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Acoustic Radiation Force of a Quasi-Gaussian Beam on an Elastic Sphere in a Fluid

A.V. Nikolaeva¹, O.A. Sapozhnikov^{1,2}, and M.R. Bailey²

¹Department of Acoustics, Physics Faculty, Moscow State University, MSU, Moscow, Russia

²Center for Industrial and Medical Ultrasound, Applied Physics Laboratory, University of Washington, CIMU APL, Seattle, USA

Abstract

Acoustic radiation force has many applications. One of the related technologies is the ability to noninvasively expel stones from the kidney. To optimize the procedure it is important to develop theoretical approaches that can provide rapid calculations of the radiation force depending in stone size and elastic properties, together with ultrasound beam diameter, intensity, and frequency. We hypothesize that the radiation force nonmonotonically depends on the ratio between the acoustic beam width and stone diameter because of coupling between the acoustic wave in the fluid and shear waves in the stone. Testing this hypothesis by considering a spherical stone and a quasi-Gaussian beam was performed in the current work. The calculation of the radiation force was conducted for elastic spheres of two types. Dependence of the magnitude of the radiation force on the beam diameter at various fixed values of stone diameters was modeled. In addition to using real material properties, speed of shear wave in the stone was varied to reveal the importance of shear waves in the stone. It was found that the radiation force reaches its maximum at the beamwidth comparable to the stone diameter; the gain in the force magnitude can reach 40% in comparison with the case of a narrow beam.

Keywords

radiation force; acoustic beam; quasi-Gaussian beam; wave scattering; shear waves

I. Introduction

One of the currently developing areas of acoustics is use of ultrasonic waves in medical and industrial technologies. Corresponding applications of ultrasound involve not only the traditional methods of ultrasonic influence on the medium such as heating or cavitation, but other phenomena, e.g. those that are based on nonlinear effects. One of them is the appearance of radiation pressure (acoustic radiation force). A new method of urolithiasis (kidney stone) management, consisting of noninvasively expelling stones from the kidney with an ultrasound beam, was proposed a few years ago [1, 2]. The ability to move and levitate small scatterers by the effect of ultrasound radiation pressure was discovered long ago [3]. A series of papers were devoted to the radiation force in the case of scatterers of size comparable to the wavelength. Various methods of the radiation force measurements have been proposed [4, 5].

In order to optimize the radiation force on a scatterer it is desirable to have an effective numerical algorithm. The modeling can provide an accurate and rapid calculation of the force and thus allows to find the radiation force dependence on the diameter and elastic properties of the scatterer, as well as the width, intensity and frequency of the acoustic beam. In this current paper an algorithm is proposed that allows radiation force calculation in the case of an elastic spherical scatterer and a quasi-Gaussian beam. The dependence of the radiation force on the ratio between the acoustic beamwidth and stone diameter is analyzed.

II. Theoretical approach

It is known that when an incident wave acts on a scatterer, radiation force appears as a result of momentum transfer from the beam to the scatterer. Accordingly, the calculation of the radiation force is based on the analysis of the acoustic beam scattering. In the case of a quasi-Gaussian beam the corresponding problem can be solved using the results obtained in Ref. [6]. Let us briefly present the calculation algorithm.

A. Quasi-Gaussian Beam

As the solution of the Helmholtz equation for the incident quasi-Gaussian beam the following expression can be used [6]:

$$p_i = p_0 \frac{z_d}{2 \sinh^2(k z_d)} \left[e^{k z_d} \frac{\sin \left(k \sqrt{r_{\perp}^2 + (z - i z_d)^2} \right)}{\sqrt{r_{\perp}^2 + (z - i z_d)^2}} - e^{-k z_d} \frac{\sin \left(k \sqrt{r_{\perp}^2 + (z + i z_d)^2} \right)}{\sqrt{r_{\perp}^2 + (z + i z_d)^2}} \right], \quad (1)$$

where p_i is the complex sound pressure amplitude in the incident wave, p_0 is the initial wave amplitude at the beam axis, $k = \omega/c$ is the wave number, c is the speed of sound in fluid, $z_d = ka_0^2$ is the diffraction divergence length of the beam, a_0 is waist radius of the beam, and $r_{\perp} = (x^2 + y^2)^{1/2}$ is the transverse coordinate (distance from the beam axis). The solution (1) is a superposition of two sources and sinks, which eliminates or suppresses the wave propagating in the opposite ($-z$) direction and eliminates singularities and branch points. Equation (1) describes beams with an arbitrary focus convergence including the case when focal waist size is on the order of the diffraction limit ($ka_0 \sim 1$). Note that, when $kz_d \gg 1$ expression (1) represents the Gaussian beam solution known in the parabolic approximation theory of wave propagation. Quasi-Gaussian beam structure represented in the form (1) depends on the numerical value of ka_0 . For a fixed value of the beam radius a_0 and $ka_0 = 1$ the wave structure barely resembles a directional beam. However, already at $ka_0 = 2$ the directionality is clearly pronounced. As ka increases, the divergence of the beam decreases.

B. Radiation Force by a Quasi-Gaussian Beam

Since the radiation force is the result of a partial transfer of the wave momentum to a scattering object, first it is necessary to solve the scattering problem. We start with the more general case of an arbitrary axially symmetric incident beam. If the problem is axisymmetric, it is convenient to represent the complex amplitude of the incident wave as a series of spherical harmonics:

$$p_i = \sum_{n=0}^{\infty} Q_n j_n(kr) P_n(\cos\theta) \quad (2)$$

where r, θ are spherical coordinates, P_n are the Legendre polynomials, j_n are spherical Bessel functions. For the incident wave described by (2), the scattered wave can be expressed as a superposition of outgoing waves represented by the following series:

$$p_s = \sum_{n=0}^{\infty} c_n Q_n h_n^{(1)}(kr) P_n(\cos\theta) \quad (3)$$

where $h_n^{(1)}$ are spherical Hankel functions and coefficients c_n characterize the scattering of the corresponding spherical harmonic. If the scatterer is an isotropic elastic sphere centered at the origin (this is the case discussed below), the expression for the scattering coefficients c_n are known analytically and described in papers [7 – 10].

In general, the radiation force vector can be written in Cartesian components and can move millimeter-sized objects in an arbitrary direction. In the axisymmetric case of a stone on axis, the radiation force has only the axial component that pushes the object along the axis of propagation. It can be expressed in terms of coefficients c_n and Q_n as follows [7, 8]:

$$F_z = \frac{2\pi}{\rho c^2 k^2} \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+1)(2n+3)} \{Q_n Q_{n+1}^* (c_n + c_{n+1}^* + 2c_n c_{n+1}^*)\} \quad (4)$$

To apply the general expression (4), to a particular type of incident beam, it is convenient to write the expansion coefficients in the following form:

$$Q_n = p_0 i^n (2n+1) g_n(kz_d) \quad (5)$$

Here the factors $g_n(kz_d)$ take into account the difference from the plane wave case (the plane wave solution corresponds to $g_n=1$). For a quasi-Gaussian type of beam (1) the coefficients $g_n(kz_d)$ can be expressed analytically [6]:

$$g_n = \frac{1 - (-1)^n e^{-2x}}{(1 - e^{-2x})^2} \cdot e^{-x} \sqrt{2\pi x} I_{n+1/2}(x) \quad (6)$$

where $I_{n+1/2}(x)$ are the Infeld functions.

III. Numerical Results

Equations (4) – (6) were used for the numerical calculation of the radiation force of the quasi-Gaussian beam on an elastic spherical scatterer in a fluid. The calculations were performed in Fortran, where the index n of (4) varied from $n = 0$ to $n = (5 \div 7) ka$.

As typical examples of scatterers, we choose two types of the elastic spherical targets. The first one was a calcium oxalate monohydrate stone (COM) with the following characteristics: the density $\rho^*=2038 \text{ kg/m}^3$, longitudinal and shear waves velocities in the stone, respectively, $c_l=4535 \text{ m/s}$ and $c_t=2132 \text{ m/s}$ [11]. The second target was a Ultracal-30 gypsum stone (U-30, $\rho^*=1700 \text{ kg/m}^3$, $c_l=2630 \text{ m/s}$, $c_t=1330 \text{ m/s}$), which is often used to model kidney stones [12]. Surrounding fluid was water with the corresponding characteristics $\rho=1000 \text{ kg/m}^3$ and $c=1470 \text{ m/s}$.

A. Normalized radiation force Y_g versus ka for a quasi-Gaussian beam and a plane wave

For the two types of scatterers described above the dependence of the normalized radiation force $Y_g=(F_z c)/W_p$ was obtained ($W_p=(\pi a^2 p_0^2)/2\rho c$) versus parameter ka . Note that ratio of beam waist radius $a_0=2 \text{ mm}$ to the scatterer radius $a=1 \text{ mm}$ is fixed (Fig. 1). Here the variable parameter is the wave number k that is proportional to the acoustic wave frequency f . The radiation force nonmonotonically depends on the ka and this dependence is related to the material elastic properties. Local minima and maxima occur because of resonant oscillations of the scatterer at certain frequencies. These types of curves calculated for different materials allow determination of the frequencies of most effective radiation force for a given output power or the least effective ones which would be chosen to minimize force. Note that in both cases of the plane wave and quasi-Gaussian beam the curves look qualitatively similar, but have some quantitative difference.

B. Averaged radiation force Y_g in the case of a plane wave for different sphere materials

To characterize the efficiency of the radiation force generation for some materials including COM and U-30 stones, the factor Y_g was averaged over different ka -intervals (Fig. 2). The incident acoustic beam is a plane wave. One can see that a reflection of the acoustic beam depends on ka and material properties. Ultracal-30 gypsum model kidney stones reflect more strongly in the interval $ka = (1-10)$ than calcium oxalate monohydrate (COM) kidney stones which behave similarly to rigid stones.

C. Radiation force dependence on the ratio of the radii of the sphere and of the beam waist

To better understanding how radiation force of a quasi-Gaussian beam depends on the ratio of the radii of the sphere and of the beam waist the following investigation was made. For the fixed values of sphere radius ($a=1 \text{ mm}$) and ka parameter ($ka=3$ or $ka=5$) the dependence of radiation force normalized on a beam's total acoustic power $W(F_g=(F_z c)/W, W=[\pi a_0^2 p_0^2 \cosh(kz_d)]/[4\rho c \sinh(kz_d)])$ was numerically calculated. The varied value in Fig. 3 is beam waist radius a_0 .

The curves look counterintuitive: at first glance, the maximum value of the force should be reached when the beamwidth is much smaller than the scatterer diameter, because then the

beam completely enters the scatterer. However, for a $ka=3$ the behavior is different: the force decreases with the increase of the beam diameter only in the region $a_0 \gg a$, when the beam is significantly wider than the scatterer, which can be explained by the fact that the majority of the beam energy misses the target. For $ka=3$, the modeling predicts for both the COM and U-30 stones that the radiation force has a maximum at $ka_0 \approx 4$, when the beam “wraps around” the sphere.

Thus, despite the fact that in the mentioned case some of the ultrasound beam energy apparently goes around the sphere, the force is larger. In various configurations the difference between the maximum force and the force at $ka_0 \rightarrow 0$ can reach 40%.

Note, that for $ka=5$ both for COM and U-30 the peak at $ka_0 \approx ka$ is not observed. Likely this is due to the fact that $ka=5$ corresponds to a minimum in force in the curves shown in Fig. 1, so stone resonance is not enhancing the radiation force.

D. Role of shear waves

The study [13] showed when the speed of sound in the surrounding fluid approximately equals the speed of shear waves in the scatterer, regions of active energy capture appear. The scatterer actively grabs the transmitted beam energy thus promoting the wave penetration to the stone and more efficient momentum transfer. This conclusion applies to the situation under consideration. The peak radiation force occurs at a beam width $ka_0=4$ on a scatterer size $ka=3$, because the wave couples into a resonance related to the shear waves.

To make sure that this effectiveness directly depends on the ratio of the speed of shear waves in stone and speed of sound in surrounding fluid the following curves were calculated (Fig. 4).

Here we calculate the averaged on a $ka=(1-10)$ interval value of radiation force $\langle F_g \rangle$ by fixing the speed of sound in water $c=1470$ m/s and varying the shear waves value within the range 100–4000 m/s to reveal the importance of shear waves in the stone.

One can see that our hypothesis is confirmed both for the COM and U-30 spheres. When the speed of sound in the fluid is close to shear wave speed in the elastic stone ($c_t \approx c=1470$ m/s), the radiation force achieves its maximum value.

IV. Discussion and Conclusions

A new model for calculating radiation force by arbitrary beams on elastic spheres was introduced. This model was used to calculate the force of a quasi-Gaussian beam on a sphere on axis. The sphere material and surrounding fluid represented those of the case of using ultrasound to reposition a kidney stones with radiation force as has been used in a clinical study [14]. The ka values are also reasonable for that application. In simulating the exact kidney stone case, we had previously found that there appears to be coupling of shear waves into the stone that increased the radiation force. In this study we showed that a maximum force can be obtained with a beam slightly larger than the stone. The energy that passes by the stone is not lost but couples into the stone. This force is further maximized when the shear wave speed of the stone matches the sound speed of the surrounding fluid. This same

effect was seen for comminuting stones where the wave on the surface coupled and led to the greatest stress in the stone [15]. Additionally, the force could be further maximized by selecting a frequency that generated a resonance of the elastic waves in the stone.

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Fig. 1. Normalized radiation force Y_g versus ka for U-30 (blue line) and COM (red line) stones. a) The case of plane wave, b) the case of quasi-Gaussian beam, $a_0=2$ mm. The radius of elastic scatterer $a=1$ mm. Local maxima and minima occur because of resonant oscillations at some frequencies.

Fig. 2.

Averaged radiation force $\langle Y_g \rangle$ in the case of plane wave for different materials: U-30, COM, aluminum, steel, silica, pressure release (a bubble) and absolutely rigid stones. Red bars represent averaged force from $ka=0.01$ to $ka=10$ by the step 0.01. Blue cones show averages from $ka=10$ to $ka=20$.

Fig. 3.

Function of force on a 1 mm sphere normalized to the total acoustic power radiation force *versus* beam waist a_0 for two materials: **a)** The case of fixed $ka=3$. The maximum force occurs when $ka_0 \approx ka$ for both for COM and U-30 stones. **b)** The case of fixed $ka=5$, where there is no peak around $ka_0 \approx ka$.

Fig. 4.

Averaged from $ka=0.01$ to $ka=10$ radiation force $\langle F_g \rangle$ versus speed of shear wave in material, $a=1$ mm, $a_0=2$ mm. The maximum value of radiation force both in U-30 and COM stones is reached when the speed of sound in fluid is close to the speed of shear wave in the elastic stone $c_t \approx c$.