The linear stability of Hunt-Rayleigh-Bénard flow

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The stability of a pressure driven flow in a duct heated from below and subjected to a vertical magnetic field (Hunt-Rayleigh-Bénard flow) is studied. We use the Chebyshev collocation approach to solve the eigenvalue problem for the small-amplitude perturbations. It is demonstrated that the magnetic field can stabilize the flow, while the temperature field can disturb the flow. There exists a threshold for the Hartmann number below which the growth rate changes with the Prandtl number non-monotonously (first increases and then decreases) with a critical Prandtl number for the maximum growth rate. By comparing the Re - α neutral curves at different Rayleigh numbers, we find that the critical Reynolds number decreases with the increase in the Rayleigh number, which has an obvious influence on the long-wave instability and a little influence on the short-wave instability. The dominant mode of the long-wave instability changes from the boundary layer instability to the inflectional instability with the increase in the growth rate, which forms a new flow map. We also compare the Ra - α curves and find that the critical Rayleigh number decreases with the increase in the Reynolds number. The obtained results gain an insight into the flow stability affected by the temperature field and the magnetic field. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4984842>]

I. INTRODUCTION

More than half a century has passed since Hartmann's study¹ on liquid metal flow under the influence of a strong magnetic field, but the MagnetoHydroDynamics (MHD) effect which can be used in many industrial production processes² has not been understood well until now. Thus, the MHD effect is still worth carrying out plenty of research, for example, liquid metal flows in an open channel or a rectangle duct. The MHD effect may become very complex if the temperature difference is introduced, as in the case of liquid metal flow with high temperature in the magnetic confinement fusion device. The Rayleigh-Bénard convection induced by the temperature difference will act on the flow field and strongly affect the stability of the flow. The heat transfer process, mass transfer process, and momentum transfer process will change correspondingly. Thus it is necessary to understand the instability mechanism of the liquid metal flow with the existence of the magnetic field and temperature field.

MHD flow instabilities have drawn much attention in the past few decades. Early works by Stuart,³ Lock,⁴ and Hunt,⁵ which focused on the electrically conducting fluid flow between parallel planes under a co-planar³ and a parallel magnetic field,^{4,5} showed that the magnetic field (the corresponding non-dimensional parameter is the Hartmann number) played an important role in the distribution of the velocity field and had a powerful stabilizing effect on the flow stability. Kakutani⁶ examined the stability of the modified Couette flow in the presence of a transverse magnetic field and found that the flow was always stable for $0 < Ha < 3.91$.

As Ha increased from 3.91, the critical Reynolds number Re_c (Re , the ratio of inertial force to viscous force) decreased rapidly to the minimum value of 3.8×10^5 when $Ha = 5.4$ and then increased steadily to a certain value. With the change of Ha , the layer perpendicular to the magnetic field known as the Hartmann layer displaces different distributions of current lines and velocity, influencing the flow stability. Gerard⁷ examined the amplification of small perturbations in the Hartmann layer, proving that some of the very small perturbations, called optimal perturbations, did not support the linear approximation. The main reason was the phenomenon of transient growth. Krasnov⁸ and Moresco *et al.*⁹ found that the flow instability arose in the Hartmann layer at a critical Reynolds number of about 380.

In the case of duct flow under a magnetic field, research on the MHD flow in a rectangular duct conducted by Hunt⁵ obtained the M-shape of the base velocity when the magnetic field was parallel to the insulating wall and perpendicular to the conducting ones. The flow is known as Hunt's flow. Tatsumi and Yoshimura¹⁰ focused on the stability of the laminar flow in a rectangular duct and investigated the four different parities distinguished by its parity of velocity. As to the most unstable mode, the flow instability depended on the value of the aspect ratio and Re_c decreased monotonically with an increasing aspect ratio. Pothérat¹¹ used the quasi-2D model to study a liquid metal flow in a rectangular duct with insulation walls. In this case, Re based on the Shercliff layer thickness became the only relevant parameter of the flow stability. Priede *et al.*¹² examined the three-dimensional stability of Hunt's flow and got two kinds of instability regimes, defined as the Helmholtz-type inflectional instability and the boundary layer instability.¹³ Furthermore,

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the linear stability of the MHD flow in a perfectly conducting duct was also carried out by Priede *et al.*¹⁴ It is demonstrated that in a sufficiently strong magnetic field, the instability was determined by the characteristic thickness $\delta \sim Ha^{-1/2}$. Apart from the linear stability analysis, some numerical results were obtained by Krasnov *et al.*,¹⁵ focusing on the turbulence, and by Sarma *et al.*,¹⁶ demonstrating that the boundary layer became narrower along with the increase in the Hartmann number.

Previous research only considered the influence of the magnetic field, however, in some industry applications, such as the electromagnetic metallurgy process, the temperature field, and the magnetic field coexist simultaneously, and the temperature field can induce the instability of the flow. Kaddeche *et al.*¹⁷ started the study of a parallel flow with a horizontal temperature gradient and different magnetic fields (vertical or horizontal with a direction transverse or longitudinal to the flow), pointing out that the vertical magnetic field had a stronger stabilizing influence on the two types of instabilities (two-dimensional stationary transverse instabilities and three-dimensional oscillatory longitudinal instabilities) than the horizontal magnetic field. In addition, Hu *et al.*^{18,19} carried out the linear stability analysis of Poiseuille-Rayleigh-Bénard flows in binary fluids. When the fluid layer was heated from below ($Ra > 0$) and the throughflow was applied, the critical instability thresholds strongly increased for positive separation factors ($\Psi > 0$). Besides, the boundary curves of the absolute instability (AI) and convective instability (CI) were also given. Furthermore, in the studies of Fakhfakh *et al.*,^{20,21} their results showed that the spanwise magnetic field could stabilize the longitudinal stationary (L) modes but had no effect on the transverse traveling (T) modes, and the (T) modes became the only dominant modes in the whole Re range when Ha exceeded a limiting value. As for the duct flow, Pelekasis²² adopted the linear stability analysis and dynamic simulations to study the free convection in a differentially heated cavity under a horizontal magnetic field. Their stability analysis demonstrated that the Hopf bifurcation was a periodic branch and internal heat generation enhanced flow instability by decreasing the critical value of the Grashof number. Through the quasi-two-dimensional (Q2D) formulation, Gelfgat and Molokov²³ indicated that the magnetic field damped the bulk flow and created thermal and Shercliff boundary layers, which became the main source of instabilities. However, the stability of a duct flow with throughflow including the temperature and magnetic fields is complicated, with the unique M-shape velocity distribution, which is the most important phenomenon of the MHD effects and can cause different instability modes. But very little literature is available so far. Based on typical Hunt's flow, we investigate the flow instability caused by the Rayleigh-Bénard convection. The neutral curves are given through the linear stability theory using the Chebyshev spectral collocation method put forward by Khorrami,²⁴ including the $Re-\alpha$ neutral curves and the $Ra-\alpha$ neutral curves. Results are analyzed through the view of energy of perturbation.

This paper is organized as follows. In Sec. II, we propose the problem formulation. The numerical method is outlined and verified in Sec. III. The energy balance of the perturbation is given in Sec. IV. Numerical results are

discussed in Sec. V. Section VI summarizes and concludes the paper.

II. PROBLEM FORMULATIONS

We consider the flow of an incompressible, viscous, electrically conducting liquid with the kinematic viscosity ν , the density ρ , and the electrical conductivity σ driven by a constant gradient of pressure p along the x direction. A square duct with half-width L is filled with the liquid and is heated from below (a temperature T_1 is applied at the upper wall and $T_2 > T_1$ is applied at the bottom). The walls of the duct perpendicular to the uniform magnetic field $\mathbf{B}$ are perfectly electrically conducting and the parallel ones are electrically insulating (see Fig. 1).

According to the Boussinesq approximation, the fluid density is restricted to the buoyancy term and is expressed as a linear law,

$$\rho = \rho_0[1 - \beta(\bar{T} - \bar{T}_0)], \quad (1)$$

where β is the thermal expansion coefficient, $\bar{T}_0$ is the reference temperature, $\bar{T}_0 = (T_2 + T_1)/2$, and ρ_0 is the density at the mean temperature. In the following, all variables are non-dimensionalised by using the maximum velocity u_{max} and the half-width L as the velocity and length scales, while the time, pressure, induced electric current density, electric potential, and temperature are scaled by L/u_{max} , ρu_{max}^2 , $\sigma u_{max} B$, $u_{max} L B$, and $\Delta\bar{T}$ ($\Delta\bar{T} = T_2 - T_1$), respectively. So the dimensionless governing equations can be written as

$$\nabla \cdot \mathbf{U} = 0, \quad (2)$$

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U} + \frac{Ha^2}{Re} \mathbf{J} \times \mathbf{e}_B + \frac{Ra}{8Re^2 Pr} T \mathbf{e}_z, \quad (3)$$

$$\frac{\partial T}{\partial t} + (\mathbf{U} \cdot \nabla) T = \frac{1}{Re Pr} \nabla^2 T, \quad (4)$$

$$\nabla \cdot \mathbf{J} = 0, \quad (5)$$

where the dimensionless variables $\mathbf{U} = (u, v, w)$, $\mathbf{J} = (J_x, J_y, J_z)$, P , and T (defined as $T = (\bar{T} - T_1)/\Delta\bar{T}$) represent the velocity,

FIG. 1. Geometry and coordinates.

induced electric current density, pressure, and temperature, respectively. The uniform magnetic field is in the z -direction, and the dimensionless electric current density $\mathbf{J}$ is given by Ohm's law for a moving fluid,

$$\mathbf{J} = -\nabla\phi + \mathbf{U} \times \mathbf{e}_B, \quad (6)$$

where ϕ is the dimensionless electric potential. Combining Eqs. (5) and (6), the dimensionless governing equation for the

electric potential ϕ can be written as

$$\nabla^2\phi = \mathbf{e}_B \cdot (\nabla \times \mathbf{U}). \quad (7)$$

There are four non-dimensional parameters in our equations: the Prandtl number $Pr = \nu/\kappa$, the Reynolds number $Re = u_{max}L/\nu$, the Rayleigh number $Ra = 8g\beta\Delta TL^3/\nu\kappa$, and the Hartmann number $Ha = BL\sqrt{\sigma/\rho\nu}$. The boundary conditions are written as

$$\text{no-slip conditions : } u = v = w = 0 \quad \text{at } z = \pm 1, y = \pm 1, \quad (8)$$

$$\text{electric potential conditions : } \phi = 0 \quad \text{at } z = \pm 1, \quad (9)$$

$$D_y\phi = 0 \quad \text{at } y = \pm 1, \quad (10)$$

$$\text{temperature conditions : } T = 1 \quad \text{at } z = -1, \quad (11)$$

$$T = 0 \quad \text{at } z = 1, \quad (12)$$

$$D_yT = 0 \quad \text{at } y = \pm 1, \quad (13)$$

where D_y denote $\frac{\partial}{\partial y}$. The non-dimensional basic steady state for the temperature can be obtained by setting the linear z profile. For z between -1 and 1 , this can be written as

$$T = 0.5(1 - z). \quad (14)$$

By solving Eq. (3), Eq. (7), and the corresponding boundary conditions Eqs. (8)–(10), we obtain the non-dimensional basic velocity field $\mathbf{U}_0 = (U_0, 0, 0)$ using the Chebyshev spectral collocation method²⁴ (the numerical resolution is 45×45),

$$\nabla^2 U_0 - Ha^2(U_0 + \frac{\partial\Phi_0}{\partial y}) = Re \frac{\partial P_0}{\partial x}, \quad (15)$$

$$\nabla^2 \Phi_0 = -\frac{\partial U_0}{\partial y}, \quad (16)$$

where $\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The temperature field has no impact on the basic velocity field, and the result is shown in Fig. 2, which

FIG. 2. The basic flow velocity profile for $Ha = 25$.

is the same as the velocity field of Hunt's analytical solution.⁵ It is obvious that there are two jet flows, which may have great influences on the instability, along the insulating side walls because of the existence of the magnetic field.¹²

III. NUMERICAL APPROACH AND VALIDATIONS

We analyze the linear stability of the Hunt-Rayleigh-Bénard (HRB) flow through adding infinitesimal disturbances to the basic state. The general solution of the problem can be written as

$$(\mathbf{U}, P, T, \phi) = (\mathbf{U}_0, P_0, T_0, \Phi_0) + (\mathbf{U}', p', \theta', \phi'), \quad (17)$$

where $\mathbf{U}_0 = (U_0, 0, 0)$, $(\mathbf{U}', p', \theta', \phi')$ are the velocity, pressure, temperature, and electric potential perturbations, respectively. After substituting Eq. (17) into Eqs. (2)–(13) and neglecting the second-order terms with respect to the small perturbations, we can obtain the linear equations,

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \quad (18)$$

$$\begin{aligned} \frac{\partial u'}{\partial t} + U_0 \frac{\partial u'}{\partial x} + v' \frac{\partial U_0}{\partial y} + w' \frac{\partial U_0}{\partial z} \\ = -\frac{\partial p'}{\partial x} + \frac{1}{Re} \nabla^2 u' - \frac{Ha^2}{Re} \left(\frac{\partial \phi'}{\partial y} + u' \right), \end{aligned} \quad (19)$$

$$\frac{\partial v'}{\partial t} + U_0 \frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} + \frac{1}{Re} \nabla^2 v' - \frac{Ha^2}{Re} (v' - \frac{\partial \phi'}{\partial x}), \quad (20)$$

$$\frac{\partial w'}{\partial t} + U_0 \frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} + \frac{1}{Re} \nabla^2 w' + \frac{Ra}{8Re^2 Pr} \theta', \quad (21)$$

$$\frac{\partial \theta'}{\partial t} + U_0 \frac{\partial \theta'}{\partial x} + w' \frac{\partial T_0}{\partial z} = \frac{1}{Re Pr} \nabla^2 \theta', \quad (22)$$

$$\nabla^2 \phi' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}. \quad (23)$$

The corresponding boundary conditions are

$$\text{no-slip conditions : } u' = v' = w' = 0 \quad \text{at } z = \pm 1, y = \pm 1, \quad (24)$$

$$\text{electric potential conditions : } \phi' = 0 \quad \text{at } z = \pm 1, \quad (25)$$

$$D_y \phi' = 0 \quad \text{at } y = \pm 1, \quad (26)$$

$$\text{temperature conditions : } \theta' = 0 \quad \text{at } z = \pm 1, \quad (27)$$

$$D_y \theta' = 0 \quad \text{at } y = \pm 1. \quad (28)$$

Furthermore, the perturbation quantities can be written as normal modes,

$$(u', v', w', p', \phi', \theta') = [\hat{u}(y, z), \hat{v}(y, z), \hat{w}(y, z), \hat{p}(y, z), \hat{\phi}(y, z), \hat{\theta}(y, z)] \exp[i(\alpha x - \omega t)], \quad (29)$$

where α is a real wavenumber and ω is a complex frequency. Substituting these expressions into the governing system, we obtain the linear stability equations expressed as

$$i\alpha \hat{u} + \partial_y \hat{v} + \partial_z \hat{w} = 0, \quad (30)$$

$$\begin{aligned} -i\omega \hat{u} + i\alpha \hat{u} \cdot U_0 + \partial_y U_0 \cdot \hat{v} + \partial_z U_0 \cdot \hat{w} \\ = -i\alpha \hat{p} + \frac{1}{Re}(D^2 - \alpha^2)\hat{u} - \frac{Ha^2}{Re}(\partial_y \hat{\phi} + \hat{u}), \end{aligned} \quad (31)$$

$$-i\omega \hat{v} + i\alpha \hat{v} \cdot U_0 = -\partial_y \hat{p} + \frac{1}{Re}(D^2 - \alpha^2)\hat{v} - \frac{Ha^2}{Re}(\hat{v} - i\alpha \hat{\phi}), \quad (32)$$

$$-i\omega \hat{w} + i\alpha \hat{w} \cdot U_0 = -\partial_z \hat{p} + \frac{1}{Re}(D^2 - \alpha^2)\hat{w} + \frac{Ra}{8Re^2 Pr} \hat{\theta}, \quad (33)$$

$$-i\omega \hat{\theta} + i\alpha \hat{\theta} \cdot U_0 + \partial_z T_0 \cdot \hat{w} = \frac{1}{PrRe}(D^2 - \alpha^2)\hat{\theta}, \quad (34)$$

$$(D^2 - \alpha^2)\hat{\phi} = i\alpha \hat{v} - \partial_y \hat{u}, \quad (35)$$

where $D^2 = \partial_y^2 + \partial_z^2$. In order to simplify the equations, we use $\hat{v}$ and $\hat{w}$ to represent $\hat{u}$ through Eq. (30), which is

$$\hat{u} = \frac{i}{\alpha}(\partial_y \hat{v} + \partial_z \hat{w}). \quad (36)$$

Then substituting Eq. (36) into Eqs. (31) and (35), we obtain

$$\begin{aligned} \frac{\omega}{\alpha}(\partial_y \hat{v} + \partial_z \hat{w}) - (\partial_y \hat{v} + \partial_z \hat{w}) \cdot U_0 + \partial_y U_0 \cdot \hat{v} + \partial_z U_0 \cdot \hat{w} \\ = -i\alpha \hat{p} + \frac{1}{Re}(D^2 - \alpha^2)\left[\frac{i}{\alpha}(\partial_y \hat{v} + \partial_z \hat{w})\right] \\ - \frac{Ha^2}{Re}[\partial_y \hat{\phi} + \frac{i}{\alpha}(\partial_y \hat{v} + \partial_z \hat{w})], \end{aligned} \quad (37)$$

$$(D^2 - \alpha^2)\hat{\phi} = i\alpha \hat{v} - \frac{i}{\alpha}(\partial_{yy} \hat{v} + \partial_{yz} \hat{w}). \quad (38)$$

Finally, we obtain a generalized eigenvalue problem, $AX = \omega BX$, where

$$X = (\hat{v}(y, z), \hat{w}(y, z), \hat{p}(y, z), \hat{\phi}(y, z), \hat{\theta}(y, z)). \quad (39)$$

In this paper, we use the Chebyshev spectral collocation method to discretize the eigenvalue problem and the well-known shift-and-invert algorithm to transform the generalized eigenvalue problem into a standard eigenvalue problem, where the desired eigenvalues are very well differentiated from the spurious eigenvalues.²⁵ The standard eigenvalue problem is solved with the implicitly restarted Arnoldi method.²⁶ We choose the eigenvalue with the largest magnitude and compare our phase velocity with Priede's¹² results in Table I. Good agreements have been obtained which implies that our simulations are reliable. The 45×45 grid has been used for the computations in the paper.

IV. ENERGY BALANCE OF THE PERTURBATION

Multiplying Eq. (31) by u' , Eq. (32) by v' , Eq. (33) by w' , and summing up the resulting equations, we obtain the kinetic energy balance of the perturbation equation. We then take integral of the above equation over a volume $(0, \lambda) \times (-1, 1) \times (-1, 1)$, and λ is the wavelength along the x direction. Focusing on its real part, we get

$$\begin{aligned} \frac{1}{2} \partial_t \int_0^\lambda \int_{-1}^1 \int_{-1}^1 (u'^2 + v'^2 + w'^2) dx dy dz \\ = \partial_t \int_0^\lambda \int_{-1}^1 \int_{-1}^1 (k_{DIS} + k_{REY} + k_{MAG1} \\ + k_{MAG2} + k_{TEM}) dx dy dz, \end{aligned} \quad (40)$$

where k_{DIS} represents the work of the viscous force,

$$\begin{aligned} k_{DIS} = -\frac{1}{Re} \left[\left(\frac{\partial u'}{\partial x} \right)^2 + \left(\frac{\partial u'}{\partial y} \right)^2 + \left(\frac{\partial u'}{\partial z} \right)^2 + \left(\frac{\partial v'}{\partial x} \right)^2 \right. \\ \left. + \left(\frac{\partial v'}{\partial y} \right)^2 + \left(\frac{\partial v'}{\partial z} \right)^2 + \left(\frac{\partial w'}{\partial x} \right)^2 + \left(\frac{\partial w'}{\partial y} \right)^2 + \left(\frac{\partial w'}{\partial z} \right)^2 \right], \end{aligned} \quad (41)$$

TABLE I. Tests of accuracy for the phase velocity when $Re = 1000$, $Ha = 100$, and $\alpha = 5$.

The present results		The Priede's results	
Resolution	$c = \omega/\alpha$	Resolution	$c = \omega/\alpha$
30 × 30	(0.490 741 6, −0.008 028 572)	30 × 30	(0.490 742 0, −0.008 028 572)
35 × 35	(0.490 741 5, −0.008 028 551)	35 × 35	(0.490 741 6, −0.008 028 551)
40 × 40	(0.490 741 5, −0.008 028 553)	40 × 40	(0.490 741 5, −0.008 028 550)
45 × 45	(0.490 741 5, −0.008 028 552)	45 × 45	(0.490 741 5, −0.008 028 550)

k_{REY} is the work of the Reynolds stress,

$$k_{REY} = -u'v'\partial_y\hat{U} - u'w'\partial_z\hat{U}, \quad (42)$$

k_{MAG1} means the change of the kinetic energy of perturbation caused by the perturbed electric current, k_{MAG2} is the energy damped by the Joule effect related with the basic current,

$$k_{MAG1} = \frac{Ha^2}{Re} \left[\left(\frac{\partial\phi'}{\partial x} \right)^2 + \left(\frac{\partial\phi'}{\partial y} \right)^2 + \left(\frac{\partial\phi'}{\partial z} \right)^2 \right], \quad (43)$$

$$k_{MAG2} = -\frac{Ha^2}{Re} (u'^2 + v'^2), \quad (44)$$

and k_{TEM} represents the production of the kinetic energy caused by buoyancy,

$$k_{TEM} = \frac{Ra}{8Re^2Pr} \theta'w'. \quad (45)$$

All energy terms can be calculated once the eigenvalue ω and the eigenfunctions are determined. Obviously, k_{DIS} and k_{MAG2} are always negative, which means that the viscous dissipation effect and Joule dissipation always damp the energy of the perturbation, thus making the flow more stable, while the other terms play different roles with the change of parameters.

V. RESULTS AND DISCUSSIONS

Through the linear stability analysis of Hunt's flow with the change of the Hartmann number, Priede *et al.*¹² concluded that the magnetic field stabilized the primary flow. In this section, we mainly consider the influence caused by the temperature field on Hunt's flow, by analyzing the neutral curves and the map of energy.^{11,13,14}

A. The Re - α neutral curve

The neutral curves for different Ra 's, plotted in the parameter plane Re - α , are given in Fig. 3 for different Ha 's. The neutral curve can be divided into two parts (for different instability modes) as put forward by Smolentsev.¹³ The distributions of the total perturbation kinetic energy of these two modes are illustrated in Fig. 4. The difference can be observed clearly that the maximum changing rate of the perturbation kinetic energy which determines the instability mode is on the various locations. One is the long-wave mode, of which the instability type is the boundary layer instability and the corresponding wave is named the Tollmien-Schlichting wave. As presented in Fig. 4(a), the maximum changing rate is along the boundary

FIG. 3. The Re - α neutral curves when $Pr = 0.01$: [(a) and (b)] $Ha = 25$; [(c) and (d)] $Ha = 35$. [(a) and (c) are the inflectional instability mode; (b) and (d) are the boundary layer instability mode.]

FIG. 4. The rate of change of total perturbation energy for $Ra = 4000$ and $Pr = 0.01$: (a) $\alpha = 0.68$ and $Re = 5985.56$ (the boundary layer instability mode); (b) $\alpha = 3.72$ and $Re = 6581.08$ (the inflectional instability mode).

layer. Another one is the short-wave mode, and its instability type is the inflectional instability, named the Helmholtz-type instability. The maximum changing rate is on the velocity inflection point [Fig. 4(b)].

1. The neutral curve

First, comparing Fig. 3(b) with Fig. 3(d), we see that with the increase in Ha from 25 to 35, Re_c of the boundary layer instability mode increases from 5985.56 to 7753.26, but Re_c of the inflectional instability mode decreases from 6581.08 to 5455.58 when $Ra = 4000$ is plotted in Figs. 3(a) and 3(c). This indicates that the most unstable mode switches from the boundary layer instability mode to the inflectional instability mode, which is in agreement with the result obtained by Priede *et al.*¹² Meanwhile, it is obvious that there exists a clear decrease in the critical Reynolds number Re_c at small wavenumbers α as Ra increases. The temperature gradient makes the flow more unstable. For example, as illustrated in Fig. 3(b), at a constant $Ha = 25$, when Ra increases from 4000 to 12 000, Re_c decreases from 5985.56 to 5576.43, and the critical wavenumber changes from $\alpha_c = 0.68$ to $\alpha_c = 0.64$. However, when the wavenumber is large, the change is different: Re_c equals 6581.18 (6577.61) at $Ra = 4000$ (12 000). The critical Reynolds number only decreases about 4 when Ra increases about 8000 [see Fig. 3(a)]. Therefore, we get the conclusion that the buoyancy effect caused by the temperature gradient plays a very important role in the flow instability at small wavenumbers, while it exerts a little influence on the flow instability at big wavenumbers.

In order to illustrate the influence of Ra on Re_c at large wavenumbers more clearly, the comparison of neutral curves is plotted in Fig. 5. $Re_c = 1166.9$ was obtained by Priede *et al.*¹² without considering the temperature effect, and our result is $Re_c = 1165.7$ when $Ra = 4000$. It elucidates that the temperature field slightly affects the critical Reynolds number for a big wavenumber α when the Rayleigh number Ra is not big enough.

Focusing on the switching of the dominant mode with the increase in the Rayleigh number, we plot the distribution of the growth rate with different Rayleigh numbers Ra and a certain wavenumber α in Fig. 6. We choose 1000 and 300 points around the most unstable point in Figs. 6(a) and 6(c) and amplify the most unstable point in Figs. 6(b) and 6(d). It

is manifested in Fig. 6(d) that when the wavenumber α equals 0.68, all points have a significant difference between $Ra = 4000$ and $Ra = 12 000$. This indicates that the temperature field does not only influence one certain unstable mode but also acts upon the four most unstable modes put forward by Tatsumil and Yoshimura.¹⁰ The result obtained when the wavenumber equals 3.72 is presented in Figs. 6(a) and 6(b). The temperature field has less impact on the four modes compared with $\alpha = 0.68$.

As plotted in Fig. 7, when the Rayleigh number equals 50 000, Re_c of the inflectional instability mode is 5420.75 at $\alpha = 2.88$, and meanwhile, Re_c of the boundary layer instability mode is 3620.23 at $\alpha = 0.60$. It is obvious that with the increase in Ra , the boundary layer instability mode (the long-wave mode) becomes the dominant mode, which means that the critical Reynolds number for this mode is smaller than the inflectional instability mode (the short-wave mode). Compared with Figs. 3(c) and 3(d), Fig. 7 indicates that the dominant mode changes under the influence of the temperature field.

To explain why the influence caused by the temperature field on the boundary layer instability mode is greater than that on the inflectional instability mode, we plot the distributions of the perturbation velocity w' in Fig. 8. As depicted in Fig. 8(a), the perturbation is a bilateral symmetry at the constant $\alpha = 0.68$, while in Fig. 8(b), the perturbation is in the form

FIG. 5. The comparison between the neutral curves with temperature and with no temperature at $Pr = 0.01$, $Ra = 4000$, and $Ha = 100$.

FIG. 6. The distribution of the growth rate with different Ra 's and a certain wavenumber when $Pr = 0.01$ and $Ha = 25$: [(a) and (b)] $\alpha = 3.72$; [(c) and (d)] $\alpha = 0.68$.

of a central symmetry at the constant $\alpha = 3.72$, and the latter amplitude is greater than the former one. But the main locations of perturbation are all at the inflection point. For the inflectional instability mode ($\alpha = 0.68$), the perturbation velocity w' , which is in the same position with the maximum changing

rate of the perturbation kinetic energy [Fig. 3(b)], accelerates the thermal diffusion, leads to the decrease in the temperature gradient, and weakens the buoyancy effect. Thus Re_c changes a little with the increase in Ra in this mode. Inversely, for the boundary layer instability mode ($\alpha = 0.68$), we find that

FIG. 7. Re - α neutral curves when $Pr = 0.01$ and $Ha = 35$: (a) the inflectional instability mode; (b) the boundary layer instability mode.

FIG. 8. The distributions of perturbation velocity w' when $Ra = 4000$, $Ha = 25$, and $Pr = 0.01$: (a) $\alpha = 0.68$ and $Re = 5985$; (b) $\alpha = 3.72$ and $Re = 6581$.

FIG. 9. The rate of change of total perturbation energy [(a), (c), and (e)] and the distributions of the perturbation energy caused by the Reynolds stress [(b), (d), and (f)] when $Ha = 25$, $Re = 5985$, $Pr = 0.01$, and $\alpha = 0.68$: [(a) and (b)] $Ra = 4000$; [(c) and (d)] $Ra = 25\,000$; [(e) and (f)] $Ra = 50\,000$.

the location of the perturbation velocity w' is different from the maximum changing rate of the perturbation kinetic energy through comparing Fig. 3(a) with Fig. 8(a). It means that the temperature gradient near the side wall is weakened slightly by the perturbation velocity w' . The effect of the temperature field on the boundary layer instability is obvious: Re_c decreases more with the increase in Ra , compared with the inflectional instability mode.

2. Flow map for instability mode

Moreover, we have known that when $Ha = 25$, $Ra = 4000$, and $\alpha = 0.68$, the critical Reynolds number is 5985.56 and the instability mode is the boundary layer instability. It is worth mentioning that if we increase Ra and keep Re at a constant value (5985), the dominant instability mode changes progressively, as shown in Fig. 9. By comparing Fig. 9(a) with Fig. 9(c), it is clear that the perturbation energy near the inflection point has a little increase from 0.05 to 0.06, but when the Rayleigh number reaches 50 000, as shown in Fig. 9(e), the inflection instability mode becomes the dominant mode instead of the boundary layer instability. The maximum perturbation energy distribution which has two parts near the center

line of $z = 0$, moving to the layer along with inflectional points, is different from the inflectional mode we mentioned before, which is in the middle of the inflectional region. The distributions of energy caused by the Reynolds stress are shown in Figs. 9(b), 9(d), and 9(f). It can be seen from these three figures that the magnitude of energy caused by the Reynolds stress increases with the increase of Ra , which means the perturbation gets more extra energy from the main flow. Besides, it is worth noting that at the critical point of instability, the dominant mode is the boundary layer instability, but with the increase in Ra , the dominant pattern will change from the boundary layer instability mode to the inflectional instability mode at a constant Re and a small wavenumber. In fact, the main factor that we care about is the growth rate. The further away from the neutral curve, the larger the growth rate is. When the growth rate is small, the dominant mode is the boundary layer instability mode, and the dominant mode changes into the inflectional mode with the increase in the growth rate. For example, the total perturbation energy of the inflectional instability will exceed the energy of the boundary layer instability at $Re = 6162$, as shown in Fig. 10(a), when $Ra = 12\,000$, $\alpha = 0.8$, and $Ha = 25$. The new flow map for the instability mode is plotted in Fig. 10(b). The left one is the neutral curve and the

FIG. 10. (a) The rate of change of total perturbation energy at $\alpha = 0.8$; (b) the flow map for the instability mode when $Ra = 12\,000$ and $Ha = 25$.

FIG. 11. The Re - α neutral curves when $Ha = 25$ and $Ra = 12\,000$ for different Prandtl numbers.

FIG. 12. Growth rate curves as a function of the wavenumber for different Prandtl numbers when $Ra = 8000$ and $Re = 7500$: (a) $Ha = 14$; (b) $Ha = 20$; (c) $Ha = 35$; (d) $Ha = 50$.

right one is the boundary curve of the instability mode. Thus, in some region of the Re - α plane, the inflectional instability mode is the only pattern which influences the flow.

Concerning the influence of the Prandtl number Pr on the flow stability, the results are presented in Fig. 11. Ha and Ra are set at constant values 25 and 8000, respectively, and Pr increases from 0.001 to 0.01. The change of Pr impinges a little on the short-wave instability (corresponding to the large wavenumber), but great impacts on the long-wave instability (corresponding to the small wavenumber, Re_c decreases from 5732 to 5519).

B. Influence of the Prandtl number

In order to investigate the influence of Pr on the flow stability in detail, we plot the growth rate curves as a function of the wavenumber for different Pr and Ha at constant $Ra = 8000$ and $Re = 7500$ in Fig. 12. The growth rate changes non-monotonically with the increase in Pr . For $Ha = 14$, in Fig. 12(a), the growth rate decreases monotonously with the increase in Pr . Besides, we find that the growth rate of $Pr = 0.005$ is smaller than that of $Pr = 0.01$, which is still bigger than the growth rate of $Pr = 0.5$ for $Ha = 20$, as plotted in Fig. 12(b). It means that the growth rate will increase first and then decrease with the increase in Pr . This conclusion

is substantiated in Fig. 13. In Fig. 12(c), we give the curves for $Ha = 35$, and a new feature appears. There exists a clear demarcation point and the growth rate curve can be divided into two parts with the increase in the wavenumber. Clearly, the growth rate decreases monotonously at the beginning and when the wavenumber reaches a certain value, the variation of the growth rate has the same characteristic mentioned when $Ha = 20$. Furthermore, from Fig. 12(d), it can be seen that the growth rate declines monotonously with the increase in the

FIG. 13. Growth rate curves as a function of the Prandtl number for different Ha when $Re = 7500$, $Ra = 8000$, and $\alpha = 0.7$.

FIG. 14. The most unstable critical Pr as a function of the Hartmann number Ha when $Ra = 8000$: (a) for different α , $Re = 7500$; (b) for different Re , $\alpha = 0.7$.

Prandtl number Pr for $Ha = 50$. Moreover, from the comparison of the growth rate between Fig. 12(c), which is positive, and Fig. 12(d), which is negative, we conclude that the magnetic field has a strong stabilizing effect on the flow.

We plot the corresponding parameters of the most unstable point in Fig. 14. There exists a certain threshold of Ha in which the growth rate changes non-monotonously with the increase in Pr , which has a critical point. The threshold changes along with Re and α . As shown in Fig. 14(a), the threshold becomes larger and the boundary value of Ha increases with the increase in α from 13–30 to 15–39, but the critical Pr decreases from 0.024 to 0.02. The same phenomenon can also be seen in Fig. 14(b) for different Re 's. The boundary value of Ha changes from 13–37 to 15–49 when Re increases from 5500 to 12 000. Thus, under a certain Ha , we can choose an appropriate Pr to make the flow more stable with different Re and α .

C. The Ra – α neutral curve

The Ra – α neutral curve is as important as the Re – α neutral curve, and we then focus on the Ra – α neutral curves with the change of Re when $Ha = 20$ and $Pr = 0.01$ in Fig. 15. As discussed in the previous part, the Rayleigh number has some minor influence on the flow instability in the big wavenumber

FIG. 15. The Ra – α neutral curves change with different Reynolds numbers, Re , when $Ha = 25$ and $Pr = 0.01$.

region which means that we will not find the critical point unless the Rayleigh number reaches 10^8 , so we only plot the curves using small wavenumbers. When the Reynolds number increases from 3000 to 4500, the critical Rayleigh number decreases from 22 858.3 to 8776.5, and the variation tendency manifests good agreement with Hu's results.¹⁸ It indicates that the flow becomes more unstable because the influence of the inertial force is stronger than the influence of the viscosity force with the increase in Re . A small perturbation will take energy from the main flow, leading to the flow instability.

VI. CONCLUSIONS

In this paper, the flow of the liquid metal in a square duct with the influence of the temperature and magnetic fields has been studied by the linear stability analysis. The walls of the duct perpendicular to the magnetic field are perfectly conducting, and the other walls are insulating. The eigenvalue problem is solved by the Chebyshev collocation method and the neutral curves have been calculated. Besides, we calculate the distributions of the kinetic energy of the perturbation.

First, the linear stability analysis indicates that the existence of the temperature field enhances the instability of the HRB flow, and we obtain the Re – α neutral curves for different Ra 's. The results show that the effect of the temperature field on the flow instability is different for different instability modes. The temperature field has a stronger destabilizing effect on the boundary layer instability mode than on the inflectional instability mode, working on the four most unstable points. As a consequence, with the increase in Ra , the boundary layer instability becomes the dominant mode replacing the inflectional instability mode. Second, for the long-wave perturbation, the dominant mode develops from the boundary instability mode to the inflectional instability mode with the increase in the growth rate, which means that on a certain Re – α plane, the inflectional instability mode is the only pattern that influences the flow. In addition, through calculating the Ra – α neutral curves, we can conclude that the critical Ra number decreases with the increase in Re for a long-wave disturbance, while the critical Ra is close to 10^8 for a short-wave disturbance and the dominant mode is the boundary layer instability

mode. However, the influence of Pr on the flow instability is complex with the change of the Hartmann number Ha . For a certain range of Ha , the instability growth rate changes non-monotonously (first increases and then decreases) with the increase in Pr , and a maximum instability growth rate exists for a fixed Re and α . Nonetheless, the instability growth rate increases monotonously with the decrease in Pr when Ha exceeds the certain range of Ha mentioned before.

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