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Direct Error in Constitutive Equation Formulation for Plane stress Inverse Elasticity Problem

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Abstract

We present a new computational formulation for inverse problems in elasticity with full field data. The formulation is a variant of an error in the constitutive equation formulation, but allows direct solution for the modulus field and accommodates discontinuous strain fields. The development of the formulation is motivated by the relatively poor performance of current direct formulations, reported so far in literature, in dealing with discontinuities in the strain and material property distribution. The formulation relies on minimizing the error in the constitutive equation, and a momentum equation constraint. Numerical results on model problems show that the formulation is capable handling discontinuous, and noisy strain fields, and also converging with mesh refinement for continuous and discontinuous material property distributions. The application to reconstruct the elastic modulus distribution in solid breast tumors is shown.

Keywords

elastography; biomechanical imaging; inverse problems; discontinuous; error in constitutive equations

1. Introduction

Pathology is often associated with alterations of the mechanical properties of tissues. Examples include fibrosis of the liver [1, 2, 3, 4], atherosclerosis [5] and solid cancers that often present as stiff masses [6, 7]. Recent studies suggest that 5 diagnosis of palpable breast masses may be improved by quantifying the elastic modulus distribution of the suspicious mass and surrounding tissues [8, 9, 10].

Using medical imaging techniques from a branch of medical imaging known as “elastography,” the deformation of breast tissue may be measured *in vivo* [11, 12, 13, 14].

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The availability of such data motivates the study and solution 10 of the following inverse problem of incompressible linear elasticity: Given a displacement field measured everywhere in an incompressible linear elastic solid, determine the heterogeneous shear modulus distribution within the solid. In addition to medical imaging applications, the resulting inverse problems are of interest in the materials characterization field broadly [15, 16, 17]. Full-field data is becoming more commonly available from digital image correlation techniques [18, 19, 20, 21, 22].

The dominant approach in the literature to solve the inverse problem is with an iterative optimization approach [23, 24, 13]. A popular objective function to minimize in such problems is based on the error in the constitutive equation, and so are called “ECE approaches” [25, 26, 27, 28, 29]. Such iterative methods tend to be relatively robust in dealing with noise, accommodate multiple measurements naturally, and can even handle missing measurements. As the objective functions in these approaches tend to be non-quadratic, finding the minimum can require many iterations, each of which requires the solution of a large forward problem. Computing the final solution can therefore be computationally expensive.

Non-iterative, i.e. direct approaches, offer a computationally efficient alternative to iterative approaches [11, 13]. In this approach, the measurements are treated as coefficients in a partial differential equation (PDE) for the desired material parameters. Several computational formulations have been proposed to solve the resulting system of PDEs [30, 31, 32, 33, 34, 35], which are capable of producing excellent results in applications where the material property distributions are smooth [36, 37]. These approaches tend to perform poorly, however, when the data contains large strain gradients, or when the sought unknown material property distribution is discontinuous [11, 36]. Since many applications involve identifying distinct inclusions of one material (e.g. a breast tumor) in a background of another (e.g. normal fibroglandular tissue), the ability to be able to solve for the unknown discontinuous material property distributions is essential.

In this work, we propose a direct approach to solve the elastic inverse problem that is based on minimizing the error in constitutive equation. This formulation is designed to address the limitations of existing direct formulations. In particular, this formulation can accommodate discontinuous strain fields, and performs well in reconstructing discontinuous material property distributions. In Section (2), we introduce the formulation. Next, we verify that the formulation is capable of correctly solving incompressible plane stress problems in section (3), and also demonstrate the convergence of the formulation. After that, we demonstrate the performance of the formulation on a model problem with noise in section (4), and then proceed to validate the formulation with ultrasound measured displacement data from a tissue mimicking phantom in section (5). We then apply the new formulation to reconstruct the elastic modulus distributions of two different breast masses, one a benign tumor and one a malignant tumor. Finally, we give a brief summary and conclusions in section (7), along with some possible extensions, and future directions.

2. Formulation

This work is motivated primarily by applications in elastography in which the material of interest is soft tissue which is well modeled as incompressible. We consider small deformation data acquired in two dimensions, and hence focus on incompressible plane stress.

We consider a 2D object, whose domain is denoted by $\Omega \subset \mathbb{R}^2$, and whose motion is characterized by the displacement field $\mathbf{u}(x, y) \equiv \mathbf{u}(\mathbf{x})$. The object is assumed to be composed of a material that is incompressible, isotropic, linearly elastic, and undergoing a plane stress deformation. In plane stress, $\sigma_{zz} = 0$ which allows us to solve for the pressure field in terms of the trace of the in-plane components of strain:

$$p = -2\mu \text{tr}(\epsilon) \quad (1)$$

Here, μ is the shear modulus of the material, and ϵ is the linearized strain tensor,

$$\epsilon = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \quad (2)$$

The in-plane constitutive equation reduces to

$$\sigma = \mu \mathbf{A} \quad (3)$$

where σ is the stress tensor, and

$$\mathbf{A}(\epsilon) = 2\text{tr}(\epsilon) \mathbf{I} + 2\epsilon. \quad (4)$$

The equilibrium equation governing quasistatic deformations in the absence of body forces is

$$\nabla \cdot \sigma = 0. \quad (5)$$

In the inverse problem of interest here, we suppose we are given the displacement field, $\mathbf{u}(\mathbf{x})$ and thus by equations (2) and (4), $\mathbf{A}(\mathbf{x})$, and we seek the shear modulus distribution $\mu(\mathbf{x})$. The exact solution for μ depends on a multiplicative constant [38], so one constraint must be imposed to complete the specification of the problem. Because of its compatibility with a variational formulation, we choose that constraint in the form of a fixed mean. Hence our problem of interest follows by substituting (3) into (5), which gives:

$$\nabla \cdot (\mu \mathbf{A}) = 0 \quad \mathbf{x} \in \Omega \quad (6.1)$$

$$\bar{\mu} = \langle \mu \rangle \equiv \frac{1}{\text{meas}(\Omega)} \int_{\Omega} \mu \, d\Omega. \quad (6.2)$$

The goal is to solve (6) for $\mu(\mathbf{x})$, given $\mathbf{A}(\mathbf{x})$.

2.1. D-ECE formulation

It is shown in [39] that a direct attack on equation (6) fails when modulus and strain fields are permitted to be discontinuous. In what follows, we will introduce a mixed weak formulation of the problem above that depends on modulus μ , stress field $\boldsymbol{\sigma}$, and a Lagrange multiplier field $\boldsymbol{\lambda}$. That formulation has the disadvantage, however, of containing multiple unknown fields (six degrees of freedom per node in $2D$). We will then show that the stress field can be eliminated exactly, and thus arrive at our final weak form which is then stabilized.

In order to develop a formulation that accommodates discontinuous strains and modulus fields, we first introduce a new problem: *Find* $(\boldsymbol{\sigma}, \mu)$ *such that*:

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \quad \mathbf{x} \in \Omega \quad (7.1)$$

$$\boldsymbol{\sigma} = \mu \mathbf{A} \quad \mathbf{x} \in \Omega \quad (7.2)$$

$$\langle \mu \rangle = \bar{\mu} \quad (7.3)$$

The difference between problems (6) and (7) is the equality (7.2), which we will enforce weakly below.

This new problem can be formulated as a constrained optimization problem. To that end, we introduce $\mathcal{S} = H(\text{div}, \Omega) \times L^2(\Omega)$, with $H(\text{div}, \Omega)$ defined as [40, 41]:

$$H(\text{div}, \Omega) := \left\{ \boldsymbol{\sigma} \in [L^2(\Omega)]^{2 \times 2} \mid \nabla \cdot \boldsymbol{\sigma} \in [L^2(\Omega)]^2 \right\}. \quad (8)$$

Now the goal is to find $(\boldsymbol{\sigma}, \mu) \in \mathcal{S}$, and $\boldsymbol{\lambda} \in [H^1(\Omega)]^2$ that minimizes the error in the constitutive equation (7.2), subject to the constraint that the equilibrium equation (7.1) is satisfied. The Lagrangian for this minimization problem is defined as:

$$\mathcal{L}(\mu, \boldsymbol{\sigma}, \boldsymbol{\lambda}) = \frac{1}{2} \|\boldsymbol{\sigma} - \mu \mathbf{A}\|^2 - (\boldsymbol{\lambda}, \nabla \cdot \boldsymbol{\sigma})_{\Omega}. \quad (9)$$

Here, we denote the inner product on two vector quantities, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and two second order tensor quantities $\mathbf{C}$ and $\mathbf{D}$ as:

$$(\alpha, \beta) = \int_{\Omega} \alpha \cdot \beta \, d\Omega, \quad (\mathbf{C}, \mathbf{D}) = \int_{\Omega} \mathbf{C} : \mathbf{D} \, d\Omega \quad (10)$$

respectively. $\boldsymbol{\lambda}$ is a Lagrange multiplier used to enforce the equilibrium equation constraint. Because this formulation depends on minimizing $\|\boldsymbol{\sigma} - \mu \mathbf{A}\|^2$, it falls under the umbrella of ECE methods [27, 42, 28]. It is quadratic in the unknowns, however, which allows us to solve for the modulus field directly. Hence, we refer to it as a “direct error in the constitutive equation” formulation, or D-ECE for short.

The stationary conditions for (9) are:

$$D_{\mu} \mathcal{L} \omega = (-\omega \mathbf{A}, \boldsymbol{\sigma} - \mu \mathbf{A})_{\Omega} = 0 \quad \forall \omega \in L_2(\Omega) \quad (11)$$

$$D_{\sigma} \mathcal{L} : \phi = (\phi, \boldsymbol{\sigma} - \mu \mathbf{A})_{\Omega} - (\boldsymbol{\lambda}, \nabla \cdot \phi)_{\Omega} = 0 \quad \forall \phi \in H(\text{div}, \Omega) \quad (12)$$

$$D_{\boldsymbol{\lambda}} \mathcal{L} \cdot \gamma = -(\gamma, \nabla \cdot \boldsymbol{\sigma})_{\Omega} = 0 \quad \forall \gamma \in [H^1(\Omega)]^2. \quad (13)$$

From these stationary conditions, we get the following Euler-Lagrange equations:

$$\nabla \boldsymbol{\lambda} = \mu \mathbf{A} - \boldsymbol{\sigma} \quad \mathbf{x} \in \Omega \quad (14.1)$$

$$\boldsymbol{\lambda} = 0 \quad \mathbf{x} \in \Gamma \quad (14.2)$$

$$\mathbf{A} : \boldsymbol{\sigma} = \mathbf{A} : \mathbf{A} \mu \quad \mathbf{x} \in \Omega \quad (14.3)$$

$$\nabla \cdot \boldsymbol{\sigma} = 0 \quad \mathbf{x} \in \Omega. \quad (14.4)$$

We note that equations (14) differ from equations (7). In particular, the fact that the residual of (7.1) is minimized rather than exactly satisfied is seen explicitly in equation (14.1). By allowing flexibility in this equation, the D-ECE formulation can better accommodate noise and discretization error than does a formulation based on (7).

The stress, $\boldsymbol{\sigma}$, can be eliminated from these equations by first computing $\nabla \cdot$ (14.1):

$$\Delta \lambda = \nabla \cdot (\mu \mathbf{A}) - \nabla \cdot \boldsymbol{\sigma} \stackrel{\text{by (14.4)}}{=} \mathbf{0} \quad (15)$$

Next we compute $\mathbf{A} : (14.1)$, which yields:

$$\mathbf{A} : \nabla \lambda = \mathbf{A} : \mathbf{A} \mu - \mathbf{A} : \boldsymbol{\sigma} \stackrel{(14.3)}{=} \mathbf{0}. \quad (16)$$

The Euler-Lagrange equations, with $\boldsymbol{\sigma}$ eliminated, reduce to:

$$\nabla \cdot (\mu \mathbf{A}) = \Delta \lambda \quad \forall x \in \Omega \quad (17a)$$

$$\mathbf{A} : \nabla \lambda = \mathbf{0} \quad \forall x \in \Omega \quad (17b)$$

$$\lambda = 0 \quad \forall x \in \Gamma. \quad (17c)$$

We refer to this problem, with $\boldsymbol{\sigma}$ eliminated, as the reduced problem.

2.2. Variational formulation of the reduced problem

Variational solutions will be sought in $\lambda \in U \subset [H^1(\Omega)]^2$, $\mu \in Q \subset L^2(\Omega)$, $\mathbf{A} \in [L^\infty(\Omega)]^{2 \times 2}$, where:

$$U := [H_0^1(\Omega)]^2, \quad X := U \times Q, \quad Y := U \times P \quad (18)$$

$$Q \equiv \left\{ \mu \in L^2(\Omega) \mid \langle \mu \rangle = \bar{\mu} \right\} \quad (19)$$

$$P \equiv \left\{ \omega \in L^2(\Omega) \mid \langle \omega \rangle = 0 \right\} \quad (20)$$

The above function spaces are equipped with the following norms:

$$\|\gamma\|_U := \|\gamma\|_{[H^1(\Omega)]^2} = \left(\sum_{i=1}^2 \|\gamma_i\|_{H^1(\Omega)}^2 \right)^{1/2}, \quad \|\omega\|_Q := \|\omega\|_{L^2(\Omega)} \quad (21)$$

$$\|(\gamma, \omega)\|_X = \|(\gamma, \omega)\|_Y := \left(\|\gamma\|_U^2 + \|\omega\|_{L^2(\Omega)}^2 \right)^{1/2}. \quad (22)$$

The variational equations are derived by multiplying (17a) by $\gamma \in U$, and integrating by parts, and also multiplying (17b) by $\omega \in P$, and integrating. Doing this results in the following weak form: *Find $(\lambda, \mu) \in X$ such that:*

$$a(\lambda, \gamma) + b(\gamma, \mu) = 0 \quad \forall \gamma \in U \quad (23a)$$

$$-b(\lambda, \omega) = 0 \quad \forall \omega \in P \quad (23b)$$

where:

$$a(\lambda, \gamma) \equiv \int_{\Omega} \nabla \gamma : \nabla \lambda \, d\Omega \quad (24)$$

$$b(\gamma, \omega) \equiv - \int_{\Omega} \nabla \gamma : (\omega \mathbf{A}) \, d\Omega. \quad (25)$$

Equations (23a) and (23b) can also be written as: *Find $(\lambda, \mu) \in X$ such that:*

$$c((\lambda, \mu), (\gamma, \omega)) = 0 \quad \forall (\gamma, \omega) \in Y \quad (26)$$

where:

$$c((\lambda, \mu), (\gamma, \omega)) = a(\lambda, \gamma) + b(\gamma, \mu) - b(\lambda, \omega). \quad (27)$$

The weak form (23) is similar to the weak form for the incompressible Stokes flow problem [43, 44, 41, 45], and reduces identically to the 2D Stokes problem when $\mathbf{A}$ is the identity. In that case, λ is the fluid velocity, and μ is the pressure. Like the Stokes problem, the numerical solution of the saddle point problem (23) often requires stabilization.

2.2.1. Stabilized problem—The weak form (26) was discretized by choosing approximation functions from the finite dimensional subspaces $\lambda_h \in U_h \subset U$, $\mu_h \in Q_h \subset Q$, $\gamma_h \in U_h \subset U$, and $\omega_h \in P_h \subset P$ where [44]:

$$Q_h \equiv \left\{ \mu_h \in C^0(\Omega) \cap L^2(\Omega) \mid \langle \mu_h \rangle = \bar{\mu} \text{ \& } \mu_{h|K} \in R_1(K) \quad \forall K \in \mathcal{T}_h \right\} \quad (28)$$

$$P_h \equiv \left\{ \omega_h \in C^0(\Omega) \cap L^2(\Omega) \mid \langle \omega_h \rangle = 0 \text{ \& } \omega_{h|K} \in R_1(K) \quad \forall K \in \mathcal{T}_h \right\} \quad (29)$$

$$U_h \equiv \left\{ \gamma_h \in [H_0^1(\Omega)]^2 \mid \gamma_{h|K} \in [R_1(K)]^2 \quad \forall K \in \mathcal{T}_h \right\} \quad (30)$$

and

$$X_h := U_h \times Q_h \quad Y_h := U_h \times P_h. \quad (31)$$

and $R_1(K)$ is a space of polynomials of degree one.

This choice of approximating functions results in a computationally unstable discrete problem. We therefore added the same type of stabilization terms typically used for the Stokes flow problem, which is a Galerkin Least Squares [43, 44] term on one of the Euler Lagrange equations (17a). The Stabilized weak form is: *Find* $(\lambda, \mu) \in X_s \subset X$, such that:

$$c_h((\lambda, \mu), (\gamma, \omega)) = 0 \quad \forall (\gamma, \omega) \in Y_s \subset Y \quad (32)$$

where

$$c_h((\lambda, \mu), (\gamma, \omega)) = c((\lambda, \mu), (\gamma, \omega)) + \sum_{K \in \mathcal{T}_h} \tau_K (\nabla \cdot (\omega \mathbf{A}) - \Delta \gamma, \nabla \cdot (\mu \mathbf{A}) - \Delta \lambda)_{\Omega_K}, \quad (33)$$

and

$$X_s := U_s \times Q_s \quad Y_s := U_s \times P_s, \quad (34)$$

and

$$U_s \equiv \left\{ \gamma \in [H_0^1(\Omega)]^2 \mid \gamma|_K \in [H^2(\Omega)]^2 \quad \forall K \in \mathcal{T}_h \right\} \quad (35)$$

$$Q_s \equiv \left\{ \mu \in L^2(\Omega) \mid \langle \mu \rangle = \bar{\mu} \ \& \ \mu|_K \in H^1(\Omega) \quad \forall K \in \mathcal{T}_h \right\} \quad (36)$$

$$P_s \equiv \left\{ \omega \in L^2(\Omega) \mid \langle \omega \rangle = 0 \ \& \ \omega|_K \in H^1(\Omega) \quad \forall K \in \mathcal{T}_h \right\}. \quad (37)$$

In the above equations, τ_K is a mesh dependent constant.

2.2.2. Discretization—We discretize (32) with piecewise linear approximation functions chosen from the function spaces defined in (28) to (30). A new element was created within an in-house finite element code to solve the discrete system of equations. The mean constraint on the modulus (7.3) is enforced with a Lagrange multiplier. The stabilization parameter, τ_K , is determined by the methods outlined in Appendix (Appendix A).

3. Verification and convergence

Here we consider two different model problems that have analytical solutions to the elasticity equations (6). The solution for the first example is continuous, and the solution for the second example is discontinuous. Both model problems are solved in a unit square domain. The goal of the analysis is to numerically evaluate convergence with mesh refinement for these specific test cases.

3.1. Continuous shear modulus

For this test, the strains are chosen to be:

$$\epsilon_{xx} = \epsilon_{yy} = 0, \quad \epsilon_{xy} = \frac{\tau_o}{\exp(\alpha y)} \quad (38)$$

where $\tau_o = 1 \times 10^{-2}$, and $\alpha = \log(3)$. The analytical solution to (6) for the problem is $\mu(\mathbf{x}) = \mu_o \exp(\alpha y)$, where μ_o is a constant. The strain coefficients are computed by evaluating (38) on meshes refined from 3×3 to 161×161 nodes in each direction. The inverse problem is solved with the coefficients, and the solution is used to evaluate the errors. A sample reconstruction on a 101×101 mesh is shown in figure (1).

The L^2 norm, and the H^1 semi-norm of the error in μ are plotted as a function of the mesh size h in figure (2). The figure demonstrates that the formulation converges at a rate of $O(h^{3/2})$ in the L^2 norm, and $O(h^{1/2})$ in the H^1 semi-norm. Given the similarity of this formulation to the incompressible Stokes flow problem, we decided to compare the convergence rate of both problems. For the incompressible Stokes flow problem with Galerkin Least squares stabilization, and a smooth pressure field $p \in H^2(\Omega)$ interpolated with C^0 piecewise linear functions, the expected convergence rate in the L^2 norm is $O(h)$ [44].

3.2. Discontinuous shear modulus

For this test, the strains are chosen to be:

$$\epsilon_{xx} = \epsilon_{yy} = 0 \quad (39)$$

$$\epsilon_{xy} = \begin{cases} 2 \times 10^{-2} & 0 \leq x \leq 1, \quad 0 \leq y \leq \tan\left(\frac{\pi x}{6}\right) \\ 1 \times 10^{-2} & 0 \leq x \leq 1, \quad \tan\left(\frac{\pi x}{6}\right) < y \leq 1. \end{cases} \quad (40)$$

Substituting these strains into (8) gives the following exact analytical solution:

$$\mu = \begin{cases} 1 & 0 \leq x \leq 1, \quad 0 \leq y \leq \tan\left(\frac{\pi x}{6}\right) \\ 2 & 0 \leq x \leq 1, \quad \tan\left(\frac{\pi x}{6}\right) < y \leq 1. \end{cases} \quad (41)$$

The mesh was refined from 3×3 to 161×161 nodes. A sample reconstruction, done on a 101×101 mesh, is shown in figure (1). The L^2 norm of the error was computed as a function of the mesh size h . The results, shown in figure (2), demonstrate that the formulation converges for this problem with discontinuous solution.

4. Simulated data with noise for an inclusion problem

A common application in elasticity imaging is the detection and characterization of tumors, which may be crudely modeled as a hard inclusion in a softer background. We therefore test the formulation on an inclusion problem. The target solution for this problem is a homogeneous circular inclusion of radius $a = 0.25$ that is bonded to a homogeneous background. The shear modulus of the inclusion and background are $\mu_I = 7$, and $\mu_B = 1$, respectively.

The elasticity equations are solved analytically, using methods described in [46], to generate the strain coefficients for (26). The equations are solved for an infinite elastic sheet with a perfectly bonded circular inclusion. This sheet is subjected to uniform shear stress at infinity as demonstrated in figure (3). This stress is chosen to be $\sigma_{xy} = \sigma_{yx} = \sigma^\infty = 6 \times 10^{-2}$. The Poisson's ratio was chosen to be $\nu = 0.5$, consistent with the incompressibility assumption.

The solution to the elasticity equation is evaluated in a 1×1 ($4a \times 4a$) subregion around the inclusion. This subregion is discretized into 20000 triangle elements. The target displacement field is evaluated at each of the 10201 nodes in this subregion. This displacement field is used as input into the D-ECE formulation to compute the target strains, and the shear modulus (SM) distribution for the 0% noise case displayed in figure (4).

The target displacement field is also used in an Adjoint Weighted Equation (AWE) formulation [34] to compute the reference SM distribution shown in the first row of figure

(4). The AWE formulation is a stable and provably convergent variational formulation that has shown robust performance in reconstructing continuous and discontinuous fields [31, 34, 36]. Here, we use it as a reference to benchmark the performance of the D-ECE formulation.

Gaussian noise was added to the target strains in order to evaluate the robustness of the D-ECE method in dealing with noise. The reconstructions obtained for different levels of noise are shown in figure (4). The percent noise displayed in the figure represents the amount of noise in the strains. The results demonstrate qualitatively that the formulation performs well for noise levels below 10%.

Diagonal lines are plotted through the inclusion region to visualize the impact of noise on the reconstructions. The plots are shown in figure (5). From the figure, we can see an overshoot of the SM values in the inclusion region even for the 0% noise level. This is partly due to the fact that the inclusion region in the reconstructions is smaller than the inclusion region in the target solution, and the average SM of the reconstructions are fixed to the average SM of the target solution. In order to satisfy the mean constraint, the SM in the inclusion region needs to be slightly larger than the target value. Even with this overshoot, the contrast between the inclusion and background SM still comes out to be about 7.12 for the 0% noise case.

The reconstructions shown in figure (4) and the line plots in figure (5) seem to indicate that the D-ECE method is better at reconstructing large contrast than the AWE method. On the line plot, we see that the AWE method recovers a contrast of about 6, and the D-ECE method recovers a contrast of about 7.2. In order to check this result, we perform reconstructions with both methods on a courser mesh with 2601 nodes, and show line plots through the inclusion region for the various mesh sizes in figure (6). The figure demonstrates that the D-ECE method is converging to the right contrast with mesh refinement, while the AWE method seems to be unable to recover a larger contrast than 6.

5. Validation

In this section, we aim to demonstrate that the D-ECE formulation is capable of producing reasonable shear modulus (SM) reconstructions from physical measurements. The displacement data used to perform the reconstructions was measured within a tissue mimicking phantom under uniaxial compression. This phantom was manufactured and processed for a different study [47]. Here, we give a brief description of how the displacement was measured and used to perform reconstructions.

5.1. Phantom manufacture

The phantom was manufactured, using agar and gelatin, to have the acoustic and mechanical properties of soft tissue using techniques described in [48]. Its geometry is a 100mm cube containing four spherical inclusions, each with a diameter of 10mm. The inclusion centers were located in a horizontal plane, and they were spaced 30mm away from each other. Each spherical inclusion had a different shear modulus value, and the background of the phantom material was created to have an approximately constant SM value [47].

5.2. Phantom imaging and displacement estimation

The phantom was imaged with a Siemens SONOLINE Antares (Siemens Medical Solutions USA, Inc, Malvern, PA) ultrasound system with a linear ultrasound transducer array (Siemens VFX9-4) pulsed at 8.89MHz. A 15cm×15cm compression plate, much larger than the phantom surface, was attached to the ultrasound transducer and used to apply a uniform static compression to the phantom. The phantom was imaged before deformation, and after it was deformed by 1.5% strain with respect to its height [47]. During the imaging, Radio Frequency (RF) data, representing the backscatter pressure field, was recorded. The RF data was used in a modified block matching algorithm to estimate the displacement field of the phantom [49].

5.3. SPREME processing

Due to the nature of ultrasound, the axial component (along the direction of sound propagation) of the measured displacement field was much better than the lateral component (perpendicular to the direction of sound propagation). The displacement therefore had to be pre-processed before it was used to reconstruct the material property distribution. This pre-processing was done with the SParse Relaxation of Momentum Equation (SPREME) formulation [50]. The problem statement for the formulation is: Given the measured displacement field $\mathbf{u}_m(\mathbf{x})$, and $\boldsymbol{\epsilon}^{(k-1)}$, find $\mathbf{u}^k$, and $\boldsymbol{\epsilon}^k$ that minimizes:

$$\pi \left[\mathbf{u}^k, \boldsymbol{\epsilon}^k \right] = \pi_O + \pi_C + \pi_R \quad (42)$$

where:

$$\pi_O = \frac{1}{2} \int_{\Omega} \left(\mathbf{u}_m - \mathbf{u}^k \right) \cdot \mathbf{T} \left(\mathbf{u}_m - \mathbf{u}^k \right) d\Omega \quad (43)$$

and:

$$\pi_C = \frac{\beta}{2} \int_{\Omega} \left(\boldsymbol{\epsilon}^k - \nabla^s \mathbf{u}^k \right)^2 d\Omega \quad (44)$$

and:

$$\pi_R = \frac{\alpha_o}{2} \int_{\Omega} \frac{\left(\nabla \cdot \mathbf{A} \left(\boldsymbol{\epsilon}^k \right) \right)^2}{\left[\left(\nabla \cdot \mathbf{A} \left(\boldsymbol{\epsilon}^{(k-1)} \right) \right)^2 + \delta \right]^n} d\Omega \quad (45)$$

Equation (42) was minimized with respect to its independent variables to compute filtered versions of the displacements and strains ($\mathbf{u}^k, \boldsymbol{\epsilon}^k$). The parameters used for the minimization are: $T_{xx} = 10^{-5}$, $T_{yy} = 1$, $\alpha_o = 10^{-3}$, $\beta = 10$, $\delta = 10^{-8}$, $n = 0.5$. The filtered displacements are used to compute the shear modulus distribution of the phantom.

5.4. Reference results

In order to evaluate the quality of the shear modulus reconstructions produced by the D-ECE formulation, we compare them to two different sets of reference results. The first set of results were from independent mechanical tests performed on the phantom material [47]. While the phantom was being manufactured, separate cylindrical samples, composed of the same material as the phantom, were also created. Six samples were created: one for each inclusion, and two for the background. During the tests, the stress strain behavior of the phantom material was measured. The measurements were then fitted to a Vernonda-Westmann model to estimate reference shear modulus value of the phantom material [47].

The second set of results were obtained by using the measured displacement field to reconstruct the shear modulus distribution with an iterative optimization algorithm. This algorithm works by iteratively updating the shear modulus distribution until one is found that produces a predicted displacement field that is consistent with the measurements [11]. The reconstructions with the iterative optimization algorithm were performed in a different study [47], and used the same modeling assumptions (e.g. isotropic plane stress) as used here. In that study, the measured displacements were downsampled to a 63×54 grid before they were used in the iterative algorithm. The same downsampling procedure was followed before filtering the measurements with SPREME.

5.5. Modulus reconstruction

The filtered displacements obtained from SPREME are used in the D-ECE formulation (26) to reconstruct the shear modulus distribution. The mean shear modulus is fixed to an arbitrary value of 2. Since the reconstructed shear modulus values are all relative, they are scaled in a post-process by dividing each value in the domain by the minimum value. This makes it easier to compare the D-ECE reconstructions to the iterative reconstructions.

The D-ECE results, along with the reference reconstructions from the iterative optimization algorithm are shown in figure (7). From the figure, we can see that the location of the inclusion in both reconstruction methods is about the same. The shape and size of the inclusion produced by both methods are different, and the background region in the D-ECE reconstruction looks more heterogeneous than the background region in the iterative reconstructions. Also the range of values on the colorbar of the D-ECE reconstruction is larger than the range for the iterative reconstructions. This difference in heterogeneity and colorbar range might be partly due to the fact that the D-ECE reconstructions are unregularized while the iterative reconstructions use a Total Variation (TV) regularization.

The shear modulus (SM) contrast between the inclusion and background are reported in table (3). The contrast is defined as the ratio of the SM value in the inclusion region to the SM value in the background region. For both the D-ECE and iterative reconstructions, the inclusion and background regions are defined by thresholding the SM images. The thresholds used are reported in table (2), and were chosen to subjectively optimize the delineation between the known shape of the circular inclusion and background in each case. Once the regions are identified, then the SM values in each region is averaged. The average values are reported in table (1). These averages are then used to compute the contrasts

displayed in table (3) for each formulation. The reference contrasts are computed using the shear modulus values obtained from independent mechanical tests as described in section (5.4).

From table (3), we see that both the D-ECE and iterative formulation underestimate the reference contrast, and the degree to which they underestimate the contrast seems to grow with the size of the reference contrast. The contrasts predicted by the iterative method are closer to the reference values than the ones predicted by the D-ECE method even though the average values in the inclusion region of the D-ECE reconstructions are larger than those of the iterative reconstructions. The reason for this is because the average SM values in the background region of the D-ECE reconstruction are larger than the corresponding SM values in the iterative reconstructions (1). One possible explanation for the larger background values is due to the fact that a uniform compression is applied to generate the displacement field, leading to conditions approximating uniaxial stress. We note that in uniaxial stress, σ has only one nonzero eigenvalue. Since $\sigma = \mu \mathbf{A}$, then the coefficient tensor $\mathbf{A}$ is also singular, with zero eigenvalue corresponding to an eigenvector in the horizontal direction. The strong form of the plane stress problem (6) is ill-posed for this type of data [38]. The variational formulation is also ill-posed for this type of data.

6. Application

In this section, we describe the application of the D-ECE method to *in vivo* data measured from patients with breast masses. This data was collected and processed for a different study [9]. In the study, displacement data, corresponding to 1% applied strain, was measured from a patient with a fibroadenoma (FA), and another patient with an invasive ductal carcinoma (IDC). The measured displacements were pre-processed with SPREME to obtain filtered displacements suitable for the D-ECE method [50]. We use the filtered displacements to reconstruct the shear modulus distribution.

6.1. Modulus reconstruction

We set the mean shear modulus (SM) to an arbitrary value of 5 for the D-ECE reconstructions, and the reconstructed shear modulus is scaled with respect to this mean. We rescale the shear modulus to have a minimum of 1 to make it easier to compare the SM reconstructions for different tumor cases.

The reconstructed SM fields computed by two methods are shown in figure (8). From the figure we see that the SM field for the fibroadenoma tumor looks like a hard inclusion with well defined boundaries, surrounded by a softer background, while the SM field for the invasive ductal carcinoma looks to have less well defined boundaries. The fibroadenoma is a benign breast tumor, and these type of tumors typically show up as localized regions with increased stiffness. The invasive ductal carcinoma on the other hand is a malignant breast tumor that typically spreads to other regions of the body. These gross features are reproduced with both direct reconstruction methods.

7. Discussion and conclusions

The direct inversion methods presented so far in the literature perform poorly with discontinuous strains, and in reconstructing discontinuous material property distributions. We introduced the D-ECE formulation as a method that is better equipped to handle discontinuities in the strains and the material property distribution. We demonstrated that this formulation converges with mesh refinement in both continuous and discontinuous model problems. We successfully validated the approach with measured data on a tissue mimicking phantom, and showed an application to reconstruct the elastic modulus of breast masses *in vivo*.

The D-ECE formulation is promising, and therefore warrants further evaluation. This includes proving that the D-ECE formulation for the incompressible plane stress problem is well-posed, and deriving *a priori* error bounds with mesh refinement for both continuous and discontinuous strains and modulus distributions. Various forms of regularization can be added to the formulation to see how it affects the solution, and identification of appropriate ways to include boundary data (e.g. measured tractions) would be valuable. In the current D-ECE formulation, the mean modulus level is set as a global condition. Fixing a constant offset is appropriate for problems that would otherwise be ambiguous to an *additive* constant. Reconstruction of the material properties with displacement-only data is ambiguous up to a *multiplicative* constant, however. One approach to modify D-ECE to accommodate this change would be to formulate the inverse problem for $\psi = \log(\mu/\mu_{ref})$, but this leads to a nonlinear pde. Alternative approaches that lead to a linear problem would be useful.

Though the D-ECE formulation was presented here in the context of quasistatic plane stress deformations of an incompressible material, it extends naturally to other cases. For all static problems, the formulation essentially remains the same as is presented in (7). The only equation that changes is the constitutive relation (7.2), and the amount of boundary data needed (7.3). So for the incompressible plane strain and 3D problems, the problem is now to find (σ, μ, p) such that:

$$\nabla \cdot \sigma = 0 \quad x \in \Omega \quad (46.1)$$

$$\sigma = -p\mathbf{I} + 2\mu\epsilon \quad x \in \Omega \quad (46.2)$$

where p is the hydrostatic pressure. For time harmonic, and transient problems, the momentum equation changes. So, for the plane stress case, (7.1) is replaced by:

$$\nabla \cdot \sigma = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad (47)$$

where ρ is the density. For the plane strain and 3D cases, (46.1) is replaced by (47).

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Appendix A: Stabilization parameter selection

The form for the stabilization parameter τ_K in (32) was determined by solving a one dimensional model problem with uniform shear strains [51]. The strains used are: $\epsilon_{xx} = \epsilon_{yy} = 0$, $\epsilon_{xy} = e$. These strains are substituted into (32), to yield the following difference equations:

$$\lambda_x^{n+1} - 2\lambda_x^n + \lambda_x^{n-1} = 0 \quad (\text{A.1})$$

$$2 \left(2\lambda_y^n - \lambda_y^{n-1} - \lambda_y^{n+1} \right) + \varepsilon h (\mu_{n+1} - \mu_{n-1}) = 0 \quad (\text{A.2})$$

$$h \left(\lambda_y^{n-1} - \lambda_y^{n+1} \right) + 2\varepsilon \tau (\mu_{n+1} - 2\mu_n + \mu_{n-1}) = 0. \quad (\text{A.3})$$

where (λ_n, μ_n) are the field variables for an arbitrary element number n , in a domain discretized into m elements. h is the mesh size, and τ is the stabilization parameter.

The solutions of the difference equations are:

$$\lambda_n = \Lambda_1 \alpha_1^n + \Lambda_2 \alpha_2^n + \Lambda_3 \alpha_3^n + \Lambda_4 \alpha_4^n \quad (\text{A.4})$$

$$\mu_n = M_1 \alpha_1^n + M_2 \alpha_2^n + M_3 \alpha_3^n + M_4 \alpha_4^n \quad (\text{A.5})$$

where:

$$\alpha_1 = \alpha_2 = 1, \quad \alpha_3 = \frac{2\bar{\tau} + 1}{2\bar{\tau} - 1}, \quad \alpha_4 = \frac{2\bar{\tau} - 1}{2\bar{\tau} + 1} \quad (\text{A.6})$$

and $\tau = \frac{-2}{\bar{\tau}} h^2$, $\alpha_p = e^{-kh}$, where k is a constant. Note that for the incompressible Stokes flow problem with Galerkin least squares stabilization, $\tau \propto h^2$. The alphas represent the characteristic numbers of the various solution modes in the discrete system. We find that α_1

and α_2 represent constant solutions and are independent of $\bar{\tau}$. Choosing $\bar{\tau} = \frac{1}{2}$ gives $\alpha_3 = \infty$

and $\alpha_4 = 0$, thus allowing these solutions to decay either infinitely rapidly, or very slowly, thus avoiding spurious oscillation that otherwise pollutes the overall solution.

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Highlights

- We present a new computational formulation to quantitatively infer mechanical properties of soft tissue in vivo, and noninvasively.
- We show application to imaging two stiff breast tumors.
- The new computational formulation allows for the direct (i.e. non-iterative) solution of an inverse problem in elasticity.
- The new computational formulation is stable and convergent for discontinuous material properties.

Figure 1.
Shear modulus reconstructions for continuous and discontinuous model problem example.
Reconstructions are done on 101×101 mesh.

Figure 2.

Convergence plots for the continuous SM, and the discontinuous SM model problems. μ_i represents a nodal interpolation of the exact solution, and μ_h represents the finite element solution. The plots show the square of the errors as a function of h .

Figure 3.

The analytical solution for a pure shear stress applied to an infinite sheet with a bonded circular inclusion is used to verify the D-ECE method. In the figure, ROI means Region Of Interest.

Figure 4.

Shear modulus reconstructions for various levels of noise in the strain fields. The reference field is reconstructed with the adjoint weighted equation formulation from data with zero noise.

Figure 5.

On the left is an AWE reconstruction, and on the right is a line plot through the inclusion region of both the AWE and D-ECE reconstructions. Various noise levels are added to the D-ECE reconstructions.

Figure 6.

On the left is a D-ECE reconstruction, and on the right are line plots through the inclusion region for both the AWE and D-ECE reconstructions at two different mesh resolutions. There is no noise in the displacement data used for all the reconstructions.

Figure 7.
Shear modulus reconstructions for tissue mimicking phantom.

Figure 8.

Shear modulus reconstructions for a fibroadenoma and an invasive ductal carcinoma breast tumor. Top row: Reconstructions obtained with D-ECE method. Bottom row: Reconstructions obtained using the AWE method [34].

Table 1

Average shear modulus (SM) values in various regions of the tissue mimicking phantom.

Average SM	Target 1	Target 2	Target 3	Target 4
Inclusion (D-ECE)	2.54	2.18	4.16	5.68
Inclusion (Iterative)	2.08	2.18	2.96	4.07
Background (D-ECE)	1.51	1.34	1.68	1.79
Background (Iterative)	1.12	1.06	1.13	1.17

Table 2

Thresholds used to identify the inclusion and background regions of the tissue mimicking phantom.

Thresholds	Target 1	Target 2	Target 3	Target 4
D-ECE	2.283	1.909	2.776	3.489
Iterative	1.599	1.517	1.971	3.321

Table 3

Reference and reconstructed contrasts for tissue mimicking phantom. The reference contrasts are obtained by performing independent mechanical tests on phantom material. The reconstructed contrasts are obtained using the average values reported in table (1).

Method	Target 1	Target 2	Target 3	Target 4
D-ECE	1.69	1.63	2.48	3.17
Iterative	1.87	2.05	2.63	3.46
Reference	2.83	2.27	3.54	5.26