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Approximate Duals of Gabor-like Frames Based on Realizable Multi-Window Spline-Type Constructions

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Abstract

In the multi-window spline-type spaces, the fast computation of the realizable dual frame could be achieved through a constructive reformulation of the biorthogonal relations. In this paper, we extend the results obtained in spline-type spaces, for the constructive realization of an approximate dual Gabor-like frame. We demonstrate the advantages of this approach in both flexibility and speed. The method allows in a natural way to handle non standard Gabor constructions like non-uniformity in frequency and the reductions of the number of used modulations. Experimental tests are presented in support of the algorithm.

Index Terms

multi-window spline-type spaces; Gabor-like frames; Realizable dual frame; Selection of modulations; Numerical algorithm

1 Introduction

Spline-type spaces (or better-known as shift-invariant spaces), are generated through shifted copies of a given atom over a 1D lattice, while Gabor analysis is using series of time-frequency shifted copies of a given atom over a 2D lattice. The efficient computation of the so-called dual atom used for reconstruction after frame analysis was a longstanding interesting topic in the literature and it was for many years a drawback in the development of the applied field due to the high computational costs involved. Several efficient algorithms exist nowadays for both the computation of the dual atom [19] and for the approximation of it [12]. Nevertheless, even today, the most efficient Gabor algorithms are affected by a restrictive constraint: the samples are taken on a lattice structure to achieve reproduction of the signal.

The purpose of this paper is to use the theory of multi-window shift invariant spaces to compute in a fast way the dual atoms of Gabor-like frames obtained in the following manner: shift in frequency will be treated as different generators of our shift invariant space in time. This approach allows us to weaken the restriction of a 2D lattice structure to 1D

lattice of time shifts and to speed up the computations, while maintaining the accuracy of the approximation.

1.1 Mathematical Preliminaries

A space $\mathcal{S}$ of complex valued functions on $\mathbb{R}^d$ is **shift invariant (SIS)**, if it is closed under the time-shift operator

$$T_x f(t) := f(t - x)$$

Formally:

$$\forall f \in \mathcal{S}, \forall \alpha \in \mathbb{Z}^d \quad T_\alpha f \in \mathcal{S}$$

We call a shift-invariant space **principal** if it is generated by only one function. Also, we are interested in writing explicitly the SI space that an atom generates. A particular example is

$$\mathcal{S}_0(\phi) := \left\{ \sum_{k=1}^n c_k T_{\alpha_k} \phi : n \in \mathbb{N}, c_k \in \mathbb{C}, \alpha_k \in \mathbb{Z}^d, \forall k=1, \dots, n \right\} \quad (1)$$

In order to properly define the projection in a Hilbert space, we introduce the closed counterpart of (1)

Definition 1. Given a Banach function space B and $\phi \in B$ we call the closure in B of $\mathcal{S}_0(\phi)$ the **principal shift-invariant subspace of B generated by ϕ** .

$$\mathcal{S}(\phi) := \overline{\text{span} \{T_\alpha \phi : \alpha \in \mathbb{Z}^d\}}^B$$

It is possible to generalise the previous notation to a multi-window setting.

Given a set of atoms $\Phi = \{\phi_i\}_{i \in I}$ we call multi-window shift-invariant space generated by Φ

$$\mathcal{S}(\Phi) := \overline{\text{span} \{T_\alpha \phi_i : \alpha \in \mathbb{Z}^d, i \in I\}}^B$$

On the other side Gabor system is build on a different structure but similar to spline-type space. The difference between shift invariant spaces and spline-type spaces is that we do not require the closure of the shift operator on the group $\mathbb{Z}^d$, spline type spaces are spaces closed under the time-shift operator, defined on a fixed subgroup of $\mathbb{R}^d$.

Definition 2. Let Λ be any discrete subgroup of $\mathbb{R}^d$, and $\Phi = \{\phi_r : r = 1 \dots R\}$ be any set of elements in some translation invariant Banach space $(B, \|\cdot\|_B)$ of functions or distributions on $\mathbb{R}^d$. Then the closed linear span of the family $\{T_x \phi_r : r = 1 \dots R, x \in \Lambda\}$ is a closed, Λ -invariant subspace of $\mathbb{R}^d$. We call it **spline-type space** generated by Φ and Λ and we will indicate as $\mathcal{S}(\Phi, \Lambda)$.

We also need a definition to distinguish between good set of shifts and ill conditioned systems as in [9]

Definition 3. A family $\Xi = \{\xi_i\}_{i \in I}$ is said to be a **Riesz projection basis** if

- 1) There is a solid Banach space of sequences in I s.t. the synthesis map is a well defined continuous bijection.
- 2) The synthesis operator has a bounded left inverse.

The Gabor system uses two different shifts, the time-shift already defined and the modulation operator. The combined shift could be interpreted as frequency-shift operator:

$$M_\omega f(t) := e^{2\pi i \omega \cdot t} f(t)$$

Together with the time shift, it forms the time-frequency shift

$$\pi(\lambda) f(t) := M_\omega T_x f(t) = e^{2\pi i \omega \cdot t} f(t - x)$$

A particular commutation rule exists between shift operator

$$M_\omega T_x = e^{2\pi i x \cdot \omega} T_x M_\omega \quad (2)$$

Another property describes how shifts compose with the Fourier transform:

$$\mathcal{F}(T_x f) = M_{-x} \mathcal{F}(f) \quad (3)$$

$$\mathcal{F}(M_\omega f) = T_\omega \mathcal{F}(f) \quad (4)$$

Definition 4. Given a function ϕ in some function space on $\mathbb{R}^d$ and a lattice $\Delta \in \mathbb{R}^d \times \hat{\mathbb{R}}^d$, we define the Gabor system (or Weyl-Heisenberg system) as

$$\mathcal{G}(\phi, \Delta) = \{\pi(\lambda) \phi \mid \lambda \in \Delta\}$$

2 Gabor-like frames in spline-type spaces with modulations selection

In our study, an important property of the Gabor system is that, despite the fact that the commutation property is not symmetric, if we consider the spline-type space generated in a time-frequency manner through an atom g , and the combined frequency-time shift operator

$T_x M_\omega$ on a lattice $\Delta \in \mathbb{R}^d \times \hat{\mathbb{R}}^d$, we always obtain a Gabor system $\mathcal{G}(\phi, \Delta)$. The connection between spline-type spaces and constructive Gabor-like systems becomes natural. Given a

set of modulations $\{M_{ig} : i \in I\}$, we can describe a frequency-generalised Gabor frame, e.g. a system of function generated by the time-frequency shifts on

$$\Delta \times I$$

where Δ is a subgroup of only time.

In this way, we are establishing a theory that can handle non-uniform sampling in frequency. It will become clearer in the next sections, why this will be really helpful once some quality of the spectrum of the signal is known.

The most common tool to analyse the local behaviour of a function is the continuous short time Fourier transform.

$$V_\phi f(s, r) := \langle f, M_r T_s \phi \rangle_{L^2(\mathbb{R}^d)} = e^{-2\pi i x \cdot \omega} \langle f, T_s M_r \phi \rangle_{L^2(\mathbb{R}^d)}$$

To measure the local behaviour of the a function, we further introduce the Wiener amalgam spaces

Definition 5. *Given a Banach space $(X, \|\cdot\|_X)$ we define the Wiener amalgam spaces*

$$W(X, l^q) := \left\{ f \in X_{loc} : \|f\|_{W(X, l^q)} := \left(\sum_k \|f \cdot \psi_k\|^q \right)^{1/q} < \infty \right\}$$

where $\{\psi_k\}$ is a regular, uniform and bounded partition of unity in the Fourier algebra

Because we choose to use only some modulations, we are not bound anymore to the dense structure of a Gabor lattice. The fastest ways to implement the Gabor transform are indeed based on the fact that the matrix of the Gabor operator has a particularly nice structure i.e. the matrix operator is block circulant, and every block is a diagonal matrix. Through the construction of a non uniform Gabor frame as proposed in this paper, we can reach an algorithm having a similar complexity but with a major flexibility given by the cancelation of all unneeded modulations. In this way, the frequency domain can be discretized with the same number of modulations as in a classical Gabor transform, but arranged in a non-lattice structure.

Definition 6. *Given a Gabor system, we call it a Hilbert frame if*

$$\exists A, B > 0 \quad A \|h\|_H \leq \|\mathcal{T} * h\|_{l^2(\Delta)} \leq B \|h\|_H$$

We call it Riesz Basis if

$$\exists A, B > 0 \quad A \|c\|_{l^2(\Delta)} \leq \|\mathcal{T} c\|_H \leq B \|h\|_{l^2(\Delta)}$$

The system is said to be tight if $A = B$.

The analysis and synthesis operators build the so called frame operator

$$Sf := \mathcal{T} \mathcal{T}^* f = \sum_{\lambda \in \Delta} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$$

If the frame operator is invertible, thank to the commutation between frame operator and shifts, we obtain the core of the Gabor theory

Theorem 1. ([9]) *Reproduction theorem for Gabor frames. Given a Gabor frame $\mathcal{G}(\phi, \cdot)$ every function in its shift invariant generated space, can be represented as*

$$f = \sum_{\lambda \in \Delta} \langle f, \pi(\lambda) \left(S^{-1}g \right) \rangle \pi(\lambda)g \quad (5)$$

$$= \sum_{\lambda \in \Delta} \langle f, \pi(\lambda)g \rangle \pi(\lambda) \left(S^{-1}g \right) \quad (6)$$

we call the function $S^{-1}g$ canonical dual of the window g .

The canonical dual is indeed a particular window that gives us the reproduction formula (5).

Any other function h such that

$$\langle h, \pi(\lambda')g \rangle = \delta_{0, \lambda'} \quad (7)$$

is called dual of the atom g and can take the place of the dual window in formula (5).

Writing a dual atom $\tilde{g}$ as an element of the shift-invariant system generated by the Gabor-

like system, we obtain $\tilde{g}_i = \sum_{j \in I} \sum_{\lambda \in \Delta} b_{\lambda}^{(i,j)} \pi(\lambda)g_j$.

Therefore,

$$\begin{aligned} \delta_{i-l} \delta_{0, \lambda'} &= \langle \tilde{g}_i, \pi(\lambda')g_l \rangle \\ &= \left\langle \sum_{j \in I} \sum_{\lambda \in \Delta} b_{\lambda}^{(i,j)} \pi(\lambda)g_j, \pi(\lambda')g_l \right\rangle \\ &= \sum_{j \in I} \sum_{\lambda \in \Delta} b_{\lambda}^{(i,j)} \langle \pi(\lambda)g_j, \pi(\lambda')g_l \rangle \\ &= \sum_{j \in I} \sum_{\lambda \in \Delta} b_{\lambda}^{(i,j)} e^{-2\pi i(\omega' - \omega)x} V_{g_l} g_j(\lambda' - \lambda) \end{aligned} \quad (8)$$

The previous expression represents the **twisted convolution** of the unknown coefficients with the short-time Fourier transform of one atom with the others. The feasibility of the method is ensured by the application of the Wiener's lemma to the Heisenberg group embedded in Wiener space.

2.1 The modulation selection algorithm

For a bandlimited signal, or a signal with known spectrum, the algorithm to pre-process the signal and to select the proper modulations is given in the Algorithm 1.

We applied this selection in 2 different ways:

- Setting to zero the modulations that were not selected.
- Using only the selected modulated atoms.

The second method forces the dual basis to have the same number of elements as the selected modulations, while in the first case we found out that our numerical scheme builds a dual basis having more non zero elements than selected modulations, but still fewer than the whole discretization in frequency.

From these results, the first method seems useful for compression, while the second to speed up the computations. Using the same stable Gabor system we used above, $L=675$, $\text{gap}_f=27$ and $\text{gap}_t=15$, we generated a thousand of random low-frequency signals having only the first and last 20 non zero components in the Fourier domain. Setting a tolerance of 10^{-5} , the algorithm selects the following modulations of the normalised Gaussian

$$\{M_0g, M_{15}g, M_{30}g, M_{45}g, M_{630}g, M_{645}g, M_{660}g\}$$

ie. the 0th modulation, the first 3 and the last 3.

With the second method, we obtained 7 poorly approximated atoms that give for every randomly generated signal an approximation error of 1%, while in the second case, we obtained 10 non null duals: the one corresponding to the 0th modulation, the duals related to the first 5 modulations (even if the 4th and the 5th were set to zero), and the duals corresponding to the last 4 modulations (even if the 4th to last were set to zero).

In Figure 4 and 5, we can understand the difference between the methods: while the second produces a small error near the boundary of the the Fourier transform of the original band-limited signal, the first produces a much more sparse error.

3 Numerical experiments

In here, we present numerical experiments carried out in order to explore the performances of Gabor-like system constructed via a multi-window spline-type system. We have performed extensive testing in 4 different cases:

- (A) Traditional Gabor frame on separable lattice (see SC)
- (B) Small deformation of Gabor frame on separable lattice

- (C) Incomplete Gabor system
- (D) Automatic selection of modulations (see Algorithm 1)

We have always used the Gaussian function as atom, because it is the a well localized function in both time and frequency.

Starting from the first standard example, we want to underline numerically how our reformulation is consistent with Gabor frames.

From table 1 it is possible to appreciate that for the standard case (SC) the loss of precision is negligible. We can notice that we are actually reproducing a Gabor frame from Figure 1. In this image the Fourier transform of the different dual is plotted. Every dual is a shift in frequency of the dual of the original, modulation 0 Gaussian. The nature of the Gabor operator, which commute with the frequency and time shift is preserved.

For the Gabor frame with parameters $L=675$, $gap_t=27$, $gap_f=15$, we have plotted in Figure 3 the product between the 45 modulations and their duals. The biorthogonality is almost perfect, up to the redundancy as constant.

In Figure 1, we have actually plotted the duals of an incomplete system, this is done to ensure legibility of the figure.

This is a reference for point (C) of our set of tests.

We have performed tests on incomplete Gabor systems defined on a rectangular lattice for both traditional Gabor algorithm and our method as it can be seen in the second line of Table 1. We applied the building of the dual for the reconstruction of low frequency signals. It is possible to appreciate from Figure 2 the fact that the biorthogonality reformulation is more stable under numerical errors than traditional Gabor dual construction.

The approximation for the traditional fast implementation of Gabor duals, gives us a constant approximation error of 2% for signals having a small support in the Fourier domain. On the other side, our algorithm gives a really small error for low frequency signals that grows exponentially with the radius of the band. The two curves seem to touch for signals having a support of radius $r \approx 70$ in the Fourier domain, this is exactly the result of the sampling theorem

$$L=4410 \quad gap_t=30 \quad \Rightarrow \quad f_{max}=73.5$$

After that the behaviour of the two methods interchange: the traditional algorithm shows instability through an exponential growth of the error, while our algorithm gives an error that is never bigger than the energy of the signal (norm 2). This example shows that through incomplete Gabor systems, we are reaching a level of compression of a signal that was not accomplished before for such classes of fast algorithms. The analysis of band limited signal, or in general sparse in frequency, was found really interesting for our work and we will come back to this topic in the next section.

In the last part of this section we want to numerically explore the continuity of our method as pointed out in (B).

The traditional inversion of Gabor frame is not continuous under the shift parameters. That is the reason why we want to explore the possibility to deform a separable lattice, once a non-uniform reformulation is available.

For our numerical tests, we used the aforementioned frame with parameters $L=675$, $\text{gap}_t=27$, $\text{gap}_f=15$. To measure the error in the approximation of a signal X through the Gabor-like expansion X_a we used the formula

$$ERR = \frac{\|X - X_a\|}{\|X\|}$$

with $\|\cdot\|$ being the norm 2.

We conclude this section with a fast observation about the continuity of the inversion of a multi-window spline type frame.

After the usual frame $L=675$ $\text{gap}_f=27$ and $\text{gap}_t=15$ was fixed, we have computed the dual of its deformed version. varying the step of the deformation in frequency, and the probability of a window to be shifted.

In Figure 6 we have plotted the distance in norm 2 from the identity matrix, with every data representing the mean of 1000 randomly generated dual bases with corresponding parameters.

The deformation seems very smooth but, of course, the error becomes high for significant deformation: around 10% for a deformation of 1/3 of the frequency step size and 1/10 of probability of deformation.

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Algorithm 1 Selection of the proper modulation**Precondition:** Signal X , Window ϕ , frequency steps b

```

1: function FIND_MODULATION( $X, \phi, b$ )
2:    $L \leftarrow \text{length}(\phi)$ 
3:    $\mathcal{F}X \leftarrow \text{FFT}(X)$ 
4:    $\mathcal{F}\phi \leftarrow \text{FFT}(\phi)$ 
5:    $S \leftarrow \text{supp}(\mathcal{F}X)$ 
6:    $S_{\text{atom}} \leftarrow \text{supp}(\mathcal{F}\phi)$ 
7:    $\Psi \leftarrow \emptyset$ 

8:   for  $r \leftarrow 0$  to  $\frac{L}{b} - 1$  do
9:     if  $S \cap T_{rb}(S_{\text{atom}}) \neq \emptyset$  then
10:       $\Psi \leftarrow \Psi \cup \{r\}$ 
11:    end if
12:  end for
13:  return  $\Psi$ 
14: end function

```

Figure 1.
Fourier transform of dual spline-type system for $L=4410$, $\text{gap}_t=30$, $\text{gap}_f=441$

Figure 2.
Approximation of bandlimited signals: $L=4410$, $\text{gap}_t=30$, $\text{gap}_f=441$

Figure 3.
Approximation of biorthogonality for the system $L=675$, $\text{gap}_t=27$, $\text{gap}_f=15$

Figure 4.
Spread of the error in modulations selection

Figure 5.
Spread of the error with unnedeed modulations set to zero

Figure 6.
Deformation from approximated biorthogonality

Table 1

Error in approximation of a signal for different cases (mean of 10000 run)

		Error in Multi-window reformulation	Error in Traditional Gabor
SC:	L=675 tg=27, #c=45	1.5733e-13	3.6865e-16
LF + FM:	L=4410 tg=30, #c=10	8.9118e-7	2.8554e-2
SF + FS:	L=4410 tg=30, #c=15	3.3580e-5	1.8194e+3

SC=Standard case, **LF**=Low frequency, **FM**=Few modulation, **SF**=Sparse frequency, **FS**=frequency selection