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Nitsche's Method For Helmholtz Problems with Embedded Interfaces

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SUMMARY

In this work, we use Nitsche's formulation to weakly enforce kinematic constraints at an embedded interface in Helmholtz problems. Allowing embedded interfaces in a mesh provides significant ease for discretization, especially when material interfaces have complex geometries. We provide analytical results that establish the well-posedness of Helmholtz variational problems and convergence of the corresponding finite element discretizations when Nitsche's method is used to enforce kinematic constraints. As in the analysis of conventional Helmholtz problems, we show that the inf-sup constant remains positive provided that the Nitsche's stabilization parameter is judiciously chosen. We then apply our formulation to several 2D plane-wave examples that confirm our analytical findings. Doing so, we demonstrate the asymptotic convergence of the proposed method and show that numerical results are in accordance with the theoretical analysis.

Keywords

Nitsche's method; Helmholtz problem; Embedded interface

1. INTRODUCTION

The numerical solution of the Helmholtz equation is of fundamental importance to the simulation of time-harmonic acoustics, electromagnetics, and steady-state elastodynamics problems [1, 2]. Numerous methods have been proposed to reduce the pollution error that encumbers the computation of wave problems. Among those are: the Galerkin/least-squares method [3], the partition of unity method [4], the ultra weak variational formulation [5], the variational theory of complex rays [6], and the discontinuous enrichment method [7].

In this article, we are interested in the simulation of steady-state wave propagation across embedded material interfaces, which arise in many problems across science and engineering. Examples include scattering in submerged bodies, composite materials, sound generation from vibrating bodies, and seismic exploration among others [8, 9, 10]. A key aspect in the simulation of wave propagation across material interfaces is the accurate and efficient

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enforcement of kinematic conditions. For instance, the acoustic pressure and normal velocity need to be continuous across an interface between different fluids.

A conventional approach to deal with such material interface in the finite element method (FEM) is to employ two meshes that match at the interface or to impose interface conditions as constraints. These constraints can be enforced either strongly through methods like multi-point constraints [11, 12] or weakly. The material interface needs to be represented by the FE meshes with sufficient accuracy to maintain optimal convergence. A significant drawback of these approaches is that meshing becomes a formidable task when the interface geometry is complex.

In recent years, embedded interface methods have gained popularity among the computational mechanics community, mainly due to their superiority to deal with interfaces with complex and evolving geometries. As their name suggests, embedded interface methods allow material interfaces to be arbitrarily embedded inside a FE mesh and cut through elements; the interface conditions are enforced weakly and incorporated as part of a variational statement. To this end, Hansbo and Hansbo [13, 14], Dolbow and Harari [15, 16] and Sanders et al. [17, 18] employed Nitsche's formulation [19, 20] for elliptic problems with embedded interfaces. The key feature of Nitsche's method is a stabilization term that is used to restore the coercivity of the resulting variational problem. The authors in [15] proposed using a local generalized eigenvalue problem to explicitly determine the Nitsche's stabilization parameter at the element level. The choice of interfacial weights and stabilization parameter in case of large material contrast and small cut elements has been further clarified by Annavarapu et al. [21] and Jiang et al. [22], leading to the so-called γ -Nitsche's method.

It is worthwhile to compare Nitsche's formulation with other methods that weakly enforce interface conditions. For instance, methods based on Lagrange multipliers [23], such as mortar methods [24, 25], can also be used to enforce interface constraints in a weak sense. However, these methods yield saddle-point problems and selecting the proper multiplier space that satisfies the Babuška-Brezzi (BB) conditions [26, 27] is nontrivial. Furthermore, introducing Lagrange multipliers increases the problem size, which demands higher computational costs. On the other hand, penalty methods violate variational consistency and suffer from suboptimal convergence [28, 29]. In comparison, Nitsche's method has the properties of being variationally consistent and maintaining optimal convergence with mesh refinement. As all embedded methods, however, Nitsche's approach is susceptible to numerical issues for degenerate geometries of intersected elements. Such issues are outside the scope of this work.

With the attractiveness of embedded interface method and Nitsche's formulation to weakly and stably enforce interface constraints, we intend to investigate the possibility of using such approach for steady-state acoustic problems with material interfaces. Similar formulations were proposed for hybrid asymptotic-numerical methods in scattering, non-reflecting boundary conditions and finite elements for exterior acoustics problems [30]. It is well known that the Helmholtz equation leads to an indefinite variational form. We will show that

a judicious choice of the Nitsche's stabilization parameter as proposed in [21, 22] leads to an asymptotically stable variational Helmholtz problem.

Several researchers have applied Nitsche's method to the simulation of steady-state dynamics problems before. In particular, Hanso and Hermansson [31] adopted a Nitsche's method to couple non-matching meshes in fluid-structure vibration problems; they reported promising results for the eigenstate analysis of a fluid domain enclosed inside an elastic container. Rabinovich et al. [32] applied Nitsche's method to couple mixed-dimensional problems considering time-harmonic response in elastic structures. A fundamental component missing in the current literature is a detailed analysis taking into consideration the non-elliptic nature of the problem. The critical role played by the wave number in Helmholtz equation needs to be investigated in the framework of Nitsche's formulation. Indeed, the authors in [32] reported large error for certain choices of angular frequencies which they characterized as resonances. Such a phenomenon can actually be connected to the inf-sup constant. This phenomenon is shown both analytically and numerically in this article. Additionally, we approach the Helmholtz material interface problem using an embedded interface method, which provides significant ease for discretization, especially when the material interface has a complex geometry.

The rest of the article is organized as follows. In section 2, we provide the analytical background on the analysis of the Helmholtz variational problem and Nitsche's method. In section 3, we show the formulation of Nitsche's method for the Helmholtz equation with an embedded interface. Particularly, we show the stability condition of the resulting variational problem and show an a priori error estimate of the FEM approximation. In section 4, we apply the embedded interface method to several 2D plane-wave examples to show the convergence behavior of the method as well as to verify our analytical findings.

2. HELMHOLTZ PROBLEM AND NITSCHÉ'S FORMULATION

To motivate the investigation into the Nitsche's method for Helmholtz problems with an embedded interface, we begin by considering a model problem that is simpler yet similar in nature. That is, we consider a scalar Helmholtz problem with Dirichlet boundary conditions weakly enforced by Nitsche's method, but without an embedded interface. To this end, we first recall the conventional variational formulation of the Helmholtz equation and state two main theorems that establish the well-posedness of the Helmholtz variational problem. We then use these theorems to establish well-posedness the Helmholtz problem using Nitsche's formulation.

2.1. Helmholtz problem

We begin by considering the following Helmholtz equation in a domain Ω . The strong form of the problem is given as

$$\begin{aligned} -\nabla \cdot (c^2 \nabla u) - \omega^2 u &= f \text{ in } \Omega \\ c^2 \nabla u \cdot \mathbf{n} &= t \text{ on } \Gamma^N \\ u &= g \text{ on } \Gamma^D \end{aligned} \quad (1)$$

where $c > 0$ is the wave speed; $\omega > 0$ is the frequency; $\mathbf{n}$ denotes a unit vector normal to the boundary and Γ^N and Γ^D indicate the Neumann and Dirichlet boundaries, respectively. Furthermore, t is a given surface source, f is a given body source, and g is a given boundary data.

The standard weak form of the above Helmholtz problem is: find $u \in U$, $U := \{u \in H^1(\Omega), u = g \text{ on } \Gamma^D\}$, such that

$$\int_{\Omega} (c^2 \nabla u \cdot \nabla \bar{v} - \omega^2 u \bar{v}) d\Omega = \int_{\Omega} f \bar{v} d\Omega + \int_{\Gamma^N} t \bar{v} d\Gamma \quad \forall v \in U_0 \quad (2)$$

where $U_0 := \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma^D\}$ and $\bar{\cdot}$ indicates complex conjugation.

We now define, respectively, a sesquilinear form $a(\cdot, \cdot)$ and an antilinear bounded functional $\ell(\cdot)$ as

$$a(u, v) = \int_{\Omega} c^2 \nabla u \cdot \nabla \bar{v} d\Omega \quad (3)$$

$$\ell(v) = \int_{\Omega} f \bar{v} d\Omega + \int_{\Gamma^N} t \bar{v} d\Gamma \quad (4)$$

Notice that $a(\cdot, \cdot)$ is Hermitian, i.e., $a(v, u) = \overline{a(u, v)}$ and coercive, i.e., $a(u, u) \geq C \|u\|_U^2 \forall u \in U_0$ for some $C > 0$.

Equation (2) now can be compactly written as

$$b(u, v) = \ell(v) \quad \forall v \in U_0 \quad (5)$$

where

$$b(u, v) = a(u, v) - \omega^2 (u, v) \quad (6)$$

is a Hermitian, sesquilinear form and $(u, v) = \int_{\Omega} u \bar{v} d\Omega$.

Using a finite element (FE) discretization, we obtain the conventional finite dimensional variational problem of the form: find $u^h \in U^h$ such that

$$b(u^h, v^h) = \ell(v^h) \quad \forall v^h \in U_0^h \quad (7)$$

where $U^h \subset U$ and $U_0^h := U^h \cap U_0$. The parameter h is the mesh size in the FE context and is related to the dimension of the subspace U^h . The coercivity of $a(\cdot, \cdot)$ restricted to U_0^h is a direct consequence of its coercivity on U_0 .

Notice that if we work with Poisson's equation ($\omega = 0$), we would end up with $a(u^h, v^h) = \ell(v^h)$ instead of Equation (7). In this case, the coercivity of $a(\cdot, \cdot)$ automatically implies the well-posedness and stability of the variational problem by the Lax-Milgram theorem [33]. However, the sesquilinear form $b(\cdot, \cdot)$ is non-coercive, in general. We show some basic results on the well-posedness of (7) in the next section.

2.2. Well-Posedness of The Helmholtz Variational Problem

Most of the results presented in this section were previously published in [34]. We reproduce the analysis of the conventional formulation herein to lay the groundwork for the analysis of the Nitsche's formulation for this problem. We would like to point out that similar results to those presented herein can be obtained for the more general case of $a(u, u) = 0 \quad \forall u \in H^1$. However, since we are interested in problems involving Dirichlet conditions, we will concentrate on the case where $a(\cdot, \cdot)$ is coercive. In addition, to simplify our subsequent analysis and without loss of generality, we shall take $g = 0$ in this subsection and shall not distinguish U_0^h from U^h henceforth.

We consider variational problems in the form of (7). The following well-known Babuška theorem [34, 35] is the foundation of the analysis.

Theorem 1—If for any $u^h \in U^h$ and $v^h \in U^h$, the sesquilinear form $b(\cdot, \cdot)$ satisfies

1. $\exists M^h > 0, |b(u^h, v^h)| \leq M^h \|u^h\|_U \|v^h\|_U$ (Continuity)
2. $\inf_{\|u^h\|_U=1} \sup_{\|v^h\|_U=1} |b(u^h, v^h)| \geq \gamma^h > 0$

Then, there exists a unique u^h . Moreover,

$$\|u^h\|_U \leq \frac{1}{\gamma^h} \|\ell\|_{U'} \quad (8)$$

where U' denotes the dual space of U and $\|\ell\|_{U'} = \sup_{u \in U} \frac{|\ell(u)|}{\|u\|_U}$ is the corresponding norm for $\ell \in U'$. Furthermore, let u be the solution to the infinite dimensional counterpart of Problem (7), then

$$\|u - u^h\|_U \leq \left(1 + \frac{M^h}{\gamma^h}\right) \|u - v^h\|_U \quad \forall v^h \in U^h \quad (9)$$

The inf-sup constant for a given subspace U^h is defined as the value γ^h for which the equality in Item 2 above is achieved. For the Helmholtz problem, $b(\cdot, \cdot)$ is given in Equation (6). Since $a(\cdot, \cdot)$ is Hermitian and coercive, we can endow the vector space U^h with the inner product $(u^h, v^h)_E := a(u^h, v^h)$ and associated norm $\|u^h\|_E^2 := a(u^h, u^h)$.

The continuity condition in Theorem 1 for this norm is

$$|b(u^h, v^h)| \leq (1 + C^h \omega^2) \|u^h\|_E \|v^h\|_E. \quad (10)$$

where C^h is some constant satisfying $\int_{\Omega} |u^h|^2 d\Omega \leq C^h \int_{\Omega} \nabla u^h \cdot \nabla u^h d\Omega$ for $\forall u^h \in U^h$ (Poincaré's inequality).

Let $\{(\lambda_i^h, \phi_i^h)\}_{i=1}^n$ be the eigenpairs of the following problem,

$$a(\phi_i^h, v^h) = \lambda_i^h (\phi_i^h, v^h) \quad \forall v^h \in U^h$$

By the properties of $a(\cdot, \cdot)$, the eigenvalues λ_i^h are real and positive, and the eigenfunctions can be chosen to satisfy

$$\begin{aligned} (\phi_i^h, \phi_j^h) &= \delta_{ij} \\ a(\phi_i^h, \phi_j^h) &= \lambda_i^h \delta_{ij} \end{aligned}$$

The following theorem is another cornerstone of our forthcoming results. The theorem was proved in Demkowicz [34]. We show our own proof in Appendix A for the sake of completeness. The proof hinges on the positivity of the eigenvalues.

Theorem 2—In terms of $\|\cdot\|_E$, the inf-sup constant for the finite dimensional Helmholtz variational problem, i.e., Equation (7), is

$$\gamma^h = \min_i \frac{|\lambda_i^h - \omega^2|}{\lambda_i^h} \quad (11)$$

Proof: See Appendix A.

The case where $\omega^2 = \lambda_i^h$ for some λ_i^h is referred to as resonance hereon. As one can see, if resonance happens, the inf-sup constant equals to 0. A critical requirement in Theorem 1 is violated.

On the other hand, if resonance does not occur, we are guaranteed to satisfy the requirements in Theorem 1 (some technical details regarding the uniform boundedness of M^h and γ^h will be addressed later in the context of Nitsche's method), we hence have the desired well-posedness and stability of the Helmholtz problem (7).

Remark 1—We now recall the following fundamental result that is of paramount importance in the analysis of Helmholtz problems. If the infinite dimensional (Helmholtz) problem has a unique solution, then there exists h_0 such that the conditions in Theorem 1 are satisfied for all $h < h_0$. See [34] for the proof.

In the next section, we develop a formulation using Nitsche's method and show that stability follows directly from the analysis presented herein.

2.3. Nitsche's formulation

In Nitsche's approach, we define a new sesquilinear operator $a_N(\cdot, \cdot)$ and an antilinear functional $\ell_N(\cdot)$ as

$$a_N(u^h, v^h) = a(u^h, v^h) - \int_{\Gamma_D} c^2 \left((\nabla u^h \cdot \mathbf{n}) \bar{v}^h + u^h (\nabla \bar{v}^h \cdot \mathbf{n}) \right) d\Gamma + \theta^h \int_{\Gamma_D} u^h \bar{v}^h d\Gamma \quad (12)$$

$$\ell_N(v^h) = \ell(v^h) - \int_{\Gamma_D} g(c^2 \nabla \bar{v}^h \cdot \mathbf{n}) d\Gamma + \theta^h \int_{\Gamma_D} g \bar{v}^h d\Gamma \quad (13)$$

where θ^h is a positive scalar that remains to be selected. Notice that $a_N(\cdot, \cdot)$ is Hermitian.

Now using Nitsche's method, we seek a solution $u^h \in V^h \in H^1(\Omega)$ such that

$$b_N(u^h, v^h) = \ell_N(v^h) \quad \forall v^h \in V^h \quad (14)$$

where

$$b_N(u^h, v^h) = a_N(u^h, v^h) - \omega^2(u^h, v^h) \quad (15)$$

The key difference between the Nitsche's method (14) and the standard method (7) is that both the trial and test functions in V^h need not satisfy the kinematic conditions on Γ^D . Instead, such constraints are built into the Nitsche's variational statement directly.

If the sesquilinear form $a_{\mathcal{N}}(\cdot, \cdot)$ were coercive in V^h , the same stability analysis presented in Section 2.2 would apply. Fortunately, as the key ingredient of the Nitsche's method [19, 20], the parameter θ^h can be selected to make $a_{\mathcal{N}}(\cdot, \cdot)$ coercive. In this sense, θ^h is also known as Nitsche's stabilization parameter.

Using Young's inequality and a discrete trace inequality, respectively, we have

$$\frac{1}{\varepsilon} \|u^h\|_{L_2, \Gamma^D}^2 + \varepsilon \|c^2 \nabla u^h \cdot \mathbf{n}\|_{L_2, \Gamma^D}^2 \geq 2 \int_{\Gamma^D} \operatorname{Re} \left((c^2 \nabla u^h \cdot \mathbf{n}) \bar{u}^h \right) d\Gamma \quad (16)$$

$$\eta^h \|c \nabla u^h\|_{L_2, \Omega}^2 \geq \|c^2 \nabla u^h \cdot \mathbf{n}\|_{L_2, \Gamma^D}^2 \quad (17)$$

where $\varepsilon > 0$ is arbitrary and $\eta^h > 0$ is some computable h -dependent constant. Using (16) and (17) with (12), we obtain

$$a_N(u^h, u^h) \geq (1 - \eta^h \varepsilon) \|c \nabla u^h\|_{L_2, \Omega}^2 + \left(\theta^h - \frac{1}{\varepsilon} \right) \|u^h\|_{L_2, \Gamma^D}^2, \forall u^h \in V_h$$

The coercivity of $a_{\mathcal{N}}(\cdot, \cdot)$ is ensured by choosing $\varepsilon < \frac{1}{\eta^h}$ and $\theta^h > \frac{1}{\varepsilon} > \eta^h$. The choice of such stabilization parameter θ^h for Helmholtz interface problems will be explained in the next section.

Once $a_{\mathcal{N}}(\cdot, \cdot)$ in (12) is made coercive (stabilized) by selecting a proper stabilization parameter θ^h , we apply the same analysis in Section 2.2 to Equation (14). When resonance does not occur, i.e., ω^2 does not coincide with any of the eigenvalues of $a_{\mathcal{N}}(\cdot, \cdot)$, we have the desired well-posedness and stability of the Nitsche's variational problem. Moreover, by Remark 1, if the infinite dimensional problem has a unique solution, optimal asymptotic convergence is guaranteed.

3. NITSCHÉ'S FORMULATION FOR HELMHOLTZ PROBLEM WITH AN EMBEDDED INTERFACE

We now turn to the main aim of this article to deal with the Helmholtz problem with embedded material interfaces. We will develop a Hermitian sesquilinear operator in the same form as Equation (15) and show that the judicious choice of Nitsche's stabilization parameter as proposed in [21, 22] provides the required stability for our interface problem as well.

3.1. Interface problem and formulation

We consider a scalar Helmholtz equation with a material interface Γ_I that partitions the domain Ω into Ω_1 and Ω_2 . The strong form for this problem is given as

$$\begin{aligned}
-\nabla \cdot c_m^2 \nabla u_m - \omega^2 u_m &= f_m \text{ in } \Omega_m \\
c_m^2 \nabla u_m \cdot \mathbf{n}_m &= t_m \text{ on } \Gamma_m^N \\
u_m &= g_m \text{ on } \Gamma_m^D
\end{aligned} \quad (18)$$

for $m = 1, 2$, where $u_m = u|_{\Omega_m}$, $\mathbf{n}_m$ denotes the unit normal vector on the boundary of Ω_m . We also consider the interface constraints

$$\begin{aligned}
[[c^2 \nabla u]] \cdot \mathbf{n} &= 0 \text{ on } \Gamma_I \\
[[u]] &= 0 \text{ on } \Gamma_I
\end{aligned}$$

where $[[\cdot]]$ indicates the jump across the interface Γ_I specifically, $[[u]] = u_2 - u_1$ and $[[c^2 \nabla u]] \cdot \mathbf{n} = (c_2^2 \nabla u_2 - c_1^2 \nabla u_1) \cdot \mathbf{n}$; $\mathbf{n}$ denotes the unit normal vector on the interface pointing from Ω_2 to Ω_1 . The different materials are distinguished by the wave speed c_m ($c_m > 0$). We restrict ourselves to the case of zero jumps; however, the method applies to the more general case where the jump terms are nontrivial.

To approximate the solution of (18), we consider a domain Ω and introduce a finite partition of the domain into non-overlapping elements. The partition is independent of the location of the interface Γ_I ; i.e., Γ_I is allowed to intersect with elements.

We denote an element domain as Ω_e^h and the collection of all elements by $\Omega^h = \cup_e \Omega_e^h$. We assume that the mesh is regular and the interface Γ_I intersects with an element only twice through two distinct edges. The part of interface inside an element is denoted by $\Gamma_e^I := \Gamma_I \cap \Omega_e^h$. Moreover, we define $h_e = \sup\{\|\mathbf{x} - \mathbf{y}\|, \mathbf{x}, \mathbf{y} \in \Omega_e^h\}$, and a characteristic mesh size h such that $h_e \leq h$.

In addition, we define the collection of elements which are intersected by Γ_I as $\mathbf{I}^h$, i.e., $\mathbf{I}^h = \{\Omega_e^h : \Omega_e^h \cap \Gamma_I \neq \emptyset\}$. For each element Ω_e^h , let $\Omega_{e,m}^h = \Omega_e^h \cap \Omega_m$ denote the part of Ω_e^h in Ω_m . We further define the collection of elements/fractional elements that belong to the two separated subdomains as $\Omega_1^h = \cup_e \Omega_{e,1}^h$ and $\Omega_2^h = \cup_e \Omega_{e,2}^h$ respectively. We have $\Omega_1^h \cup \Omega_2^h = \Omega^h$, $\Omega_1^h \cap \Omega_2^h = \emptyset$.

Using Nitsche's approach to enforce interface conditions, the variational form of (18) is cast as: Find $u^h \in U^h$ such that

$$a_B(u^h, v^h) + a_I(u^h, v^h) - \omega^2 \sum_m \int_{\Omega_m^h} u_m^h \bar{v}_m^h d\Omega = L(v^h) \forall v^h \in U^h \quad (19)$$

where $U^h = U_1^h \times U_2^h$,

$$U_m^h = \{u_m^h : u_m^h \in H^1(\Omega_m^h), u_m^h|_{\Omega_{e,m}^h} \in \mathcal{P}_p(\Omega_{e,m}^h) \forall \Omega_{e,m}^h \in \Omega_m^h, u_m^h(\Gamma_m^D) = 0\}$$

and $\mathcal{P}_p$ is the space of polynomials of degree at most p . We again assumed $g_m = 0$ for simplicity.

The bulk contribution $a_B(u^h, v^h)$ is given as

$$a_B(u^h, v^h) := \sum_m \int_{\Omega_m^h} c_m^2 \nabla u_m^h \cdot \nabla \bar{v}_m^h d\Omega \quad (20)$$

and the interface contribution $a_I(u^h, v^h)$ is

$$a_I(u^h, v^h) := \sum_{e \in \mathcal{I}^h} \left(- \int_{\Gamma_e^I} \langle c^2 \nabla u^h \rangle_\gamma \cdot \mathbf{n} [[\bar{v}^h]] d\Gamma - \int_{\Gamma_e^I} \langle c^2 \nabla \bar{v}^h \rangle_\gamma \cdot \mathbf{n} [[u^h]] d\Gamma + \theta_e^I \int_{\Gamma_e^I} [[u^h]] [[\bar{v}^h]] d\Gamma \right) \quad (21)$$

and $L(\cdot)$ is a bounded antilinear functional defined as

$$L(v^h) := \sum_m \int_{\Omega_m^h} f_m \bar{v}_m^h d\Omega + \sum_m \int_{\Gamma_m^N} t_m \bar{v}_m^h d\Gamma \quad (22)$$

$\langle \cdot \rangle_\gamma = \gamma_1(\cdot)_1 + \gamma_2(\cdot)_2$ is the weighted average quantity on the interface; $0 < \gamma_1, \gamma_2 < 1$ are the weighting parameter satisfying $\gamma_1 + \gamma_2 = 1$. The variable θ_e^I denotes the element-wise stabilization parameters, which remain to be selected to recover the coercivity of the sesquilinear form $a_B(u^h, v^h) + a_I(u^h, v^h)$. The choice of γ_i turns out to be critical for the conditioning of the resulting system matrix. Robust choices for such parameters were proposed in [21, 22], which we will explain in a subsequent section.

Remark 2—It can be easily checked that the Nitsche's variational problem is consistent in the sense that the u solving (18) satisfies (19). Also, notice that the functions belonging to U^h are not required to be continuous across the interface Γ_I . The continuity constraints are instead enforced in the weak statement of Nitsche's method. To represent the potential discontinuity of the solution across the interface, local enrichment of basis functions for fractional elements can be employed [13, 15].

To relate to our previous analysis in Section 2.2, we define the following sesquilinear forms.

$$a(u^h, v^h) := a_B(u^h, v^h) + a_I(u^h, v^h) \quad (23)$$

$$b(u^h, v^h) := a(u^h, v^h) - \omega^2(u^h, v^h) \quad (24)$$

where $(u^h, v^h) = \sum_m \int_{\Omega_m} u_m^h \bar{v}_m^h d\Omega$. It is evident that $a(\cdot, \cdot)$ is Hermitian. The analysis in Section 2.2 follows once we make $a(\cdot, \cdot)$ coercive as we show next.

3.2. Stability and error bound

To analyze the above variational problem, we first define a mesh-dependent norm as follows,

$$\|u^h\|_h^2 = \sum_m \left(\|c_m \nabla u_m^h\|_{L_2, \Omega_m^h}^2 + \frac{1}{D^2} \|u_m^h\|_{L_2, \Omega_m^h}^2 \right) + \sum_{\Omega_e \in \mathcal{I}^h} \left(h \|\langle c^2 \nabla u^h \rangle_\gamma \cdot \mathbf{n}\|_{L_2, \Gamma_e^I}^2 + \frac{1}{h} \|[[u^h]]\|_{L_2, \Gamma_e^I}^2 \right) \quad (25)$$

where D is the diameter of the domain Ω defined as $D := \sup \|x - y\|, x, y \in \Omega$.

We adopt the procedure proposed in [15] to select θ_e^I so that $a(\cdot, \cdot)$ satisfies coercivity in U^h .

In particular, for each $\Omega_e^h \in \mathcal{I}^h$, we can compute a h -independent constant $\beta_e^I > 0$, such that if $\theta_e^I > \frac{\beta_e^I}{h}$ (we assume $\theta_e^I = 2 \frac{\beta_e^I}{h}$) for all $\Omega_e^h \in \mathcal{I}^h$, $a(\cdot, \cdot)$ satisfies coercivity in U^h . That is

$$a(u^h, u^h) > C_1 \|u^h\|_h^2, \forall u^h \in U^h \quad (26)$$

for some $C_1 > 0$, where

$$a(u^h, u^h) = \sum_m \|c_m \nabla u_m^h\|_{0, \Omega_m^h}^2 - 2 \sum_{\Omega_e \in \mathcal{I}^h} \int_{\Gamma_e^I} \text{Re} \left(\langle c^2 \nabla u^h \rangle_\gamma \cdot \mathbf{n} [[\bar{u}^h]] \right) d\Gamma + \sum_{\Omega_e \in \mathcal{I}^h} \theta_e^I \|[[u^h]]\|_{0, \Gamma_e^I}^2 \quad (27)$$

Different approaches to compute θ_e^I have been given in [15, 21, 22]. For the sake of completeness, we briefly describe one of these methods in the next subsection.

After establishing the coercivity of $a(\cdot, \cdot)$, we can associate with $a(\cdot, \cdot)$ a norm $\|\cdot\|_E$ defined as $\|\cdot\|_E = a(\cdot, \cdot)^{\frac{1}{2}}$. The latter norm is equivalent to $\|\cdot\|_h$ because U^h is a finite dimensional subspace of U . That is

$$C_1 \|u^h\|_h^2 \leq \|u^h\|_E^2 \leq C_2 \|u^h\|_h^2 \quad \forall u^h \in U^h \quad (28)$$

for some other constant $C_2 > 0$. The constant C_1 is the same as in (26). It is straightforward to check that, by virtue of the definition of $\|\cdot\|_h$, C_1 and C_2 can be chosen to be h -independent.

The continuity of $b(u^h, v^h)$ can be easily shown as follows. After applying Cauchy-Schwarz inequality to (24) and breaking the interface integral in (21), it is easy to show that

$$|b(u^h, v^h)| \leq \max(1, h\theta_e^I, h^2\omega^2) \|u^h\|_h \|v^h\|_h \quad (29)$$

Noticing the equivalence of norms in (28), Equation (29) now implies continuity with respect to $\|\cdot\|_E$ with a multiplicative constant $M^h = \max(1, h\theta_e^I, h^2\omega^2)/C_1$. It is important to note that because $h\theta_e^I = 2\beta_e^I$, which is h -independent, M^h is uniformly bounded as $h \rightarrow 0$ given a fixed ω .

Since $a(\cdot, \cdot)$ is Hermitian and coercive, applying Theorem 2 yields

$$\inf_{\|u^h\|_E=1} \sup_{\|v^h\|_E=1} |b(u^h, v^h)| \geq \gamma^h = \min_i \frac{|\lambda_i^h - \omega^2|}{\lambda_i^h} \quad (30)$$

where λ_i^h is the i -th eigenvalue associated with $a(\cdot, \cdot)$. The existence and uniqueness of the solution u^h follows from Theorem 1 under the condition that ω^2 does not coincide with any of the eigenvalues λ_i^h . We also have an error bound

$$\|u^h - u\|_E \leq \left(1 + \frac{M^h}{\gamma^h}\right) \|v^h - u\|_E \quad \forall v^h \in U^h \quad (31)$$

Using (28), a more useful error bound in terms of $\|\cdot\|_h$ can also be derived as

$$|||u^h - u|||_h \leq \sqrt{\frac{C_2}{C_1}} \left(1 + \frac{M^h}{\gamma^h}\right) |||v^h - u|||_h \quad \forall v^h \in U^h \quad (32)$$

Finally, using an interpolation error estimate (e.g., see [13, 16]), we obtain

$$|||u^h - u|||_h \leq \sqrt{\frac{C_2}{C_1}} \left(1 + \frac{M^h}{\gamma^h}\right) Ch^p \quad (33)$$

where $C > 0$ is independent of h , and p is the order of polynomial completeness in U^h .

3.3. Numerical implementation

Our implementation follows closely that described in [21]. In particular, for each element cut by the interface, we create a duplicate set of nodes, called ghost nodes, for the computations on the partial elements (see Figure 1).

In Figure 1, the trial and test functions defined on the lower triangle (Ω_1) are interpolated with the shape functions associated with the nodes $\{1, 2', 3', 4'\}$, whereas for the trial and test function defined on the upper octagon (Ω_2), we use the shape functions associated with $\{1', 2, 3, 4\}$. Hence in each interface element in I^h , we have the following expressions:

$$u^h = \sum_{j \in \{1, 2', 3', 4'\}} \chi^1(\mathbf{x}) N_j(\mathbf{x}) u_j + \sum_{j \in \{1', 2, 3, 4\}} \chi^2(\mathbf{x}) N_j(\mathbf{x}) u_j \quad (34)$$

$$v^h = \sum_{j \in \{1, 2', 3', 4'\}} \chi^1(\mathbf{x}) N_j(\mathbf{x}) v_j + \sum_{j \in \{1', 2, 3, 4\}} \chi^2(\mathbf{x}) N_j(\mathbf{x}) v_j \quad (35)$$

where u_j and v_j are the unknown displacement at node j , $N_j(\mathbf{x})$ are shape functions at node j , and $\chi^m(\mathbf{x})$ is the characteristic function given by

$$\chi^m(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega_m, \\ 0 & \text{otherwise.} \end{cases} \quad (36)$$

The standard FEM interpolations are adopted for all the other elements.

$$u^h = \sum_{j \in \Omega_e^I} N_j(\mathbf{x}) u_j \quad v^h = \sum_{j \in \Omega_e^I} N_j(\mathbf{x}) v_j \quad (37)$$

Upon introducing the test and trial functions into the discrete variational form (19), the following matrix equation is derived

$$\begin{bmatrix} K_b^1 + K_n^1 + K_s^1 & K_c \\ K_c^T & K_b^2 + K_n^2 + K_s^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix} \quad (38)$$

where u^m and f^m are the displacement vector and force vector of the two subdomains (with enriched degree of freedom), respectively. The bulk matrix K_b^m is assembled from each (partial) element e using,

$$k_{b,e}^m = \int_{\Omega_{e,m}^h} B^T c_m^2 B d\Omega - \omega^2 \int_{\Omega_{e,m}^h} N^T N d\Omega$$

while the other matrices arise only for interface elements $\Omega_e^h \in I^h$,

$$\begin{aligned} k_{n,e}^m &= -\gamma_{e,m} \int_{\Gamma_e^I} N^T c_m^2 (n_m)^T B d\Gamma - \gamma_{e,m} \int_{\Gamma_e^I} B^T n_m c_m^2 N d\Gamma \\ k_{s,e}^m &= \theta_e^I \int_{\Gamma_e^I} N^T N d\Gamma \\ k_{c,e} &= -\theta_e^I \int_{\Gamma_e^I} N^T N d\Gamma + \gamma_{e,2} \int_{\Gamma_e^I} B^T n_2 c_2^2 N d\Gamma + \gamma_{e,1} \int_{\Gamma_e^I} B^T n_1 c_1^2 N d\Gamma \end{aligned}$$

The element stabilization parameter θ_e^I and the weights $\gamma_{e,m}$ play a fundamental role in the performance of Nitsche's method. We adopt the γ -Nitsche method developed in [21, 22] for computing these parameters. For each $\Omega_e^h \in I^h$, we consider two one-sided inequalities on the two partial elements as follows

$$\int_{\Gamma_e^I} (\nabla u^h \cdot n_1) (\nabla \bar{u}^h \cdot n_1) d\Gamma \leq C_{e,1} \int_{\Omega_{e,1}^h} \nabla u^h \cdot \nabla \bar{u}^h d\Omega \quad (39)$$

$$\int_{\Gamma_e^I} (\nabla u^h \cdot n_2) (\nabla \bar{u}^h \cdot n_2) d\Gamma \leq C_{e,2} \int_{\Omega_{e,2}^h} \nabla u^h \cdot \nabla \bar{u}^h d\Omega \quad (40)$$

where $C_{e,m}$ are some positive constants satisfying the inequalities. In fact, these inequalities are discrete versions of the generalized inverse inequality in (17) at the element level, which is critical to recover the coercivity of $a(\cdot, \cdot)$.

To practically compute viable $C_{e,m}$, we solve the following generalized eigenvalue problem

$$A_m x = \lambda_{e,m} B_m x \quad (41)$$

where

$$\begin{aligned} \mathbf{A}_m &= \int_{\Gamma_e^I} (\nabla u^h \cdot \mathbf{n}_m) (\nabla \bar{u}^h \cdot \mathbf{n}_m) d\Gamma \\ \mathbf{B}_m &= \int_{\Omega_{e,m}^h} \nabla u^h \cdot \nabla \bar{u}^h d\Omega \end{aligned}$$

and choose $C_{e,m} = \max(\lambda_{e,m})$, where $\max(\lambda_{e,m})$ is the largest generalized eigenvalue computed in Equation (41). It can be easily checked that such a choice satisfies the inequalities in (39) and (40).

With $C_{e,m}$ computed, the weights $\gamma_{e,m}$ and the element stabilization parameter θ_e^I are determined by

$$\gamma_{e,m} = \frac{1/(C_{e,m}c_m^2)}{1/(C_{e,1}c_1^2) + 1/(C_{e,2}c_2^2)} \quad (42)$$

$$\theta_e^I = \frac{2}{1/(C_{e,1}c_1^2) + 1/(C_{e,2}c_2^2)} \quad (43)$$

These choices of $\gamma_{e,m}$ and θ_e^I are designed to be robust against the potential scenarios of large material contrast, e.g., $\frac{c_1}{c_2} \gg 1$, or small cut element, e.g., $\frac{C_{e,1}}{C_{e,2}} \gg 1$. They have been shown to preserve the optimal convergence rates. We refer the readers to [22] for more details on the γ -Nitsche method.

4. NUMERICAL EXAMPLES

We employ simple examples using 2D plane-wave problems to showcase our implementation and verify our analytical findings. We use Q4 bilinear elements for all of our problems.

4.1. An equivalent 1D example of wave propagation across material interface

In the first example, we employ a plane-wave problem in a rectangular domain, see Figure 2, with a vertical embedded interface (red line) dividing the domain into two identical subdomains. Notice that the interface cuts through the center layer of elements. These are the elements on which enrichment of basis function applies, see Section 3.3. The subdomains have different wave speeds, i.e., $c_1 = 1$ and $c_2 = \sqrt{5}$.

By setting Dirichlet boundary conditions $u = 0$ and $u = 1$ on the left and right end of the domain respectively, the steady-state wave is set to propagate in the horizontal direction. These simple settings allow the 2D problem to be reduced to an equivalent 1D counterpart, thereby enabling an analytical solution of the waveform.

Figure 3a shows the field of numerical solution with $\omega^2 = 400$ and $h = 0.003$. Figure 3b compares the numerical wave form with the analytical one. The error and convergence rate in both the L_2 norm and H^1 seminorm are shown in Figure 4 for two different frequencies, i.e., $\omega^2 = 100$ and $\omega^2 = 400$. The error is normalized by the corresponding norm of the analytical solution. Particularly, the convergence rate with h refinement is, as expected, quadratic in the L_2 norm and linear in the H^1 seminorm.

4.2. 2D circular interfaces

In this example, we consider 2D domains with an embedded circular interface in the center. We shall study two cases. In the first case, the materials inside and outside the interface have the same wave speed, i.e., $c_1 = c_2$; in the second case, the wave speed of the material inside the interface is different from that of the background, i.e., $c_1 \neq c_2$.

Our goal in the first case is to verify the behavior of the numerical approximation in the context of the analysis presented in Section 2.2. In particular, we study the connection between the inf-sup constant and the behavior of the FE approximation. In the second case, we compare the embedded interface method with the conventional meshed interface method in simulating more complex steady-state wave propagation problems including wave deflection and scattering.

4.2.1. Homogeneous domain—We consider a two dimensional version of Equation (18) with homogeneous wave speed $c_1 = c_2 = 1$ and a specified Dirichlet boundary condition as follows,

$$u(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} \text{ for } \mathbf{x} \in \Gamma \quad (44)$$

where Γ indicates the boundary of the domain, $\mathbf{k}$ is vector-valued and composed of the wave numbers in x and y direction respectively, i.e., $\mathbf{k} = [k_1, k_2]^T$ with $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$.

This problem has a unique solution $u(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}}$ if $\|\mathbf{k}\|^2 \neq \lambda_{m,n}$, where $\lambda_{m,n} = (m\pi)^2 + (n\pi)^2$ for $m, n = 1, 2, 3, \dots$ are the eigenvalues of the Dirichlet problem for the Laplacian.

If $\|\mathbf{k}\|^2 = \lambda_{m,n}$, on the other hand, $u(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} + C \sin(m\pi x) \sin(n\pi y)$ with arbitrary constant C is also a solution. The problem becomes ill-posed.

We first solve this problem using the embedded interface method, for which an exemplary FE mesh and the embedded interface are shown in Figure 5. For comparison purposes, we also solve the same problem using the conventional FEM without modeling the interface in the domain since we have homogeneous wave speed.

It is well known that as $h \rightarrow 0$, the eigenvalues of the discretized coercive operator $\mathcal{A}(\cdot, \cdot)$ converge to the analytical ones from above [34, 38, 39]. Given a $\|\mathbf{k}\|^2$ slightly larger than an analytical eigenvalue λ_i of $\mathcal{A}(\cdot, \cdot)$, λ_i^h crosses the borderline $\|\mathbf{k}\|^2$ as we refine the mesh (at some choice of h_0), causing the inf-sup constant in (30) to become very close to zero and the solution error to increase suddenly. Stable convergence of the numerical solution can,

therefore, be expected only when $h < h_0$ and all eigenvalues λ_i^h have located on the proper side of k^2 .

To demonstrate such a phenomenon, we show two examples, in which the k^2 is deliberately chosen to be close to one of the eigenvalues of the Nitsche's stabilized coercive operator $a(\cdot, \cdot)$, see (27). Since these eigenvalues can not be determined analytically, we take the eigenvalues of the discretized operator using a very fine mesh as the "true" eigenvalues.

The first eight eigenvalues of the stabilized operator $a(\cdot, \cdot)$ are shown in Figure 6. As is seen from Figure 6, as the mesh is refined, the spectrum of eigenvalues indeed shifts to the left. When an eigenvalue crosses k^2 , the solution error jumps, as expected. Actually, when $h > 0.01$, the solution error hardly decreases with mesh refinement due to the decreasing inf-sup constant, which is reflected in Equation (33). On the other hand, once all eigenvalues are on the correct sides of k^2 , stable convergence with mesh refinement is observed. Meanwhile, both the method with artificial embedded interface and the conventional FEM without interface perform identically.

To provide a more thorough comparison between the embedded interface method and the conventional FEM without the interface, we compare the two methods for a range of frequencies (with k^2 ranging from 200 to 5,000). Figure 7 shows that both methods are equivalent when the mesh is sufficiently refined so that stable convergence is achieved. The comparison highlights the efficacy of the embedded interface method for the 2D plane-wave Helmholtz equation. Meanwhile, it is also seen that the solution error of plane-wave equation highly depends on k , as expected (see [36, 37] for more details).

4.2.2. Inhomogeneous domain with circular inclusion—We conclude with a more complex 2-D steady-state wave propagation problem with $c_1 \neq c_2$. Again, only Dirichlet boundary conditions are specified, as shown in Figure 8. We set $c_1 = 1$ and vary the wave speed c_2 of the inclusion to produce different material contrast with ratio $r = c_2^2/c_1^2$. The frequency is set to be $\omega = 60$. We model the problem using both conventional FEM with a meshed interface and the embedded interface method.

Due to the lack of an analytical solution for the current problem, we take a highly refined conventional FEM with a meshed interface as reference (referred to as the reference solution). To this end, we use a very fine mesh composed of more than 1,600,000 elements (with an average mesh size of $h = 0.0012$). Meanwhile, we take a series of embedded interface models with mesh size ranging from 0.02 to $h = 0.00167$ and investigate the behavior of the error between the embedded interface approximation and the reference solution.

Figure 9, 10, 11 show a qualitative comparison of the reference wave fields and that produced by the embedded interface approach with a mesh size of $h = 0.00167$ (720,000 structured elements). Notice that both approaches produce nearly identical fields for all three different material contrasts ($r = 0.7, 1.5$ and 5).

The difference between the embedded interface approximation and the reference solution is shown in Figure 12 in terms of the H^1 seminorm. We can observe the expected convergence rate of these approximations with mesh refinement. Moreover, under the current problem settings, the approximation error is also seen to be sensitive to the material contrast, as expected. The nonmonotonic decrease of the error shown for the case $r=5$ is typical of Helmholtz-type problems. The theoretical foundations for the latter behavior were explained in Section 4.2.1.

5. CONCLUSION

In this work, we have explored the potential of using Nitsche's method for solving Helmholtz problems with an embedded interface. We have shown that by choosing Nitsche's stabilization parameter as originally proposed in [21, 22], well-posedness of the variational problem can be guaranteed except for cases when resonance occurs. That is, we have shown that the inf-sup constant remains uniformly bounded provided that the Nitsche's stabilization parameter is judiciously chosen and that the driving frequency does not coincide with the eigenvalues of the stabilized coercive operator $a(\cdot, \cdot)$.

Through numerical examples and parametric studies using simple 2D plane-wave problems, we have demonstrated the accuracy and convergence behavior of Nitsche's method for the Helmholtz equation with an embedded a material interface. In particular, it is shown that the embedded interface method has an equivalent performance with the conventional FEM with a meshed interface, provided that the FE mesh is fine enough to achieve asymptotic convergence, as in the conventional formulation. However, the embedded interface method has more flexibility and can provide significant ease for discretization when solving Helmholtz problems with a geometrically complex material interface. Our potential future work includes extending the current method to acoustic-structure interaction (ASI) problems.

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APPENDIX

A. Proof of Theorem 2

Proof

Because the eigenfunctions of $a(\cdot, \cdot)$ form a basis in U^h , we can represent any function $u^h \in U^h$ as $u^h = \sum_{j=1}^n u_j^h \phi_j^h$, where $u_j^h = (u^h, \phi_j^h)$. Therefore,

$$a(u^h, v^h) = \sum_{j=1}^n \lambda_j^h u_j^h \bar{v}_j^h$$

Hence,

$$b(u^h, v^h) = \sum_{j=1}^n u_j^h \bar{v}_j^h (\lambda_j^h - \omega^2)$$

Now, consider

$$\sup_{\|v^h\|_E=1} |b(u^h, v^h)|$$

Notice that for a given u^h , the sesquilinear form induces a antilinear functional on U^h . That is

$$\ell_u(v^h) := b(u^h, v^h) \quad (45)$$

By the definition of the norm of a functional, we have

$$\sup_{\|v^h\|_E=1} |b(u^h, v^h)| = \|\ell_u\|_{U^{h'}} \quad (46)$$

From the Riesz representation theorem, we know that for each $\ell_u \in U^{h'}$ there is a unique $w^h \in U^h$ such that

$$\begin{aligned} (w^h, v^h)_E &= \ell_u(v^h) \\ &= b(u^h, v^h) \quad \forall v \in U^h \end{aligned} \quad (47)$$

and that $\|\ell_u\|_{U^{h'}} = \|w^h\|_E$. Hence,

$$\sup_{\|v^h\|_E=1} |b(u^h, v^h)| = \|w^h\|_E. \quad (48)$$

Now using the eigenfunction expansions of u^h , v^h , and w^h in (47), we get

$$\sum_{j=1}^n \lambda_j^h w_j^h \bar{v}_j^h = \sum_{j=1}^n u_j^h \bar{v}_j^h (\lambda_j^h - \omega^2) \quad \forall v^h \in U^h \quad (49)$$

Using the fact that (49) holds for arbitrary v^h , we obtain the following result for the spectral components of the Riesz element w^h .

$$w_j^h = \frac{u_j^h (\lambda_j^h - \omega^2)}{\lambda_j^h} \quad (50)$$

The inf-sup constant is found by solving the following problem

$$\begin{aligned}\gamma^h &= \inf_{\|u^h\|_E=1} \|w^h\|_E \\ &= \inf_{\|u^h\|_E=1} \left[\sum_{j=1}^n \lambda_j^h \frac{|u_j^h|^2 (\lambda_j^h - \omega^2)^2}{|\lambda_j^h|^2} \right]^{\frac{1}{2}}\end{aligned}\quad (51)$$

Using the last expression on the right, we have

$$\begin{aligned}\sum_{j=1}^n \frac{|u_j^h|^2 \lambda_j^h (\lambda_j^h - \omega^2)^2}{(\lambda_j^h)^2} &\geq \min_i \frac{(\lambda_i^h - \omega^2)^2}{(\lambda_i^h)^2} \sum_{j=1}^n \lambda_j^h |u_j^h|^2 \\ &= \min_i \frac{(\lambda_i^h - \omega^2)^2}{(\lambda_i^h)^2} \|u^h\|_E^2\end{aligned}\quad (52)$$

Now, using (52) in (51), we get

$$\gamma^h \geq \min_i \frac{|\lambda_i^h - \omega^2|}{\lambda_i^h} \quad (53)$$

The converse inequality is trivial by picking a u^h that has components vanishing along all but the eigenvector associated with λ_i^h , which minimizes $\frac{|\lambda_i^h - \omega^2|}{\lambda_i^h}$.

Figure 1.

Treatment for interface elements: black circles are physical nodes, hollow circles are ghost nodes.

Figure 2.
FEM mesh with embedded interface.

Figure 3.

(a) Numerical solution with embedded interface (b) Comparison of numerical and analytical wave forms.

Figure 4. Convergence with h refinement, (a) normalized L_2 norm of the error (b) normalized H^1 seminorm of the error.

Figure 5.
FEM mesh with embedded circular interface.

Figure 6.

(a) k^2 close to the first eigenvalue of $a(\cdot, \cdot)$, the borderline is crossed when mesh size is around 0.01 (b) k^2 close to the 7th and 8th (same) eigenvalue of $a(\cdot, \cdot)$, the borderline is crossed when mesh size is around 0.01.

Figure 7.
Comparison of the embedded interface method and the conventional FEM without interface.

Figure 8.
Model for circular inclusion.

Figure 9. Wave fields with soft inclusion, material contrast ratio $r = 0.7$ (a) reference solution (b) solution using embedded interface method.

Figure 10. Wave fields with stiff inclusion, material contrast ratio $r = 1.5$ (a) reference solution (b) solution using embedded interface method.

Figure 11.

Wave fields with stiff inclusion, material contrast ratio $r = 5$ (a) reference solution (b) solution using embedded interface method.

Figure 12.
Convergence plot of the embedded interface solution towards the reference solution.