set. Then $\{M(s, \mathbf{T}; \theta) : s \in \mathcal{S}\}$ is a $|\mathbf{T}|$ -dimensional zero-mean square-integrable martingale with predictable quadratic covariation process*

$$\langle M(\cdot, \mathbf{T}; \theta) \rangle(s) = \left[(A(s, \min(t_1, t_2); \theta))_{t_1, t_2 \in \mathbf{T}} \right], \quad s \in \mathcal{S}.$$

Consequently, $\mathbf{P}N(s, \mathbf{T}) = \mathbf{P}A(s, \mathbf{T}; \theta)$ and $\mathbf{V}M(s, \mathbf{T}; \theta) = \mathbf{P}\langle M(\cdot, \mathbf{T}; \theta) \rangle(s)$.

Proof: Follows from the boundedness and predictability of $s \mapsto Z(s, \mathbf{T})$, the fact that $Z(s, t_1)Z(s, t_2) = Z(s, \min(t_1, t_2))$, by stochastic integration theory, and since

$$M(s, \mathbf{T}; \theta) = \int_0^s Z(v, \mathbf{T}) M^\dagger(dv; \theta).$$

Let $s \in \mathcal{S}$ and denote by

$$0 \equiv S_0 < S_1 < S_2 < \dots < S_{N^\dagger(s-)} < S_{N^\dagger(s-)+1} \equiv \min(s, \tau)$$

the $N^\dagger(s-)$ successive event occurrence times for the unit. Define the (random) functions $\mathcal{E}_j$: $\mathcal{S} \rightarrow \mathcal{R}$ via

$$\mathcal{E}_j(v) = \mathcal{E}(v) I_{(S_{j-1}, S_j]}(v)$$

for $j = 1, 2, \dots, N^\dagger(s^-) + 1$. By condition, on $(S_{j-1}, S_j]$, $\mathcal{E}_j(\cdot)$ is nondecreasing and differentiable. We denote by $\mathcal{E}_j^{-1}(\cdot)$ its inverse function and by $\mathcal{E}_j'(\cdot)$ its derivative. Define the (random) functions $\varphi_j: \mathcal{S} \rightarrow \mathcal{R}$ according to

$$\varphi_j(v; \alpha, \beta) = \frac{\rho(v, j-1; \alpha) \psi[\mathbf{X}(v) \beta]}{\mathcal{E}_j'(v)} I_{(S_{j-1}, S_j]}(v),$$

for $j = 1, 2, \dots, N^\dagger(s^-) + 1$. Next, we define the doubly-indexed process $\mathbf{Y} = \{Y(s, t; \alpha, \beta) : (s, t) \in \mathcal{Q}\}$ according to

$$Y(s, t; \alpha, \beta) = \sum_{j=1}^{N^\dagger(s^-)+1} \varphi_j[\mathcal{E}_j^{-1}(t); \alpha, \beta] I_{(\mathcal{E}_j(S_{j-1}), \mathcal{E}_j(S_j)]}(t). \quad (5)$$

This is a generalized at-risk process. The importance of these doubly-indexed processes arise from the representation of the $\mathbf{A}$ -process in Proposition 2, which de-couples the effective age process $\mathcal{E}(\cdot)$ from the baseline hazard function $\Lambda_0(\cdot)$, and the change-of-variable identity in Proposition 3. Restricted forms of these results were used in the IID recurrent event model considered in [15, 13].

Proposition 2—For $(s, t) \in \mathcal{Q}$, $A(s, t; \theta) = \int_0^t Y(s, w; \alpha, \beta) \Lambda_0(dw)$.

Proof: Partition the region of integration $(0, s]$ into the disjoint union

$(0, s] = \bigcup_{j=1}^{N^\dagger(s^-)+1} (S_{j-1}, S_j]$; do a variable transformation on each region; manipulate; and then simplify.

Proposition 3—Let $\{H(s, t) : (s, t) \in \mathcal{Q}\}$ be a bounded vector-valued process such that for each t , $s \mapsto H(s, t)$ is predictable. For $(s, t) \in \mathcal{Q}$, we have

$$\int_0^s H(s, \mathcal{E}(v)) M(dv, t; \theta) = \int_0^t H(s, w) M(s, dw; \theta).$$

Proof: Start with the left-hand side, write the M process into its $N^\dagger$ and $A^\dagger$ components, then perform the same manipulations as in the proof of Proposition 2.

3.2 Estimation of Λ_0

Propositions 1 and 2 now combine to suggest the stochastic differential equation, for an observable $\mathbf{D}$,

$$N(s^*, dt) = Y(s^*, t; \alpha, \beta) \Lambda_0(dt) + M(s^*, dt; \theta).$$

When data $\mathfrak{D}_n$ is available from n units, we therefore obtain the differential form

$$\mathbb{P}N(s^*, dt) = \{\mathbb{P}Y(s^*, t; \alpha, \beta)\} \Lambda_0(dt) + \mathbb{P}M(s^*, dt; \theta). \quad (6)$$

Define

$$S^{(0)}(s, t; \alpha, \beta) = \mathbb{P}Y(s, t; \alpha, \beta) \equiv \frac{1}{n} \sum_{i=1}^n Y_i(s, t; \alpha, \beta) \quad (7)$$

and $\mathcal{I}(s, t; \alpha, \beta) = \mathbb{I}\{S^{(0)}(s, t; \alpha, \beta) > 0\}$. With the convention that $0/0 = 0$, we obtain from (6) the stochastic integral identity

$$\int_0^t \frac{J(s^*, w; \alpha, \beta)}{S^{(0)}(s^*, w; \alpha, \beta)} \mathbb{P}N(s^*, dw) = \int_0^t J(s^*, w; \alpha, \beta) \Lambda_0(dw) + \int_0^t \frac{J(s^*, w; \alpha, \beta)}{S^{(0)}(s^*, w; \alpha, \beta)} \mathbb{P}M(s^*, dw; \theta). \quad (8)$$

Let us consider the last term in (8). We have

$$\begin{aligned} \int_0^t \frac{J(s^*, w; \alpha, \beta)}{S^{(0)}(s^*, w; \alpha, \beta)} \mathbb{P}M(s^*, dw; \theta) &= \frac{1}{n} \sum_{i=1}^n \int_0^t \frac{J(s^*, w; \alpha, \beta)}{S^{(0)}(s^*, w; \alpha, \beta)} M_i(s^*, dw; \theta) \\ &= \frac{1}{n} \sum_{i=1}^n \int_0^{s^*} \frac{J(s^*, \mathcal{E}_i(v); \alpha, \beta)}{S^{(0)}(s^*, \mathcal{E}_i(v); \alpha, \beta)} M_i(dv, t; \theta) \end{aligned} \quad (9)$$

where the last equality is obtained by invoking Proposition 3. The integrand in each summand in (9) is bounded and predictable, so it follows from stochastic integration theory that, for $i = 1, 2, \dots, n$,

$$\mathbb{P} \int_0^{s^*} \frac{J(s^*, \mathcal{E}_i(v); \alpha, \beta)}{S^{(0)}(s^*, \mathcal{E}_i(v); \alpha, \beta)} M_i(dv, t; \theta) = 0. \quad (10)$$

It therefore follows from (8) and (10) that

$$\mathbf{P} \int_0^t \frac{J(s^*, w; \alpha, \beta)}{S^{(0)}(s^*, w; \alpha, \beta)} \mathbb{P}N(s^*, dw) = \mathbf{P} \int_0^t J(s^*, w; \alpha, \beta) \Lambda_0(dw).$$

Analogously to Aalen's idea [1], if for the moment we assume that (α, β) is known, we may propose a method-of-moments estimator for $\Lambda_0(\cdot)$ given by

$$\tilde{\Lambda}_0(t; \alpha, \beta) = \int_0^t \frac{J(s^*, w; \alpha, \beta)}{S^{(0)}(s^*, w; \alpha, \beta)} \mathbb{P}N(s^*, dw) = \int_0^t \frac{\mathbb{P}N(s^*, dw)}{S^{(0)}(s^*, w; \alpha, \beta)}. \quad (11)$$

However, (α, β) is not known, hence $\tilde{\Lambda}_0$ is not an estimator. We now therefore find an estimator for (α, β) , which will then be plugged-in (11) to obtain a legitimate estimator of Λ_0 .

3.3 Estimator of (α, β)

For the purpose of estimating (α, β) , we form a generalized likelihood process, based on $\mathfrak{D}_n$, denoted by $\mathbf{L} = \{L(s, t; \theta) : (s, t) \in \mathfrak{A}\}$. We define

$$L(s, t; \theta) = \prod_{i=1}^n \prod_{v=0}^s [A_i(dv, t; \theta)]^{N_i(\Delta v, t)} [1 - A_i(dv, t; \theta)]^{1 - N_i(\Delta v, t)},$$

with the understanding that when the product operation is over a continuous index, such as v in the second product operation, then it means product-integral; see [10]. By property of the product-integral and re-writing in an expanded form, we have that

$$L(s, t; \theta) = \left\{ \prod_{i=1}^n \prod_{v=0}^s [Z_i(v, t) A_i^\dagger(dv; \theta)]^{Z_i(v, t) N_i^\dagger(\Delta v)} \right\} \exp \{-n \mathbb{P}A(s, t; \theta)\}.$$

This likelihood process involves the functional parameter $\Lambda_0(\cdot)$, for which we have an estimator given in (11) if (α, β) is known. We can therefore obtain a profile likelihood for (α, β) by replacing the $\Lambda_0(\cdot)$ in $L(s^*, t^*; \theta)$ by the $\tilde{\Lambda}_0(s^*, \cdot; \alpha, \beta)$ in (11). Doing so yields a profile likelihood function given by

$$L_P(s^*, t^*; \alpha, \beta) = \prod_{i=1}^n \prod_{v=0}^{s^*} \left[\frac{\rho(v, N_i^\dagger(v-); \alpha) \psi[\mathbf{X}_i(v) \beta]}{S^{(0)}(s^*, \mathcal{E}_i(v); \alpha, \beta)} \right]^{N_i(\Delta v, t^*)}. \quad (12)$$

This function may also be viewed as a generalized partial likelihood function for (α, β) being very much reminiscent of the Cox partial likelihood function; see [7, 8, 4, 9, 3, 2].

From this partial likelihood function we obtain its maximizer as our estimator of (α, β) , that is,

$$(\hat{\alpha}, \hat{\beta}) \equiv (\hat{\alpha}(s^*, t^*), \hat{\beta}(s^*, t^*)) = \arg \max_{(\alpha, \beta) \in \mathbb{R}^q \times \mathbb{R}^p} L_P(s^*, t^*; \alpha, \beta). \quad (13)$$

Numerical methods, such as the Newton-Raphson algorithm, are needed to obtain the values of $(\hat{\alpha}, \hat{\beta})$, as has been done in [14].

Having obtained an estimator of (α, β) , now replace (α, β) in $\tilde{\Lambda}_0(s^*, t; \alpha, \beta)$ to obtain an estimator of $\Lambda_0(\cdot)$. This resulting estimator of $\Lambda_0(\cdot)$ is given by

$$\hat{\Lambda}_0(s^*, t) = \tilde{\Lambda}_0(s^*, t; \hat{\alpha}, \hat{\beta}) = \int_0^t \frac{\mathbb{P}N(s^*, dw)}{S^{(0)}(s^*, w; \hat{\alpha}, \hat{\beta})}, \quad t \in \mathcal{T} \quad (14)$$

Observe that the form of this estimator is analogous to the estimator of the baseline hazard function in the Cox PH model [7, 6, 4], hence it seems appropriate to refer to this as a generalized Aalen-Breslow-Nelson (ABN) estimator.

Denoting by F_0 the distribution function associated with the baseline hazard function Λ_0 , then dictated by the product-integral representation of F_0 by Λ_0 , we are able to obtain a product-limit type estimator of the survivor function $\bar{F}_0(t) = 1 - F_0(t)$ given by

$$\hat{\bar{F}}_0(s^*, t) = \prod_{w=0}^t \left[1 - \frac{\mathbb{P}N(s^* dw)}{S^{(0)}(s^*, w; \hat{\alpha}, \hat{\beta})} \right], \quad t \in \mathcal{T}. \quad (15)$$

Small to moderate sample size properties of the estimators presented above were examined through simulation studies in [14] for specific forms of the effective age process $\mathcal{E}$, for a function ρ which was made to depend on s only through $N^\dagger(s-)$, and for an exponential link function ψ . Applications of these estimators to some real data sets were also presented in that paper. However, general asymptotic properties of these estimators are still unavailable, and establishing the large-sample properties of these semiparametric estimators is the *raison d'être* of the current paper.

4 Preliminaries for Asymptotics

For studying the large-sample properties of our semiparametric estimators, it is first convenient to deal with the model where $A^\dagger$ in (4) is of form

$$A^\dagger(s; \eta) = \int_0^s Y^\dagger(v) \lambda_0[\mathcal{E}(v)] \kappa(v; \eta) dv. \quad (16)$$

Here $\kappa = \{\kappa(s; \eta) : s \in \mathcal{S}\}$ is a bounded and predictable process and $\eta \in \Gamma$ with Γ an open subset of $\mathfrak{R}^k$. We assume that $\eta \mapsto \kappa(s; \eta)$ is twice-differentiable and we let

$$\dot{\kappa}(s; \eta) = \nabla_{\eta} \kappa(s; \eta) \quad \text{and} \quad \ddot{\kappa}(s; \eta) = \nabla_{\eta\eta^T} \kappa(s; \eta).$$

Later to obtain the specific results for the model in (4), we then simply identify η with (α, β) and with

$$\kappa(s; \eta) = \rho(s, N^{\dagger}(s-); \alpha) \psi[\mathbf{X}(s)\beta].$$

With the above simplification, for one unit monitored over $\mathcal{S} = [0, s^*]$, we will then define

$$\begin{aligned} \varphi_j(v; \eta) &= \frac{\kappa(v; \eta)}{\mathcal{E}^{\dagger}(v)} I_{(s_{j-1}, s_j]}(v), \quad j=1, 2, \dots, N^{\dagger}(s^*-)+1; \\ Y(s^*, t; \eta) &= \sum_{j=1}^{N^{\dagger}(s^*-)+1} \varphi_j[\mathcal{E}_j^{-1}(t); \eta] I_{(\mathcal{E}(s_{j-1}), \mathcal{E}(s_j)]}(t), \end{aligned}$$

so that with n units, we will then have

$$S^{(0)}(s^*, t; \eta) = \mathbb{P}Y(s^*, t; \eta) = \frac{1}{n} \sum_{i=1}^n Y_i(s^*, t; \eta)$$

where in this last function the κ functions may also depend on i .

We denote by (η^0, Λ_0^0) the true parameter vector, and to simplify notation, we suppress writing these true parameter vector in our functions if no confusion could arise. Thus, $A_i(s^*, t) \equiv A_i(s^*, t; \eta^0, \Lambda_0^0)$, $Y_i(s^*, t) \equiv Y_i(s^*, t; \eta^0)$, and $M_i(s^*, t) \equiv M_i(s^*, t; \eta^0, \Lambda_0^0)$.

In establishing consistency and weak convergence properties of the estimators, we will need a general weak convergence result of processes formed as stochastic integrals of the processes $M_i(s^*, t)$, $i = 1, 2, \dots, n$, which we recall are martingales with respect to s^* but not with respect to t .

Given an n and an (s^*, t, η) , let us define a random discrete probability measure $\mathbb{Q}_n(\cdot; s^*, t, \eta)$ on the (random) set

$$\mathcal{H}_n(s^*) = \{(i, j) : j=1, 2, \dots, N_i^{\dagger}(s^*-)+1; i=1, 2, \dots, n\}$$

according to the probabilities

$$\mathbb{Q}_n((i, j); s^*, t, \eta) = \frac{1}{n} \left\{ \frac{Y_i(s^*, t; \eta)}{S^{(0)}(s^*, t; \eta)} \right\} \left\{ \frac{\varphi_{ij}[\mathcal{E}_{ij}^{-1}(t); \eta]}{Y_i(s^*, t; \eta)} I_{(\mathcal{E}_i(S_{ij-1}), \mathcal{E}_i(S_{ij}])}(t) \right\}.$$

For a function $g: \mathcal{K}_n(s^*) \rightarrow \mathbb{R}^r$, which could be random and also depending on (s^*, t, η) ,

$$\mathbb{E}_{\mathbb{Q}_n(s^*, t, \eta)} g \equiv \mathbb{Q}_n(s^*, t, \eta) g$$

will denote its expectation with respect to the p.m. $\mathbb{Q}_n$ and

$$\begin{aligned} \mathbb{V}_{\mathbb{Q}_n(s^*, t, \eta)} g &\equiv \mathbb{Q}_n(s^*, t, \eta) [g - \mathbb{Q}_n(s^*, t, \eta) g]^{\otimes 2} \\ &= \mathbb{Q}_n(s^*, t, \eta) g^{\otimes 2} - [\mathbb{Q}_n(s^*, t, \eta) g]^{\otimes 2} \end{aligned}$$

will denote its variance-covariance matrix with respect to $\mathbb{Q}_n$.

Let us also define

$$\mathbb{Q}_n(i; s^*, t, \eta) = \frac{1}{n} \left\{ \frac{Y_i(s^*, t; \eta)}{S^{(0)}(s^*, t; \eta)} \right\} = \frac{Y_i(s^*, t; \eta)}{\sum_{l=1}^n Y_l(s^*, t; \eta)}, i=1, 2, \dots, n.$$

Thus, when the function $g: \mathcal{K}_n(s^*) \rightarrow \mathbb{R}^r$ is such that $g(i, j) = g^*(i)$ for some g^* , then

$$\mathbb{Q}_n(s^*, t, \eta) g = \sum_{i=1}^n g^*(i) \mathbb{Q}_n(i; s^*, t, \eta) = \sum_{i=1}^n g^*(i) \left[\frac{Y_i(s^*, t; \eta)}{\sum_{l=1}^n Y_l(s^*, t; \eta)} \right].$$

In this case, the variance-covariance matrix of g with respect to $\mathbb{Q}_n$ is also in more simplified form.

Theorem 1—Let $\{H_i^{(n)}(s, t): (s, t) \in \mathcal{I} = [0, s^*] \times [0, t^*]\}$ for $i = 1, 2, \dots, n; n = 1, 2, \dots$ be a triangular array of vector processes, and assume the following conditions:

- a. $\forall i, H_i^{(n)}$ is bounded and $\forall v \in [0, s], H_i^{(n)}(s, \mathcal{E}_i(v))$ is $\mathbb{F}$ -predictable;
- b. There exists a deterministic function $s^{(0)}: \mathcal{Q} \rightarrow \mathbb{R}_+$ such that

$$|S^{(0)}(s^*, t) - s^{(0)}(s^*, t)| \xrightarrow{up} 0$$

and $\inf_{t \in \mathcal{T}} s^{(0)}(s^*, t) > 0$; and

- c. There exists a deterministic matrix function $\mathbf{v}: \mathcal{Q} \rightarrow \mathbb{R}_+$ such that

$$\|\mathbb{Q}_n(s^*, w)\{[H^{(n)}(s^*, w)]^{\otimes 2}\} - \mathbf{v}(s^*, w)\| \xrightarrow{up} 0,$$

and for every $t \in (0, t^*]$,

$$\sum(s^*, t) = \int_0^t \mathbf{v}(s^*, w) s^{(0)}(s^*, w) \Lambda_0^0(dw)$$

is positive definite.

Defining the stochastic integrals, for $n = 1, 2, \dots$,

$$W^{(n)}(s^*, t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t H_i^{(n)}(s^*, w) M_i(s^*, dw),$$

then $\{W^{(n)}(s^*, t) : t \in \mathcal{T}\}$ converges weakly on Skorokhod's space $D[0, t^*]$ to a zero-mean Gaussian process $\{W^{(\infty)}(s^*, t) : t \in \mathcal{T}\}$ whose covariance function is

$$\text{Cov}\{W^{(\infty)}(s^*, t_1), W^{(\infty)}(s^*, t_2)\} = \sum(s^*, \min(t_1, t_2)).$$

Proof: The proof of this result is analogous to the proof of the general theorem in [15].

5 Consistency Properties

In this section we will establish the consistency of the sequence of estimators $\hat{\eta}_n$ and $\hat{\Lambda}_n(s^*, \cdot)$ as the number of units n increases to infinity. We shall assume the following set of “regularity conditions.” These are somewhat analogous conditions to those in [4] and [5].

(C1) For each $(s, t) \in \mathcal{Q}$, $\eta \mapsto \kappa(s, t; \eta)$ is twice-continuously differentiable with

$$\dot{\kappa}(s, t; \eta) = \nabla_{\eta} \kappa(s, t; \eta) \quad \text{and} \quad \ddot{\kappa}(s, t; \eta) = \nabla_{\eta\eta^T} \kappa(s, t; \eta).$$

Furthermore, the operations of differentiation (with respect to η) and integration could be interchanged.

(C2) There exists a deterministic function $s^{(0)} : \mathcal{Q} \times \Gamma \rightarrow \mathcal{R}_+$ such that

$$\sup_{t \in \mathcal{T} : \eta \in \Gamma} |S^{(0)}(s^*, t; \eta) - s^{(0)}(s^*, t; \eta)| \xrightarrow{p} 0,$$

and with $\inf_{t \in \mathcal{T}} s^{(0)}(s^*, t; \eta) > 0$ and with $\Lambda_0^0(t^*) = \int_0^{t^*} \lambda_0^0(w) dw < \infty$.

(C3) There exist deterministic functions $s^{(1)} : \mathcal{Q} \times \Gamma^2 \rightarrow \mathcal{R}^k$ and $s^{(2)} : \mathcal{Q} \times \Gamma^2 \rightarrow (\mathcal{R}^k)^{\otimes 2}$ such that with

$$\begin{aligned} Q_n^{(1)}(s^*, t; \eta_1, \eta_2) &= Q_n(s^*, t; \eta_1) \left[\frac{\dot{\kappa}}{\kappa}(\mathcal{E}^{-1}(t); \eta_2) \right]; \\ Q_n^{(2)}(s^*, t; \eta_1, \eta_2) &= Q_n(s^*, t; \eta_1) \left[\frac{\ddot{\kappa}}{\kappa}(\mathcal{E}^{-1}(t); \eta_2) \right], \end{aligned}$$

and

$$\begin{aligned} q^{(1)}(s^*, t; \eta_1, \eta_2) &= \frac{s^{(1)}}{s^{(0)}}(s^*, t; \eta_1, \eta_2); \\ q^{(2)}(s^*, t; \eta_1, \eta_2) &= \frac{s^{(2)}}{s^{(0)}}(s^*, t; \eta_1, \eta_2), \end{aligned}$$

we have

$$\begin{aligned} \sup_{t \in \mathcal{T}; (\eta_1, \eta_2) \in \Gamma^2} \|Q_n^{(1)}(s^*, t; \eta_1, \eta_2) - q^{(1)}(s^*, t; \eta_1, \eta_2)\| &\xrightarrow{p} 0; \\ \sup_{t \in \mathcal{T}; (\eta_1, \eta_2) \in \Gamma^2} \|Q_n^{(2)}(s^*, t; \eta_1, \eta_2) - q^{(2)}(s^*, t; \eta_1, \eta_2)\| &\xrightarrow{p} 0. \end{aligned}$$

(C4) With $\mathbf{v}(s^*, t)$ satisfying

$$\sup_{t \in \mathcal{T}} \|\mathbb{V}_{Q_n(s^*, t)} \left[\frac{\dot{\kappa}}{\kappa}(\mathcal{E}^{-1}(t)) \right] - \mathbf{v}(s^*, t)\| \xrightarrow{pr} 0,$$

the matrix

$$\sum(s^*, t) = \int_0^t \mathbf{v}(s^*, w) s^{(0)}(s^*, w) \Lambda_0^0(dw)$$

is positive definite for each $t \in (0, t^*]$.

(C5) For each $s \in [0, s^*]$, the mappings

$$\begin{aligned} (v, \eta) &\mapsto \frac{\dot{\kappa}}{\kappa}(v; \eta) - Q_n^{(1)}(s, \mathcal{E}(v); \eta, \eta); \\ (v, \eta) &\mapsto \frac{\ddot{\kappa}}{\kappa}(v; \eta) - Q_n^{(2)}(s, \mathcal{E}(v); \eta, \eta), \end{aligned}$$

are bounded and $\mathfrak{F}_{s-}$ -measurable for each $v \in [0, s]$.

We first establish an intermediate result.

Lemma 1—For $w \in \mathcal{T}$ and $\eta \in \Gamma$, we have

$$\begin{aligned} \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, w; \eta) &= Q_n^{(1)}(s^*, w; \eta, \eta); \\ \frac{\ddot{S}^{(0)}}{S^{(0)}}(s^*, w; \eta) &= Q_n^{(2)}(s^*, w; \eta, \eta). \end{aligned}$$

Proof: The proofs are straightforward and hence omitted.

For notational brevity, let us define

$$\Psi_n(s^*, t^*; \eta) = \nabla_\eta \left\{ \frac{1}{n} l_P(s^*, t^*; \eta) \right\};$$

$$\Psi(s^*, t^*; \eta) = \int_0^{t^*} [q^{(1)}(s^*, w; \eta^0, \eta) - q^{(1)}(s^*, w; \eta, \eta)] s^{(0)}(s^*, w) \Lambda_0^0(dw),$$

where $l_P(s^*, t^*; \eta) = \log L_P(s^*, t^*; \eta)$ is the logarithm of the partial likelihood function. We are now in position to state a result concerning the consistency of the partial MLE of η . Without loss of generality, we shall assume that the maximizer of the partial likelihood can be obtained as a zero of $\eta \mapsto \Psi_n(s^*, t^*; \eta)$.

Theorem 2—If $\hat{\eta}_n$ is such that $\Psi_n(s^*, t^*; \hat{\eta}_n) = 0$ and if, for every $\varepsilon > 0$, we have that

$$\inf_{\{\eta: \|\eta - \eta^0\| \geq \varepsilon\}} \|\Psi(s^*, t^*; \eta)\| > 0,$$

then, under the regularity conditions (C1)–(C5), $\hat{\eta}_n \xrightarrow{P} \eta^0$.

Proof: From (12), (C1), and Lemma 1, we have

$$\begin{aligned} \Psi_n(s^*, t^*; \eta) &= \mathbb{P}_n \int_0^{s^*} \left[\frac{\dot{\kappa}}{\kappa}(v; \eta) - Q_n^{(1)}(s^*, \mathcal{E}(v); \eta, \eta) \right] N(dv, t^*) \\ &= \mathbb{P}_n \int_0^{s^*} \left[\frac{\dot{\kappa}}{\kappa}(v; \eta) - Q_n^{(1)}(s^*, \mathcal{E}(v); \eta, \eta) \right] M(dv, t^*) + \end{aligned} \quad (17)$$

$$\mathbb{P}_n \int_0^{s^*} \left[\frac{\dot{\kappa}}{\kappa}(v; \eta) - Q_n^{(1)}(s^*, \mathcal{E}(v); \eta, \eta) \right] A(dv, t^*). \quad (18)$$

By (C5) and Theorem 1, the term in (17) is $o_P(1)$. On the other hand, the term in (18) becomes, after splitting the region of integration into the disjoint intervals $(S_{j-1}, S_j]$ for $j = 1, 2, \dots, N^+(s^*) + 1$ and then doing a variable transformation,

$$\begin{aligned} \text{Term (18)} &= \int_0^{t^*} \mathbb{P}_n \left\{ \sum_{j=1}^{N^+(s^*)+1} \left[\frac{\dot{\kappa}}{\kappa}(\mathcal{E}_j^{-1}(w); \eta) - Q_n^{(1)}(s^*, w; \eta, \eta) \right] \times \varphi_j[\mathcal{E}_j^{-1}(w); \eta] I_{(\mathcal{E}(S_{j-1}), \mathcal{E}(S_j)]}(w) \right\} \Lambda_0^0(dw) \\ &= \int_0^{t^*} S^{(0)}(s^*, w) \left[Q_n^{(1)}(s^*, w; \eta^0, \eta) - Q_n^{(1)}(s^*, w; \eta, \eta) \right] \Lambda_0^0(dw). \end{aligned}$$

By conditions (C2) and (C3), this last term will converge uniformly in probability to $\Psi(s^*, t^*; \eta)$, so that we will have the result

$$\sup_{\eta \in \Gamma} \|\Psi_n(s^*, t^*; \eta) - \Psi(s^*, t^*; \eta)\| \xrightarrow{p} 0. \quad (19)$$

Finally, observe that $\Psi(s^*, t^*; \eta^0) = 0$, so by the condition of the theorem and coupling with (19), it follows from Theorem 5.9 of van der Vaart [18]) that $\hat{\eta}_n \xrightarrow{p} \eta^0$.

Indeed, there is more to be said based on the following Lemma 2 which will also be used in the weak convergence result proof in Section 6. Since $\Sigma(s^*, t^*)$ is positive definite, this lemma implies that, in fact, η^0 is a maximizer of the limit in probability of the log-partial likelihood $[I_P(s^*, t^*; \eta) - I_P(s^*, t^*)]/n$.

Lemma 2—Under conditions (C1)–(C5),

$$\begin{aligned} \dot{\Psi}_n(s^*, t^*) &\equiv \nabla_{\eta\eta^T} \left\{ \frac{1}{n} I_P(s^*, t^*; \eta) \right\} \Big|_{\eta=\eta^0} \\ &= - \int_0^{t^*} \mathbb{V}_{Q_n(s^*, w)} \left[\frac{\dot{\kappa}}{\kappa}(\mathcal{E}^{-1}(w)) \right] S^{(0)}(s^*, w) \Lambda_0^0(dw) + o_p(1) \\ &\xrightarrow{p} -\Sigma(s^*, t^*). \end{aligned}$$

Proof: Straightforward, though tedious, calculations show that

$$\begin{aligned} \dot{\Psi}_n(s^*, t^*; \eta) &= \mathbb{P}_n \int_0^{s^*} \left[\frac{\dot{\kappa}}{\kappa}(v; \eta) - \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, \mathcal{E}(v); \eta) \right] N(dv, t^*) - \\ &\quad \mathbb{P}_n \int_0^{s^*} \left\{ \left[\frac{\dot{\kappa}}{\kappa}(v; \eta) \right]^{\otimes 2} - \left[\frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, \mathcal{E}(v); \eta) \right]^{\otimes 2} \right\} N(dv, t^*) \\ &= \int_0^{t^*} \{ Q_n^{(2)}(s^*, w; \eta^0, \eta) - Q_n^{(2)}(s^*, w; \eta, \eta) \} S^{(0)}(s^*, w; \eta^0) \Lambda_0^0(dw) - \\ &\quad \int_0^{t^*} \left\{ Q_n(s^*, w; \eta^0) \left[\frac{\dot{\kappa}}{\kappa}(\mathcal{E}^{-1}(w); \eta) \right]^{\otimes 2} - \left[Q_n^{(1)}(s^*, w; \eta, \eta) \right]^{\otimes 2} \right\} \times \\ &\quad S^{(0)}(s^*, w; \eta^0) \Lambda_0^0(dw) + o_p(1). \end{aligned}$$

Evaluating at $\eta = \eta^0$, and noting that

$$Q_n(s^*, w; \eta^0) \left[\frac{\dot{\kappa}}{\kappa}(\mathcal{E}^{-1}(w); \eta^0) \right] = Q_n^{(1)}(s^*, w; \eta^0, \eta^0)$$

then yields the representation given in the statement of the lemma. Letting $n \rightarrow \infty$, the limiting matrix is $-\Sigma(s^*, t^*)$.

Theorem 3—Under conditions (C1)–(C5), $\hat{\Lambda}_{0n}(s^*, \cdot)$ converges uniformly in probability to $\Lambda_0^0(\cdot)$ on $[0, t^*]$, that is,

$$\sup_{t \in [0, t^*]} \left| \hat{\Lambda}_{0n}(s^*, t) - \Lambda_0^0(t) \right| \xrightarrow{p} 0.$$

Proof: With $\Lambda_0^*(s^*, t) = \int_0^t I\{S^{(0)}(s^*, w; \hat{\eta}) > 0\} \Lambda_0^0(dw)$, we have that

$$\begin{aligned} |\hat{\Lambda}_0(s^*, t) - \Lambda_0^0(t)| &\leq |\hat{\Lambda}_0(s^*, t) - \Lambda_0^*(s^*, t)| + |\Lambda_0^*(s^*, t) - \Lambda_0^0(t)| \\ &\leq \left| \hat{\Lambda}_0(s^*, t) - \int_0^t \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} \mathbb{P}N(s^*, dw) \right| + \end{aligned} \quad (20)$$

$$\left| \int_0^t \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} \mathbb{P}M(s^*, dw) \right| + \quad (21)$$

$$\left| \int_0^t I\{S^{(0)}(s^*, w; \hat{\eta}) = 0\} \Lambda_0^0(dw) \right|. \quad (22)$$

Term (22) is bounded above by

$$\left| \int_0^{t^*} I\{S^{(0)}(s^*, w; \hat{\eta}) = 0\} \Lambda_0^0(dw) \right|,$$

which is $o_p(1)$ since $S^{(0)}(s^*, w; \hat{\eta}) \xrightarrow{p} s^{(0)}(s^*, w)$ and by (C2) we have $\Lambda_0^0(t^*) < \infty$ and $\inf_{w \in [0, t^*]} s^{(0)}(s^*, w) > 0$. Term (20) is bounded above by

$$\left\{ \sup_{w \in [0, t^*]} \left| \frac{I\{S^{(0)}(s^*, w; \hat{\eta}) > 0\}}{S^{(0)}(s^*, w; \hat{\eta})} - \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} \right| \right\} \mathbb{P}N(s^*, t^*).$$

But $\mathbb{P}N(s^*, t^*) = \mathbb{P}M(s^*, t^*) + \mathbb{P}A(s^*, t^*)$. By Theorem 1, $\mathbb{P}M(s^*, t^*) = o_p(1)$, while

$\mathbb{P}A(s^*, t^*) = \int_0^{t^*} S^{(0)}(s^*, w) \Lambda_0^0(dw)$, which converges in probability to $\int_0^{t^*} s^{(0)}(s^*, w) \Lambda_0^0(dw)$, a finite quantity by (C2). Thus, $\mathbb{P}N(s^*, t^*) = O_p(1)$. Since

$$\sup_{w \in [0, t^*]} \left| \frac{I\{S^{(0)}(s^*, w; \hat{\eta}) > 0\}}{S^{(0)}(s^*, w; \hat{\eta})} - \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} \right| = o_p(1)$$

it therefore follows that term (20) is $o_p(1)$. Finally, by Theorem 1, we have that the process

$$\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} M_i(s^*, dw) : t \in [0, t^*] \right\}$$

converges weakly to a zero-mean Gaussian process $\tilde{G}$ whose covariance function is

$$\text{Cov}(\tilde{G}(t_1), \tilde{G}(t_2)) = \int_0^{\min(t_1, t_2)} \frac{\Lambda_0^0(dw)}{s^{(0)}(s^*, w)}$$

for $t_1, t_2 \in [0, t^*]$. As a consequence,

$$\sup_{t \in [0, t^*]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} M_i(s^*, dw) \right|$$

converges weakly to $\sup_{t \in [0, t^*]} |\tilde{G}(t)|$, which is $O_p(1)$. It follows that

$$\begin{aligned} \sup_{t \in [0, t^*]} \left| \int_0^t \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} \mathbb{P}M(s^*, dw) \right| &= \frac{1}{\sqrt{n}} \sup_{t \in [0, t^*]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} M_i(s^*, dw) \right| \\ &= o_p(1). \end{aligned}$$

This completes the proof of the theorem.

6 Distributional Properties

In this section we establish the limiting distributional properties of $\{\sqrt{n}[\hat{\eta}_n - \eta^0], n = 1, 2, \dots\}$ and $\{W_n(s^*, t): t \in \mathcal{T}; n = 1, 2, \dots\}$, where

$$W_n(s^*, t) = \sqrt{n} \left[\hat{\Lambda}_0^{(n)}(s^*, t) - \Lambda_0^0(t) \right].$$

Define the process $\{B_n(s^*, t): t \in \mathcal{T}; n = 1, 2, \dots\}$ according to

$$B_n(s^*, t) = \int_0^t I\{S^{(0)}(s^*, w) > 0\} \frac{\dot{S}^{(0)}(s^*, w)}{[\dot{S}^{(0)}(s^*, w)]} \mathbb{P}_n N(s^*, dw).$$

Let us also define the process $\{V_n(s^*, t): t \in \mathcal{T}; n = 1, 2, \dots\}$ via

$$V_n(s^*, t) = \sqrt{n} \left[\hat{\Lambda}_0^{(n)}(s^*, t) - \Lambda_0^0(t) \right] + \sqrt{n}(\hat{\eta}_n - \eta^0)^T B_n(s^*, t).$$

Furthermore, we shall assume that $\hat{\eta}_n$ solves the equation

$$U_P^{(n)}(s^*, t^*; \eta) = 0 \quad \text{with} \quad U_P^{(n)}(s^*, t; \eta) = \nabla_\eta l_P(s^*, t; \eta).$$

We now present and prove a result from which the asymptotic properties follow.

Theorem 4—Under conditions (C1)–(C5), we have the representations

$$\sqrt{n}(\hat{\eta}_n - \eta^0) = \left[\sum (s^*, t^*) \right]^{-1} \times \left\{ \sqrt{n} \mathbb{P}_n \int_0^{t^*} \left[\frac{\dot{\kappa}}{\kappa} [\mathcal{E}^{-1}(w)] - \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, w) \right] M(s^*, dw) \right\} + o_p(1);$$

(23)

and

$$V_n(s^*, t) = \sqrt{n} \int_0^{t^*} I(w \leq t) \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} \mathbb{P}_n M(s^*, dw) + o_p(1). \quad (24)$$

Furthermore, $\{\sqrt{n}(\hat{\eta}_n - \eta^0)\}$ and $\{V_n(s^*, t); t \in \mathcal{T}\}$ are asymptotically independent with each weakly converging to Gaussian limits.

Proof: From the definition of $\hat{\eta}_n$, we have by first-order Taylor expansion that

$$\sqrt{n}(\hat{\eta}_n - \eta^0) = \left[-\dot{\Psi}_n(s^*, t^*; \tilde{\eta}_n) \right]^{-1} \left[\sqrt{n} \Psi_n(s^*, t^*; \eta^0) \right]$$

where $\tilde{\eta}_n$ is in a neighborhood centered at η^0 and whose radius is $\|\hat{\eta}_n - \eta^0\|$. It is easy to see that

$$\begin{aligned} \sqrt{n} \Psi_n(s^*, t^*; \eta^0) &= \sqrt{n} \mathbb{P}_n \int_0^{s^*} \left\{ \frac{\dot{\kappa}}{\kappa}(v) - \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, \mathcal{E}(v)) \right\} M(dv, t^*) \\ &= \sqrt{n} \mathbb{P}_n \int_0^{t^*} \left\{ \frac{\dot{\kappa}}{\kappa}[\mathcal{E}^{-1}(w)] - \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, w) \right\} M(s^*, dw). \end{aligned}$$

Furthermore, since $\hat{\eta}_n \xrightarrow{p} \eta^0$, and by virtue of Lemma 2, we have that

$$\left[-\dot{\Psi}_n(s^*, t^*; \tilde{\eta}_n) \right]^{-1} = \left[\sum (s^*, t^*) \right]^{-1} + o_p(1).$$

As such we obtain the representation for $\sqrt{n}(\hat{\eta}_n - \eta^0)$.

Once again, by first-order Taylor expansion, we have that on the set where $S^{(0)}(s^*, w; \hat{\eta}_n) > 0$,

$$\frac{1}{S^{(0)}(s^*, w; \hat{\eta}_n)} = \frac{1}{S^{(0)}(s^*, w; \eta^0)} - (\hat{\eta}_n - \eta^0)^T \frac{\dot{S}^{(0)}(s^*, w; \tilde{\eta}_n)}{[S^{(0)}(s^*, w; \tilde{\eta}_n)]^2}$$

with $\tilde{\eta}_n$ inside the ball centered at η^0 with radius $\|\hat{\eta}_n - \eta^0\|$. Defining

$$\Lambda_0^*(s^*, t) = \int_0^t I\{S^{(0)}(s^*, w; \hat{\eta}_n) > 0\} \Lambda_0^0(dw),$$

and recalling that

$$\hat{\Lambda}_0^{(n)}(s^*, t) = \int_0^t \frac{I\{S^{(0)}(s^*, w; \hat{\eta}_n) > 0\}}{S^{(0)}(s^*, w; \hat{\eta}_n)} \mathbb{P}_n N(s^*, dw),$$

we obtain

$$\begin{aligned} \sqrt{n} \left[\hat{\Lambda}_0^{(n)}(s^*, t) - \Lambda_0^*(s^*, t) \right] &= \int_0^t \frac{I\{S^{(0)}(s^*, w; \hat{\eta}_n) > 0\}}{S^{(0)}(s^*, w; \hat{\eta}_n)} \sqrt{n} \mathbb{P}_n M(s^*, dw) - \\ &\quad \sqrt{n} (\hat{\eta}_n - \eta^0)^T \int_0^t I\{S^{(0)}(s^*, w; \hat{\eta}_n) > 0\} \frac{[\dot{S}^{(0)}(s^*, w; \hat{\eta}_n)]}{[S^{(0)}(s^*, w; \hat{\eta}_n)]^2} \mathbb{P}_n N(s^*, dw). \end{aligned}$$

The representation for $V_n(s^*, t)$ given in the statement of the lemma now follows by noting that

$$\begin{aligned} \sup_{0 \leq t \leq t^*} \|\sqrt{n}[\Lambda_0^*(s^*, t) - \Lambda_0^0(t)]\| &= o_p(1); \\ \sup_{0 \leq t \leq t^*} \|S^{(0)}(s^*, t; \hat{\eta}_n) - S^{(0)}(s^*, t; \eta^0)\| &= o_p(1); \\ \sup_{0 \leq t \leq t^*} \|\dot{S}^{(0)}(s^*, t; \hat{\eta}_n) - \dot{S}^{(0)}(s^*, t; \eta^0)\| &= o_p(1). \end{aligned}$$

Finally, let $\mathbf{t} = (t_1, t_2, \dots, t_p)^T \subset \mathcal{T}$. From the just-established representations, with $I\{w = \mathbf{t}\} = (I\{w = t_1\}, \dots, I\{w = t_p\})^T$, we have

$$\begin{bmatrix} \sqrt{n}(\hat{\eta}_n - \eta^0) \\ V_n(s^*, \mathbf{t}) \end{bmatrix} = \begin{bmatrix} \Sigma(s^*, t^*)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \times \sqrt{n} \mathbb{P}_n \int_0^{t^*} \begin{bmatrix} \frac{\dot{\kappa}}{\kappa} [\mathcal{E}^{-1}(w)] - \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, w) \\ I(w \leq \mathbf{t}) \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} \end{bmatrix} M(s^*, dw) + o_p(1).$$

By the main weak convergence theorem or by invoking the Martingale Central Limit Theorem after a time transformation, this converges weakly to the random vector

$$\begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix} = \begin{bmatrix} \Sigma(s^*, t^*)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix}$$

where $(\mathbf{Z}_1^T, \mathbf{Z}_2^T)^T$ is a $(k + p)$ -dimensional zero mean multivariate normal random vector with covariance matrix

$$\begin{aligned} & \text{Cov}[(\mathbf{Z}_1, \mathbf{Z}_2), (\mathbf{Z}_1, \mathbf{Z}_2)] \\ &= \text{plim}_{n \rightarrow \infty} \int_0^{t^*} \mathbb{Q}_n(s^*, w) \\ & \times \left[\frac{\dot{\kappa}}{\kappa} [\mathcal{E}^{-1}(w)] - \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, w) I(w \leq \mathbf{t}) \frac{I\{S^{(0)}(s^*, w) > 0\}}{S^{(0)}(s^*, w)} \right] \\ & S^{(0)}(s^*, w) \Lambda_0^0(dw). \end{aligned}$$

However, the covariance matrix between $\mathbf{Z}_1$ and $\mathbf{Z}_2$ equals $\mathbf{0}$ since, for every $w \in \mathcal{T}$,

$$\mathbb{Q}_n(s^*, w) \left[\frac{\dot{\kappa}}{\kappa} [\mathcal{E}^{-1}(w)] - \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, w) \right] = \mathbf{0}.$$

Because of the Gaussian limits, this then establishes that $\sqrt{n}(\hat{\eta}_n - \eta^0)$ and $V_n(s^*, \cdot)$ are asymptotically independent.

The following two corollaries are then immediate consequences of the preceding theorem and elements of its proof.

Corollary 1—Under the conditions of Theorem 4, as $n \rightarrow \infty$,

$$\sqrt{n}(\hat{\eta}_n - \eta^0) \xrightarrow{d} N\left(0, \Sigma(s^*, t^*)^{-1}\right).$$

Proof: This is immediate from the fact that $\mathbf{Z}_1$ in the proof of Theorem 4 is a k -dimensional zero-mean normal vector with covariance matrix $\Sigma(s^*, t^*)$.

Corollary 2—Under the conditions of Theorem 4, as $n \rightarrow \infty$, the process

$W_n(s^*, \cdot) = \sqrt{n} [\Lambda_0^{(n)}(s^*, \cdot) - \Lambda_0^0(\cdot)]$ converges weakly in Skorokhod's $D[\mathcal{T}]$ -space to a zero-mean Gaussian process with covariance function given by

$$c(s^*, t_1, t_2) = \int_0^{\min(t_1, t_2)} \frac{\Lambda_0^0(dw)}{s^{(0)}(s^*, w)} + b(s^*, t_1)^T \left\{ \sum (s^*, t^*) \right\}^{-1} b(s^*, t_2), \quad (25)$$

for $t_1, t_2 \in \mathcal{T}$ and with $b(s^*, t) = \int_0^t q^{(1)}(s^*, w) \Lambda_0^0(dw)$.

Proof: From Theorem 4 we have the results that

$$\sqrt{n}(\hat{\eta}_n - \eta^0) \xrightarrow{d} \mathbf{W}_1(s^*, t^*)$$

where $\mathbf{W}_1(s^*, t^*) \sim N(\mathbf{0}, [\Sigma(s^*, t^*)]^{-1})$. Also, we have that

$$\{V_n(s^*, t): t \in \mathcal{T}\} \Rightarrow \{Z_2(s^*, t): t \in \mathcal{T}\}$$

where $\{Z_2(s^*, t): t \in \mathcal{T}\}$ is a zero-mean Gaussian process with covariance function

$$\text{Cov}\{Z_2(s^*, t_1), Z_2(s^*, t_2)\} = \int_0^{\min(t_1, t_2)} \frac{\Lambda_0^0(dw)}{S^{(0)}(s^*, w)}.$$

In addition, $\mathbf{W}_1(s^*, t^*)$ and $\{Z_2(s^*, t): t \in \mathcal{T}\}$ are independent. It is also evident that

$$\sup_{t \in \mathcal{T}} \|B_n(s^*, t) - b(s^*, t)\| \xrightarrow{p} 0.$$

From the representations in Theorem 4, it follows that $\{W_n(s^*, t): t \in \mathcal{T}\}$ converges weakly to the process $W_\infty \equiv \{W_\infty(s^*, t): t \in \mathcal{T}\}$ with

$$W_\infty(s^*, t) = Z_2(s^*, t) - b(s^*, t)^T \mathbf{W}_1(s^*, t^*).$$

As such W_∞ is a zero-mean Gaussian process and its covariance function is

$$\begin{aligned} c(s^*, t_1, t_2) &= \text{Cov}\{W_\infty(s^*, t_1), W_\infty(s^*, t_2)\} \\ &= \int_0^{\min(t_1, t_2)} \frac{\Lambda_0^0(dw)}{s^{(0)}(s^*, w)} + b(s^*, t_1)^T \left[\sum (s^*, t^*) \right]^{-1} b(s^*, t_2). \end{aligned}$$

This completes the proof of the corollary.

Possible consistent estimators of the covariance functions are then easily obtained. For the covariance matrix $\Sigma(s^*, t^*)$, this could be estimated by

$$\begin{aligned} & \hat{\Sigma}(s^*, t^*) \\ &= \int_0^{t^*} \mathbb{Q}_n(s^*, w; \hat{\eta}_n) \left[\frac{\dot{\kappa}}{\kappa} [\mathcal{E}^{-1}(w); \hat{\eta}_n] - \frac{\dot{S}^{(0)}}{S^{(0)}}(s^*, w; \hat{\eta}_n) \right]^{\otimes 2} \\ & \times S^{(0)}(s^*, w; \hat{\eta}_n) \hat{\Lambda}_0^{(n)}(s^*, dw; \hat{\eta}_n). \end{aligned}$$

For the covariance function of $Z_2(s^*, \cdot)$, a consistent estimator is given by

$$\widehat{Cov}[Z_2(s^*, t_1), Z_2(s^*, t_2)] = \int_0^{\min(t_1, t_2)} \frac{\hat{\Lambda}_0^{(n)}(s^*, dw)}{S^{(0)}(s^*, w; \hat{\eta}_n)}.$$

On the otherhand, an estimator of $b(s^*, t)$ is given by

$$\hat{b}(s^*, t) = \int_0^t \frac{\dot{S}^{(0)}(s^*, w; \hat{\eta}_n)}{S^{(0)}(s^*, w; \hat{\eta}_n)} \hat{\Lambda}_0^{(n)}(s^*, dw).$$

From these estimators, we are then able to obtain a consistent estimator of the covariance function $\alpha(s^*, t_1, t_2)$ of the limiting Gaussian process $W_\infty(s^*, \cdot)$. This estimator is

$$\hat{c}(s^*, t_1, t_2) = \widehat{Cov}[Z_2(s^*, t_1), Z_2(s^*, t_2)] + \hat{b}(s^*, t_1)^T \left[\hat{\Sigma}(s^*, t^*) \right]^{-1} \hat{b}(s^*, t_1).$$

Observe that the results in Corollaries 1 and 2 are highly analogous to those in [4] pertaining to the estimators of the parameters of the Cox proportional hazards model. However, one need to be cautious since under the setting being considered, the limit functions appearing in the above results are more complicated as they must reflect aspects of the sum-quota accrual scheme and the dynamics of the performed interventions or repairs after each event occurrence.

Through these asymptotic results, large-sample confidence intervals and bands, large-sample hypothesis testing procedures, and goodness-of-fit or model validation methods for the infinite-dimensional parameters may now be constructed for this general dynamic model for recurrent events. We note, however, that the results presented in this paper are still limited to the general dynamic recurrent event model *without* frailties. It remains an open problem to obtain large-sample results for the general dynamic model incorporating frailties.

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