Published in final edited form as:

J Appl Mech. 1980 June ; 47(2): 247–253. doi:10.1115/1.3153650.

A Model of Pressure and Flow Distribution in Branching Networks

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Abstract

The objective of this work is to provide a mechanical description of steady-state flow of Newtonian fluid in a branching network that consists of rigid vessels of different diameters. Solution of this problem is of importance for better understanding of the mechanics of blood flow within the microcirculation. The developed branching network model predicts a wide distribution of the hydrodynamic pressure and flow in the vessels of the same caliber (flow heterogeneity). The obtained results are compared with predictions of a simple series-parallel network model. It is shown that this model provides an accurate approximation to the values of the mean pressure and flow given by the branching network model.

Introduction

Studies of the architecture of the microcirculation reveal rather complex branching structure of the microcirculatory networks (e.g., [1–3]). During the past 15 years new techniques have been developed that made it possible to measure the velocities of red blood cells and the hydrodynamic pressure in single microvessels [4–6]. The results obtained with these techniques indicate that the flow and pressure distributions in the microcirculation are heterogeneous since the characteristics vary substantially from one vessel to another vessel of the same caliber (e.g., [7–10]). These results pose a number of fundamental questions regarding the relationship between the whole organ (“macroscopic”) data and the data pertaining to single micro-vessels. Indeed, one can envision various processes at the microcirculation level that would not be simply translated to the macroscopic level; for example, redistribution of flow in the microvessels without affecting the mean flow. In most cases, the large number of vessels in a given area of the tissue and the complex three-dimensional structure of the system make it impossible to provide a complete experimental description of each individual vessel. A more adequate description of such a system can be given in probabilistic terms, for example, in terms of the frequency (probability) distribution functions. Some examples of experimentally obtained frequency distributions of the velocity of blood and hydrodynamic pressure can be found in [7, 10].

The distribution of pressure and flow in the network is affected by the topology of the network, dimensions of the vessels and many other factors. In order to assess the information contained in these distributions it would be desirable to develop a mathematical

model of branching networks that would make it possible to calculate the frequency distributions of pressure and flow in different generations of vessels. This would provide a basis for theoretical investigation of mechanical processes in microvascular networks.

A number of models have been developed to simulate blood flow in the microcirculation. Lee and Nellis [11] considered vascular pattern of the mesentery and calculated the mean blood flow in different generations of vessels. Lipowsky and Zweifach [12] and Lipowsky [13] utilized the data on vessel geometry to develop a more detailed mathematical model of the mesenteric microcirculation, and calculated hemodynamic parameters for each of the considered vessels. Mayrovitz, et al. [14], and Mayrovitz [15] considered a comprehensive model of branching networks and made calculations of resistances and hemodynamic parameters for the vascular network of the bat wing. The calculations resulted in certain mean characteristics of flow and pressure for different generations of vessels [16].

Besides specific branching structure of the network, various other factors may contribute to the complex behavior of flow in the microvasculature among which are

1. Heterogeneous distribution of vessel dimensions (lengths, diameters) within the vessels of the same branching order.
2. Tapering of vessels along the length.
3. Non-Newtonian viscosity of blood.
4. Uneven distribution of red blood cells in the microvessels.
5. Compliance of the vascular walls.

Neither of these factors have been accounted for in the present study. It is clear that taking into account the aforementioned factors would lead to a very sophisticated mathematical model. As a starting point, however, it appears useful to investigate a more simple case of flow of Newtonian fluid in a branching network composed of rigid vessels. The results of the present work show that even such a simple system exhibits certain interesting properties, the mechanical interpretation of which is important for understanding of the microcirculatory function.

Geometry of the Network and Governing Equations

Consider a branching network composed of several generations of vessels (vessels of different “branching orders”). The geometry of a vessel of an i th branching order ($i = 1, 2, \dots, n$) is illustrated in Fig. 1. It is assumed that all vessels of the same branching order are geometrically identical, and each vessel of the i th branching order consists of m_i segments (a segment is a portion of the vessel situated between two adjacent bifurcations). In this work it is assumed that vessels of the branching orders $i = 1, 2, \dots, n - 1$ terminate in two vessels of the next branching order. This geometry appears the most realistic, however, other geometries could easily be considered within the framework of this model. The vessels of the last branching order ($i = n$) consist of a single segment, hence $m_n = 1$. This description completely defines the topology of the network. The network, therefore, contains $N_i = (m_1$

+ 1) $(m_2 + 1) \dots (m_{i-1} + 1)$ vessels of the i th order, and consists of the total of $N = \sum_{i=1}^n N_i$ vessels.

Each node of the i th-order vessel, except for the terminal node, can be associated with a sequence $(i, k_i, k_{i-1}, \dots, k_1)$, where k_i is the number of nodes preceding the given node in given vessel (including the node from which the vessel originates), and the sequence $(i-1, k_{i-1}, k_{i-2}, \dots, k_1)$ refers to the node from which the vessel originates considered as a node pertaining to the $(i-1)$ st branching order. Terminal nodes are considered as “double” nodes and two sequences are assigned to them: $(i, m_i, k_{i-1}, \dots, k_1)$ and $(i, m_i + 1, k_{i-1}, \dots, k_1)$. The two terminal vessels of the $(i+1)$ st branching order are considered as originating from different nodes of the double node; for example, the sequences $(i+1, 1, m_i, k_{i-1}, \dots, k_1)$ and $(i+1, 1, m_i + 1, k_{i-1}, \dots, k_1)$ identify the second nodes of these vessels. Therefore, the introduced procedure will assign a unique sequence $(i, k_i, k_{i-1}, \dots, k_1)$ to each node of an i th branching order if terminal nodes are considered as double nodes; in this sequence $i = 1, 2, \dots, n$; $k_i = 1, 2, \dots, m_i + 1$ for $1 \leq i \leq n$; $k_i = 0, 1, \dots, m_i + 1$. Different sequences are assigned to any two nodes of the network. It should be noted, however, that each node is an intersection of vessels of two successive branching orders, and, therefore, can be identified by either $(i, k_i, k_{i-1}, \dots, k_1)$ or $(i+1, 0, k_i, k_{i-1}, \dots, k_1)$ with the former referring to the node of the i th branching order, and the latter referring to the node of the $(i+1)$ st branching order. Therefore, the introduced notation is unambiguous if the branching order is specified. If a segment of the i th-order vessel has the node $(i, k_i, k_{i-1}, \dots, k_1)$ at the end then the same sequence is assigned to this segment; the terminal segment is denoted $(i, m_i, k_{i-1}, \dots, k_1)$.

If the segments are sufficiently long compared to their diameters, and the Reynolds numbers for flow in each segment are small then the hydrodynamic pressure is nearly constant in a cross section of the vessel, and the pressure difference between two adjacent nodes is proportional to the flow through the segment situated between the nodes; the coefficient of proportionality is the hydrodynamic resistance of the segment, [17].

The hydrodynamic pressure at the node $(i, k_i, k_{i-1}, \dots, k_1)$ is denoted as

$$p_{i,k_i,k_{i-1},\dots,k_1} \quad (1)$$

and the volumetric flow rate through the segment $(i, k_i, k_{i-1}, \dots, k_1)$ is denoted as

$$Q_{i,k_i,k_{i-1},\dots,k_1} \quad (2)$$

The introduced notation implies the relationships

$$p_{i,k_i,k_{i-1},\dots,k_1} = p_{i+1,0,k_i,\dots,k_1},$$

and

$$p_{i,m_i,k_{i-1},\dots,k_1} = p_{i,m_i+1,k_{i-1},\dots,k_1}.$$

Using the notations (1), (2) one can express the conservation of mass at each node in the form

$$Q_{i,k_i,k_{i-1},\dots,k_1} = Q_{i,k_i+1,k_{i-1},\dots,k_1} + Q_{i+1,1,k_i,\dots,k_1} \quad (3)$$

where

$$k_j = 1, 2, \dots, m_j + 1 \quad \text{for } 1 \leq j \leq i - 1;$$

$$k_i = 1, 2, \dots, m_i - 1$$

and

$$Q_{i,m_i,k_{i-1},\dots,k_1} = 2Q_{i+1,1,m_i k_{i-1},\dots,k_1} = 2Q_{i+1,1,m_i+1,k_{i-1},\dots,k_1} \quad (4)$$

where

$$k_j = 1, 2, \dots, m_j + 1 \quad \text{for } 1 \leq j \leq i - 1.$$

Using the definition of the hydrodynamic resistance, ζ_{i,k_i} and the assumption that it is independent of pressure and flow, one can write for each segment

$$p_{i,k_i-1,k_{i-1},\dots,k_1} - p_{i,k_i,k_{i-1},\dots,k_1} = \zeta_{i,k_i} Q_{i,k_i,k_{i-1},\dots,k_1} \quad (5)$$

where

$$k_j = 1, 2, \dots, m_j + 1 \quad \text{for } 1 \leq j \leq i - 1;$$

$$k_i = 1, 2, \dots, m_i.$$

Suppose that the pressure at the entrance of the first branching order, and at the end of the last branching order are specified: $p_{1,0} = p_a$ and

$$p_{n,1,k_{n-1},\dots,k_1} = p_v,$$

respectively, for all possible $k_{n-1}, k_{n-2}, \dots, k_1$, where p_a is the “arterial” pressure and p_v is the “venous” pressure. The problem is to determine the flow in every segment and the pressure at every node providing the resistances of the segments, ζ_{i,k_i} are given. The problem (3)–(5) is linear due to the assumption that the resistance ζ_{i,k_i} is independent of the flow and pressure in the segment.

In order to obtain the solution of the problem, it is convenient to introduce auxiliary input hydrodynamic resistances analogous to those used by Mayrovitz, et al. [14], and Mayrovitz [15] in modeling vascular networks

$$p_{i,k_i-1,k_{i-1},\dots,k_1} - p_v = R_{i,k_i} Q_{i,k_i,k_{i-1},\dots,k_1} \quad (6)$$

where

$$k_j = 1, 2, \dots, m_j + 1 \quad \text{for } 1 \leq j \leq i - 1;$$

$$k_i = 1, 2, \dots, m_i.$$

For the two cases illustrated in Fig. 2, the following recurrency relationships are formulated that make it possible to compute the matrix R_{i,k_i} :

$$R_{i,k_i} = \zeta_{i,k_i} + \frac{R_{i,k_i+1} R_{i+1,1}}{R_{i,k_i+1} + R_{i+1,1}}$$

$$\text{for } k_i = 1, 2, \dots, m_i - 1 \quad (7)$$

$$R_{i,m_i} = \zeta_{i,m_i} + 0.5 R_{i+1,1} \quad (8)$$

$$R_{n,1} = \zeta_{n,1} \quad (9)$$

Therefore, starting with known quantities $R_{n,1}$ and doing calculations in the “reverse” order using (7) and (8) one can determine $\sum_{j=1}^{n-1} m_j$ terms of the sequence

$$\begin{aligned} & R_{n-1,m_{n-1}}; R_{n-1,m_{n-1}-1}; \dots; R_{n-1,2}; R_{n-1,1}; \\ & R_{n-2,m_{n-2}}; R_{n-2,m_{n-2}-1}; \dots; R_{n-2,2}; R_{n-2,1}; \\ & \dots \dots \dots \\ & R_{1,m_1}; R_{1,m_1-1}; \dots; R_{1,2}; R_{1,1} \end{aligned} \quad (10)$$

The knowledge of the input hydrodynamic resistance matrix R_{i,k_i} makes it possible to determine the flow and pressure distribution in the network using the relationships (3)–(5). Thus, for the first segment,

$$Q_{1,1} = \frac{p_a - p_v}{R_{1,1}}, \quad p_{1,1} = p_a - \zeta_{1,1} Q_{1,1} \quad (11)$$

Generally, if the flow $Q_{i,k_{j-1},k_{j-1},\dots,k_1}$ and the pressure $p_{i,k_{j-1},k_{j-1},\dots,k_1}$ have already been determined, the flow and pressure further downstream can be found from (4) and the following relationships:

$$Q_{i,k_i,k_{i-1},\dots,k_1} = \frac{R_{i+1,1}}{R_{i,k_i} + R_{i+1,1}} Q_{i,k_{i-1},k_{i-1},\dots,k_1} \quad (12)$$

where

$$k_j = 1, 2, \dots, m_j + 1 \quad \text{for } 1 \leq j \leq i - 1;$$

$$k_i = 1, 2, \dots, m_i$$

$$Q_{i+1,1,k_i,\dots,k_1} = \frac{R_{i,k_i+1}}{R_{i,k_i+1} + R_{i+1,1}} Q_{i,k_i,k_{i-1},\dots,k_1} \quad (13)$$

where

$$k_j = 1, 2, \dots, m_j + 1 \quad \text{for } 1 \leq j \leq i - 1;$$

$$k_i = 1, 2, \dots, m_i - 1$$

$$p_{i,k_i,k_{i-1},\dots,k_1} = p_{i,k_{i-1},k_{i-1},\dots,k_1} - \zeta_{i,k_i} Q_{i,k_i,k_{i-1},\dots,k_1} \quad (14)$$

The values of indices in (14) are the same as in (12).

Certain mean values of the pressure and flow will be calculated. The mean pressure at the entrance of the vessels of the i th branching order is

$$p_{i,0} = \frac{1}{(m_{i-1}+1) \dots (m_1+1)} \sum_{k_{i-1}=1}^{m_{i-1}+1} \dots \sum_{k_1=1}^{m_1+1} p_{i,0,k_{i-1},\dots,k_1} \quad (15)$$

where the ranges of sums account for the two daughter vessels at the end of given parent vessels. The mean volumetric flow rate in the first segments of the vessels of the i th branching order is

$$Q_{i,1} = \frac{1}{(m_{i-1}+1) \dots (m_1+1)} \sum_{k_{i-1}=1}^{m_{i-1}+1} \dots \sum_{k_1=1}^{m_1+1} Q_{i,1,k_{i-1},\dots,k_1} \quad (16)$$

and the mean volumetric flow rate in the vessels of the i th branching order averaged over all segments is

$$Q_i = \frac{1}{m_i(m_{i-1}+1) \dots (m_1+1)} \sum_{k_i=1}^{m_i} \sum_{k_{i-1}=1}^{m_{i-1}+1} \dots \sum_{k_1=1}^{m_1+1} Q_{i,k_i,k_{i-1},\dots,k_1} \quad (17)$$

The conservation of mass yields

$$Q_{i,1,k_{i-1},\dots,k_1} = \sum_{k_i=1}^{m_i+1} Q_{i+1,1,k_i,\dots,k_1} \quad (18)$$

where

$$k_j = 1, 2, \dots, m_j+1 \quad \text{for } 1 \leq j \leq i.$$

Thus, from (16) and (18), it follows that

$$Q_{i+1,1} = \frac{1}{(m_i+1)} Q_{i,1} \quad (19)$$

The relationship (19) shows that the mean flow in the first segment of an $(i+1)$ st-order vessel equals the mean flow in the first segment of the i th-order vessel divided by the number of daughter vessels of the $(i+1)$ st-order.

Application of the averaging operators defined in (15) and (16) to equation (6) with $k_j = 1$ yields simple relationships between the mean pressure and flow and the input hydrodynamic resistance

$$p_{i,0} - p_v = R_{i,1} Q_{i,1} \quad (20)$$

Relationships (19) and (20) imply

$$p_{i,0} - p_{i+1,0} = (R_{i,1} - R_{i+1,1}/(m_i+1)) Q_{i,1} \quad (21)$$

Therefore, the quantity $R_{i,1} - R_{i+1,1}/(m_i + 1)$ may be considered as an effective resistance of an “average” i th-order vessel.

Numerical Examples

For the sake of simplicity, it is assumed that the segments of the vessels are long cylindrical tubes so that the Poiseuille law is applicable; then the hydrodynamic resistance can be expressed in the form

$$\zeta_{i,k_i} = \frac{128\mu L_{i,k_i}}{\pi D_{i,k_i}^4} \quad (22)$$

Here L_{i,k_i} is the length of the segment, D_{i,k_i} is the internal diameter of the segment, and μ is the viscosity of the fluid.

A detailed experimental study of vessel dimensions was performed for the vascular network of the bat wing [1, 15]; these data are utilized here in the numerical examples. Table 1 lists the diameters D_i , the lengths L_i and the number of daughter branches $m_i + 1$ for four successive branching orders [1, 14, 15]. These quantities are assumed to be connected to the characteristics of the segment by means of relationships: $D_{i,k_i} = D_i$, $L_{i,k_i} = L_i/m_i$ i.e., the diameters of all segments of the i th branching order are identical, and the lengths of all segments of the i th branching order are identical.

The resistances of the vessels located downstream from the forth branching order (terminal arterioles, capillaries and the venous part of the circulation) are “lumped” together, and the effective resistance of this part of the network, $R_{5,1}$, is specified. This terminal resistance is varied within the range $R_{5,1} = 10^{17} - 5 \times 10^{17}$ N/m⁵•s. The range $\mu = 0.002 - 0.005$ Pa•s is chosen for viscosity of the fluid (cf. [18]).

Dimensional analysis of the problem suggests [19] that a dimensionless variable

$$\bar{p}_{i,k_i,k_{i-1},\dots,k_1} = \frac{p_{i,k_i,k_{i-1},\dots,k_1} - p_v}{p_a - p_v} \quad (23)$$

is independent of a particular choice of the arterial pressure p_a and the venous pressure p_v . The form (23) is used in this study for presentation of the pressure distributions.

Some comments should be made regarding the form of presentation of the results. The number of vessels considered in this particular example is substantial; indeed, there are $N_5 = 13 \times 12 \times 10 \times 5 = 7800$ vessels of the 5th branching order. However, the large volume of data makes it difficult to assess the information pertaining to each individual vessel. Therefore, it is convenient to introduce frequency distribution functions in the following manner. Consider a sequence $\{A_j\}$, $j = 1, 2, \dots, M$ and let $A_{\min}$ and $A_{\max}$ be the minimum and maximum values of A_j in this sequence. If M_j is the number of terms that satisfy the inequality

$$A_{\min} + \frac{A_{\max} - A_{\min}}{10}(l-1) \leq A_j < A_{\min} + \frac{A_{\max} - A_{\min}}{10}l, \quad l=1, 2, \dots, 10 \quad (24)$$

with a fixed value of l (the term $A_j = A_{\max}$ is included into the last interval) then the frequency distribution of the quantity A is defined as

$$\Phi(A_j) = 100M_l/M \quad (25)$$

The calculations are done in the following order. First, the resistances ζ_{i,k_j} are calculated using the relationships (22), then the matrix of input resistances R_{i,k_j} is computed using the relationships (7)–(9). Finally, the hydrodynamic pressures at all nodes of the network, and the volumetric flow rates through all segments were calculated using the relationships (12)–(14) and (4) and stored in the computer; then these data were utilized in calculations of the frequency distribution functions for pressure and flow.

Table 2 presents the calculated values of $R_{i,1}$ for three different values of the terminal resistance $R_{5,1}$, and the viscosity $\mu = 0.003 \text{ Pa}\cdot\text{s}$. Knowing $R_{1,1}$ one can calculate the total flow through the network, $Q_{1,1}$, in terms of the difference of the input and output pressures using the relationship (11), and then calculate the mean flow for all branching orders, $Q_{i,1}$, using the relationship (19).

Fig. 3 shows the frequency distributions of the dimensionless hydrodynamic pressure at the entrance of the vessels,

$$\Phi(\bar{p}_{i,0,k_{i-1},\dots,k_1}),$$

for $\mu = 0.003 \text{ Pa}\cdot\text{s}$, and $R_{5,1} = 10^{17}$, 3×10^{17} , and $5 \times 10^{17} \text{ N/m}^5\cdot\text{s}$. Arrows indicate the mean values $\bar{p}_{i,0}$ defined by the relationships (15) and (23). The frequency distributions for an i th branching order are based on $N_i = (m_1 + 1)(m_2 + 1)\dots(m_{i-1} + 1)$ values of pressure. The range of variation of the entrance hydrodynamic pressures in the vessels of the same branching order is substantial and reaches as much as 0.5 on the dimensionless scale. The frequency distribution of pressure may undergo qualitative changes with variation of the terminal resistance $R_{5,1}$. Indeed, the distribution

$$\Phi(\bar{p}_{5,0,k_{i-1},\dots,k_1})$$

in Fig. 3 (left panel) is skewed to the left which means that a large fraction of vessels have low entrance pressure, whereas the distribution corresponding to a higher terminal resistance (right panel), is more “symmetrical” with respect to the mean pressure $\bar{p}_{5,0}$.

Fig. 4 depicts the quantity $\bar{p}_{5,0}$. It shows that variation of viscosity of fluid significantly affects the pressure distribution. Thus it could be expected that non-Newtonian behavior of fluid would have an impact on the pressure and flow distribution in the network.

The relationship (6) yields

$$Q_{i,1,k_{i-1},\dots,k_1} = \frac{p_{i,0,k_{i-1},\dots,k_1} - p_v}{R_{i,1}} \quad (26)$$

which implies that the shape of the frequency distribution of flow through the first segments of the vessels is similar to the shape of the corresponding frequency distributions of the entrance pressure. Fig. 5 shows the frequency distribution of dimensionless flow at the entrance of the vessels

$$\Phi(Q_{i,1,k_{i-1},\dots,k_1}/Q_{i,1})$$

normalized for each branching order by the mean flow $Q_{i,1} = (p_a - p_v)/R_{i,1}N_b$ for $R_{5,1} = 3 \times 10^{17} \text{ N/m}^5\text{s}$, and $\mu = 0.003 \text{ Pa}\cdot\text{s}$. Note that with such scaling of flow, the distribution of flow becomes wider as the branching order i increases; the dimensionless mean flow is equal to unity. Clearly, the distribution of flow

$$Q_{i,k_i,k_{i-1},\dots,k_1}$$

in all segments k_i is substantially more heterogeneous than the distribution of flow at the entrance of the vessels,

$$Q_{i,1,k_{i-1},\dots,k_1}$$

because the loss of fluid into the daughter branches leads to diminution of flow in successive segments of the parent vessel. Fig. 6, which depicts the frequency distribution

$$\Phi(Q_{i,k_i,k_{i-1},\dots,k_1}/Q_{i,1}),$$

illustrates this difference (compare with Fig. 5). The ratio $Q/Q_{i,1}$, where the mean flow Q_i is defined by (17), is also shown for each distribution.

Comparison Between the Branching Network Model and a Series-Parallel Network Model

The results of the previous paragraph manifest that the calculated flow and pressure in the vessels of the same branching order vary within rather wide range from one vessel to another. Of course, this variation cannot be predicted if one considers a model illustrated in Fig. 7 with series-parallel arrangement of the vessels, where the vessels of the same branching order are arranged in parallel, and the vessels of different orders are arranged in series. Such a model yields homogeneous flow in vessels of the same branching order. Nevertheless, this model has long been used as an aid in interpretation of experimental data (e.g., [20]). At this point, it is of interest to find out if the values of the mean pressure and

flow predicted by the branching network model as described in the preceding paragraphs, could be predicted with a reasonable accuracy by the series-parallel network model.

Consider a network shown in Fig. 7 with the following parameters pertaining to a branching order i : diameter of the vessel D_i , length of the vessel $L_i^* = \alpha L_i$, and number of “parallel” vessels N_i . Therefore, instead of the actual length L_i , an effective length L_i^* is used, where the ratio $\alpha = L_i^*/L_i$ is independent of the branching order i . A condition will be formulated below leading to determination of the parameter α . The hydrodynamic resistance of a single vessel of the i th branching order is calculated as

$$\zeta_i^* = 128\mu L_i^* / \pi D_i^4, \quad i=1, 2, 3, 4 \quad (27)$$

The total flow through the network is then given by

$$Q_1^* = \frac{p_a - p_v}{128\mu\alpha\pi^{-1} \sum_{j=1}^4 L_j D_j^{-4} N_j^{-1} + R_{5,1} N_5^{-1}} \quad (28)$$

The flow through a single vessel of the i th branching order is

$$Q_i^* = Q_1^* / N_i \quad (29)$$

The pressure at the entrance of the i th branching order is

$$p_i^* = p_a - (p_a - p_v) \frac{128\mu\alpha\pi^{-1} \sum_{j=1}^{i-1} L_j D_j^{-4} N_j^{-1}}{128\mu\alpha\pi^{-1} \sum_{j=1}^4 L_j D_j^{-4} N_j^{-1} + R_{5,1} N_5^{-1}} \quad (30)$$

It is proposed here to determine the parameter α from the condition that the total flow through the network Q_1^* be equal to the total flow $Q_{1,1}$ given by equation (11), or, in other words, that the total resistance of the series-parallel network be equal to the input resistance $R_{1,1}$ of the branching network as defined by (6). This condition yields

$$\alpha = \frac{R_{1,1} - R_{5,1} N_5^{-1}}{128\mu\pi^{-1} \sum_{j=1}^4 L_j D_j^{-4} N_j^{-1}} \quad (31)$$

As follows from the relationships (19) and (31), the equality $Q_1^* = Q_{1,1}$ implies $Q_i^* = Q_{i,1}$ for $i = 2, 3, 4, 5$, thus the mean flow in the vessels of any branching order for the branching network model coincides with the flow in the same branching order for the series-parallel network model. The important result is that the parameter α determined from the

relationship (31) remains nearly constant despite variation of $R_{5,1}$ and μ ; Fig. 8 shows that fivefold elevation of $R_{5,1}$ causes only 4 percent increase of α when $\mu = 0.002 \text{ Pa}\cdot\text{s}$, and 6 percent increase of α when $\mu = 0.005 \text{ Pa}\cdot\text{s}$.

Now, the question can be raised how well the pressure distributions (30) approximate the mean pressure distributions predicted by the branching network model. Fig. 9 shows the values of $\bar{p}_i^* = (p_i^* - p_v)/(p_a - p_v)$, together with the previously computed characteristics of the branching network model: the dimensionless mean pressure $\bar{p}_{i,0} = (p_{i,0} - p_v)/(p_a - p_v)$, the maximum pressure

$$\bar{p}_{i,0}^{\max} = \max \bar{p}_{i,0,k_{i-1},\dots,k_1},$$

and the minimum pressure

$$\bar{p}_{i,0}^{\min} = \min \bar{p}_{i,0,k_{i-1},\dots,k_i}.$$

The agreement between $\bar{p}_i^*$ and $\bar{p}_{i,0}$ is rather remarkable; indeed, the difference between these two quantities does not exceed 0.04 for the considered values of parameters. Equation (20) yields

$$p_{5,0} = p_v + R_{5,1} Q_{5,1} \quad (32)$$

which can be shown to coincide with p_5^* given by (30).

Therefore, the series-parallel network model with effective lengths of the vessels $L_j^* = \alpha L_j$ (with $\alpha \approx 0.40 - 0.44$) yields the value of total flow that coincides with the total flow predicted by the branching network model, and also yields the distribution of pressure that is very close to the distribution of the mean pressure predicted by the branching network model.

Conclusion

A model of steady-state flow of Newtonian fluid in a branching network of rigid vessels has been developed. Numerical examples have shown that a specific branching structure of the network causes the pressure and flow in the vessels of the same caliber to vary within a wide range. The calculated values of the mean pressure and flow can be accurately predicted within the framework of a simple series-parallel network model. The results of this work may be useful for qualitative assessment of the experimental data on flow and pressure distribution in branching networks. The present model also provides a basis for further development of the theory, in particular accounting for non-Newtonian properties of the fluid.

Acknowledgments

This work was supported by the National Institutes of Health Grants HL-17421 and HL-23362. The author wishes to thank Shen Min Liang for the help in numerical calculations.

Nomenclature

D_{i,k_i}	segment diameter
L_i	vessel length
L_{i,k_i}	segment length
N_i	number of vessels of an i th branching order
p_a	input pressure
p_v	output pressure
$p_{i,0}$	mean hydrodynamic pressure at the entrance of the i th branching order, equation (15)
$Q_{i,1}$	mean flow at the entrance of the i th branching order, equation (16)
Q_i	mean flow, equation (17)
R_{i,k_i}	input hydrodynamic resistance
α	ratio of the lengths, L_i^*/L_i
ξ_{i,k_i}	hydrodynamic resistance
μ	viscosity
Φ	frequency distribution

Subscripts

i	branching order
k_j	node number

Superscript

*	pertaining to the series-parallel network model
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Fig. 1.
Topology of the branching network; each vessel of an i th branching order consists of m_i segments

Fig. 2.
A scheme illustrating the definition of the input hydrodynamic resistances for two types of bifurcations

Fig. 3.

Frequency distributions of the dimensionless hydrodynamic pressure at the entrance of the vessels,

$$\Phi(\bar{P}_{I,0,k_{I-1},\dots,k_1}),$$

$i = 1, 2, 3, 4, 5$ for $\mu = 3 \times 10^{-3} \text{ Pa}\cdot\text{s}$ and $R_{5,1} = 10^{17}, 3 \times 10^{17}$ and $5 \times 10^{17} \text{ N/m}^5\cdot\text{s}$. Vertical arrows indicate the mean values of pressure $\bar{p}_{i0}$; also shown are the minimum and maximum values of the pressure for each branching order.

Fig. 4.

The dimensionless mean pressure at the entrance of the 5th-order vessels for different values of the viscosity μ , and of the terminal resistance $R_{5,1}$

Fig. 5.
Frequency distributions of the dimensionless volumetric flow rate at the entrance of the vessels,

$$\Phi(Q_{I,1,k_{I-1},\dots,k_1}/Q_{I,1})$$

for $\mu = 0.003 \text{ Pa}\cdot\text{s}$ and $R_{5,1} = 3 \times 10^{17} \text{ N/m}^5\cdot\text{s}$. Vertical arrows indicate the dimensionless mean flow, which is equal to unity.

Fig. 6.
Frequency distributions of the dimensionless volumetric flow rate,

$$\Phi(Q_{I,k_I,k_{I-1},\dots,k_1}/Q_{I,1}),$$

for $\mu = 0.003 \text{ Pa}\cdot\text{s}$ and $R_{5,1} = 3 \times 10^{17} \text{ N/m}^5\cdot\text{s}$. Vertical arrows indicate the dimensionless mean flow, which is equal to unity.

Fig. 7.
Topology of the series-parallel network; vessels of the same branching order are arranged in parallel, and vessels of different branching orders are arranged in series

Fig. 8.
Dimensionless parameter $\alpha = L_i^*/L_i$ for different values of μ and $R_{5,1}$

Fig. 9. Distributions of the dimensionless mean pressure $\bar{p}_{i,0}$, the maximum pressure $\bar{p}_{i,0}^{\max}$ and the minimum pressure $\bar{p}_{i,0}^{\min}$ throughout the network (solid lines) calculated within the framework of the branching network model, and the distributions of $\bar{p}_i^*$ (dotted lines) calculated within the framework of the series-parallel network model, for $\mu = 3 \times 10^{-3} \text{ Pa} \cdot \text{s}$

Table 1

Geometrical parameters of the network

Branching order, i	Length, $L_i(m)$	Diameter, $D_i(m)$	Number of daughter vessels, $m_i + 1$
1	4.0×10^{-2}	97×10^{-6}	13
2	1.7×10^{-2}	52×10^{-6}	12
3	3.5×10^{-3}	19×10^{-6}	10
4	9.5×10^{-4}	7×10^{-6}	5

Table 2Input hydrodynamic resistances, $\text{N/m}^5\cdot\text{s}$

$R_{1,1}$	6.58×10^{13}	9.33×10^{13}	1.20×10^{14}
$R_{2,1}$	5.86×10^{14}	9.37×10^{14}	1.28×10^{15}
$R_{3,1}$	5.68×10^{15}	9.88×10^{15}	1.39×10^{16}
$R_{4,1}$	4.34×10^{16}	8.51×10^{16}	1.26×10^{17}
$R_{5,1}$	1.00×10^{17}	3.00×10^{17}	5.00×10^{17}