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Compensation of Corneal Oblique Astigmatism by Internal Optics: a Theoretical Analysis

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Abstract

Purpose—Oblique astigmatism is a prominent optical aberration of peripheral vision caused by oblique incidence of rays striking the refracting surfaces of the cornea and crystalline lens. We inquired whether oblique astigmatism from these two sources should be expected, theoretically, to have the same or opposite signs across the visual field at various states of accommodation.

Methods—Oblique astigmatism was computed across the central visual field for a rotationally-symmetric schematic-eye using optical design software. Accommodative state was varied by altering the apical radius of curvature and separation of the biconvex lens's two aspheric surfaces in a manner consistent with published biometry. Oblique astigmatism was evaluated separately for the whole eye, the cornea, and the isolated lens over a wide range of surface curvatures and asphericity values associated with the accommodating lens. We also computed internal oblique astigmatism by subtracting corneal oblique astigmatism from whole-eye oblique astigmatism.

Results—A visual field map of oblique astigmatism for the cornea in the Navarro model follows the classic, textbook description of radially-oriented axes everywhere in the field. Despite large changes in surface properties during accommodation, intrinsic astigmatism of the isolated human lens for collimated light is also radially oriented and nearly independent of accommodation both in theory and in real eyes. However, the magnitude of ocular oblique astigmatism is smaller than that of the cornea alone, indicating partial compensation by the internal optics. This implies internal oblique astigmatism (which includes wavefront propagation from the posterior surface of the cornea to the anterior surface of the lens and intrinsic lens astigmatism) must have tangentially-oriented axes. This non-classical pattern of tangential axes for internal astigmatism was traced to the influence of corneal power on the angles of incidence of rays striking the internal lens.

Conclusions—Partial compensation of corneal astigmatism by internal optics is due mainly to the highly converging nature of wavefronts incident upon the lens resulting from corneal refraction. The degree of compensation is quadratically dependent on eccentricity but is expected to diminish as the eye accommodates. Neutralising the cornea by index-matching defeats internal compensation, revealing classical, radially-oriented oblique astigmatism in the isolated lens.

Keywords

Accommodation; Oblique astigmatism; Peripheral Vision; Compensation

Introduction

The human eye is a compound optical system consisting of two refractive elements (cornea and crystalline lens) on opposite sides of an aperture stop (iris). Both refractive elements have positive focusing power, which gives the whole eye greater power than either element acting alone. However, some ocular aberrations have opposite signs, causing the whole eye to be less aberrated than would be predicted from the cornea alone. For example, spherical aberration is positive for the cornea but negative for the crystalline lens.[1, 2] As the eye accommodates for near targets, lenticular spherical aberration becomes even more negative which cancels corneal spherical aberration and, with sufficient accommodation, eventually reverses the sign of ocular spherical aberration from positive to negative. [3–6] Coma caused by lateral displacement of the pupil relative to the cornea and lens is another example of aberration compensation. Corneal and lenticular sources of coma have opposite signs, which allows the eye to maintain relatively good optical quality despite misalignment of its optical elements. These examples provide evidence that the human eye has a robust optical design that produces relatively constant optical quality for foveal vision despite large variations in the structural properties of the system.[7]

Although the work cited above has emphasised aberration compensation for foveal vision, it raises the question of possible compensation also for aberrations in peripheral vision. For example, oblique astigmatism is a prominent optical aberration in the peripheral visual field that arises from the oblique incidence of rays striking the refracting surfaces of the cornea and crystalline lens.[8, 9] Recently we have reported evidence that oblique astigmatism of the cornea and lens do indeed have opposite signs, which produces better image quality for the eye as a whole than for the cornea by itself even when the eye accommodates.[10] In this report we offer an optical model to account for this empirical result across an area of central visual field sufficient to demonstrate the variation of oblique astigmatism, and its compensation, with eccentricity.

Methods

Oblique astigmatism was investigated by implementing a rotationally-symmetrical schematic-eye[5] in the ZEMAX OpticStudio (<http://www.zemax.com/os/opticstudio>) optical design software. As illustrated in Fig. 1, the relaxed model consists of a meniscus lens cornea (refractive index =1.3700, anterior apical radius =7.72mm, anterior conic constant =0.74, posterior apical radius =6.50mm, posterior conic constant =1, thickness=0.55mm) and a biconvex lens (refractive index =1.4129, anterior apical radius =10.20mm, anterior conic constant =−2.13, posterior apical radius =−6.00mm, posterior conic constant =0, and 4.00mm thickness) separated by 3.05mm. Accommodation was implemented by altering the apical radius of curvatures and thickness of the biconvex lens. The anterior chamber (aqueous) has refractive index = 1.3312, the posterior chamber (vitreous) has refractive index = 1.3302, the stop aperture (iris) rests at the anterior vertex of the lens, and the spherical retina has radius=12mm.

All four refracting surfaces are aspheric with shape described by the conic equation:

$$z(r, \theta) = \frac{cr^2}{1 + \sqrt{1 - Pc^2r^2}} \quad (1)$$

where z is the surface sag in mm, P is the unitless conic constant, c is curvature in mm^{-1} , r is radial distance from surface apex in mm. In this equation, conic constant P ($P=1+k$, where k is conic constant used in Navarro's equation) has value 1 for a circle, with other positive P -values being ellipses and negative values for hyperbolas.[11] To show the impact of asphericity on lenticular oblique astigmatism, the conic constants for both lenticular surfaces were adjusted while keeping apical radii fixed. Astigmatism of the isolated lens for collimated light was computed numerically by tracing rays from an infinitely distant object point (wavelength = 850nm commonly used in wavefront sensing and retina imaging) to the sagittal and tangential image points that demarcate the interval of Sturm. The magnitude of astigmatism for any given field location is defined as half the dioptric width of this interval between image shells. Astigmatism of the isolated lens was also computed for strongly converging wavefronts, as occurs in the natural eye.

Oblique astigmatism associated with off-axis object points arises from several sources in the Navarro eye model. The first source is the cornea (C), where refraction of a sheet of rays in the meridional/tangential plane containing the optical axis and object point (the plane of Fig. 1) is greater than refraction in the sagittal plane containing the chief ray and lying orthogonal to the meridional/tangential plane of the diagram. The second source is wavefront propagation (P), which changes the wavefront's mean and astigmatic vergence as light travels from the posterior surface of the cornea to the anterior surface of the lens.[12] The third source is the crystalline lens (L). In this paper we treat the cornea and lens as thick systems when reporting ocular astigmatism, taking into account wavefront propagation between their respective anterior and posterior surfaces. In this report corneal astigmatism refers to the anterior cornea surface only, which is readily measured with clinical video keratographers. Oblique astigmatism of the posterior corneal surface is one of several factors represented by the term "internal astigmatism", which is computed by subtracting corneal astigmatism from whole-eye astigmatism. Internal astigmatism also includes the combined effects of wavefront propagation through the anterior chamber and refraction of this converging, astigmatic wavefront by the lens. Internal astigmatism also takes into account the large increase in wavefront convergence at the anterior surface of the lens caused by refraction at the cornea.

Astigmatism is specified in power vector format,[13] which may be expressed in either polar or Cartesian forms. Polar form $[J, \phi]$ is most convenient for displaying visual field maps of astigmatism since axis ϕ (the meridian of maximum positive power) is typically parallel to the visual meridian of the object point. However, the Cartesian form $[J_0, J_{45}]$ is preferred for vector algebra computations such as subtracting corneal astigmatism from total astigmatism to determine internal astigmatism.[14] Conversion from Cartesian to polar form is achieved by applying the Pythagorean theorem in double-angle space,

$$\begin{cases} J = \sqrt{J_0^2 + J_{45}^2} \\ \phi = 0.5 \tan^{-1}(J_{45}/J_0) \end{cases} \quad (2)$$

Unlike clinical conventions, astigmatism in this report refers to the eye rather than the lens used to correct the eye. For example, a positive value of J_0 for the eye's axial astigmatism indicates "against-the-rule" astigmatism in clinical terminology.

Results

Oblique astigmatism of any optical system, including the Navarro eye model, arises when the chief ray from an off-axis object point encounters a refracting surface that lacks rotational symmetry locally near the intersection point. Textbook explanations typically show imaging of a radially symmetric object (e.g. a bicycle wheel) coaxial with a single refracting surface.[15] Clear images of the wheel's spokes and tire are formed on separate surfaces with different curvatures, rather like a teacup on a saucer, suggesting the alliterative mnemonic "sagittal rays from spokes focus on the saucer, but tangential rays from the tire focus on the teacup".[11]

A visual field map of oblique astigmatism (in polar format) for the cornea in the Navarro model follows this classic, textbook example as shown in Fig. 2B. In power vector format, the axis of astigmatism is radially oriented everywhere in the field (except the coordinate origin, where oblique astigmatism vanishes) and the magnitude of astigmatism is radially symmetric. Oblique astigmatism of the whole eye (Fig. 2A) has these same features, but the magnitude is smaller because of partial compensation by the internal optics. Vector subtraction of the corneal map from the whole eye map gives a map of internal astigmatism (Fig. 2C). Note that the visual field map for internal optics does not follow the classic pattern: the axes are tangentially oriented rather than radially oriented, just as we have observed in human eyes.[10] Surprisingly, this unexpected feature of internal astigmatism is not shared by the isolated lens when analysed with collimated light, which has the classic form of oblique astigmatism with radially oriented axes (Fig. 2D).

To better appreciate the quantitative aspects of these field maps, it is helpful to examine a cross section as shown in Fig. 3 for the superior vertical meridian. Ocular, corneal and lens astigmatism all become increasingly negative as eccentricity in the visual field increases, and yet internal astigmatism changes in the positive direction indicating the axis of internal astigmatism is tangentially oriented rather than the conventional radial orientation. To explain this surprising change in sign (which implies a 90° rotation of axis) we devote the remainder of this report to uncovering the source of unconventional, tangentially-oriented oblique-astigmatism of the eye's internal optical system that runs counter to the textbook mnemonic.

Initial computations indicated that refraction by the corneal posterior surface is a minor factor, compensating for less than 7% of the oblique astigmatism produced by the anterior corneal surface. Therefore, we concentrated our attention on the crystalline lens as the

presumed source of compensation that would be expected if corneal and lenticular oblique astigmatism have opposite signs.

The importance of surface asphericity for determining the intrinsic oblique astigmatism of the isolated lens suspended in aqueous (index of refraction=1.3312). Since the field map in Fig. 2D is radially symmetric and monotonic with eccentricity, it was sufficient to study in detail a single object location (13.5° in the superior visual field) where ocular J_0 is negative (with-the-rule astigmatism). We anticipated that ocular J_0 will become even more negative if J_0 of the lens is also negative, but will become less negative (indicating partial compensation) if J_0 of the lens is positive. Restating these predictions in polar form permits generalization of this result for any field location: the magnitude J of ocular oblique astigmatism should increase when the axes of corneal and lens astigmatism are parallel (i.e. both are radial) but decrease when their axes are orthogonal (e.g. corneal is radial, internal is tangential). Below we describe how surface asphericity affects the sign of J_0 for the isolated lens (Fig. 4A) with the understanding that the results are easily generalised in polar form by omitting the subscript zero.

Computed values of J_0 for the isolated lens with an object placed at infinity over a range of anterior (P_a) and posterior (P_p) asphericities are shown graphically in Fig. 4A. For this analysis, surface curvatures, lens thickness and refractive indices were held fixed at values specified in the Navarro model. Horizontal colour bands in this figure indicate that asphericity of the anterior lenticular surface has no significant influence on oblique astigmatism, which is to be expected since the stop aperture is located at this surface.[16] Asphericity of the posterior surface, however, has a major influence on lens astigmatism. J_0 of the lens is zero for a hyperbolic posterior surface with $P_p = -0.9$, regardless of anterior asphericity, as indicated in Fig 4A by the dot-dashed null-astigmatism line. Oblique astigmatism is tangentially oriented ($J_0 > 0$) only for hyperbolic posterior surfaces with $P_p < -0.9$. Oblique astigmatism is radially oriented ($J_0 < 0$) for hyperbolas in the range $-0.9 < P_p < 0$ and for ellipses ($P_p > 0$). Thus the domain of interest, where the axis of oblique astigmatism is tangentially oriented for the lens thereby compensating partially for corneal astigmatism, lies below the dashed line in Fig. 4A for the case of collimated light.

The behaviour illustrated in Fig. 4A is for a real source at infinity, whereas *in vivo* the light striking the crystalline lens is strongly converging and thus appears to arise from a virtual point source at a finite distance. Optical theory (e.g. Coddington's equations) predicts that the depth of Sturm's interval will be very different in these two situations. To show this difference quantitatively, we computed the oblique astigmatism of the isolated lens for spherical wavefronts with vergence = +42.1D measured at the corneal plane. In this case compensatory oblique astigmatism with tangential axis is more prevalent as shown in Fig. 4B. As in Fig 4A, anterior asphericity has no influence on J_0 . However, tangential astigmatism now occurs for the full range of hyperbolic posterior surfaces, not just those with $P_p < -0.9$. For the parabolic posterior surface ($P_p = 0$) of the Navarro model, oblique astigmatism is radially oriented for collimated light (Fig. 4A) but vanishes for +42D converging light (Fig. 4B).

Surface asphericity is not the only factor that influences lenticular oblique astigmatism. Optical designers commonly manipulate oblique astigmatism by bending lenses into different shapes, as quantified by shape factor q computed from the apical radius of curvature of the anterior (r_a) and posterior (r_p) surfaces,

$$q = \frac{r_p + r_a}{r_p - r_a} \quad (3)$$

Several examples of lenses with various shape factors[16] with their associated names are illustrated in Fig. 5. Since the crystalline lens is biconvex ($r_a > 0, r_p < 0$) at all states of accommodation with flatter anterior surface ($|r_a| > |r_p|$), it has a shape factor between -1 and zero. It is well-known that spherical aberration and coma depend on the shape factor,[17] but here we investigate the effect of shape on oblique astigmatism at different accommodative states of the lens. To reduce the number of free parameters, the asphericity of both surfaces (anterior $P_a = -2.13$, posterior $P_p = 0$) of the Navarro model eye were held fixed.

The effect of surface shape on the intrinsic oblique astigmatism of the isolated lens for collimated light is illustrated graphically in Fig. 6A for the same object location (13.5° in the superior visual field) used above in connection with Fig. 4A. The range of values for anterior radius of curvature (abscissa) and posterior radius of curvature (ordinate) in Fig. 6 includes all possible lenticular shapes that can achieve approximately 10 D range of lens powers associated with accommodation. A thick dashed line indicates biconvex shape with equal radii of curvature ($q=0$) and the domain of human crystalline lens shapes occupies the area below this $q=0$ line. The curved grid of thin solid lines show equal-power contours (along which shape factor varies) and equal shape contours (along which power varies). The small region of the graph below the curve labelled $J_0=0$ is the domain of partial compensation where the axes of lenticular and corneal astigmatism are orthogonal.

Figure 6A is a parametrically rich nomogram that may be used to determine the astigmatic effect of shape changes that accompany accommodation by the lens for collimated light. For example, parameters of the relaxed crystalline lens are shown by the black cross symbol for the Navarro eye model, which may be compared with the red squares representing two empirical studies of human eyes.[18, 19] By tracing an equal power contour through each of these symbols we see that J_0 changes in the positive direction as the anterior surface becomes flatter (longer radius of curvature) and the posterior surface becomes more curved (shorter radius). In fact, both of these changes are required to position the lens in the compensatory domain below the curve labelled $J_0=0$ for collimated light. As the lens accommodates, a trajectory through the space of Fig. 6A is mapped out by the black circles (Rosales' 2006 data[18]) and by the black diamonds (Dubbelman's 2005 data[19]), culminating in a fully accommodated eye shown by the red triangles. This trajectory lies very close to the $J_0 = -0.17$ contour line for Rosales (2006) and the $J_0 = -0.08$ contour for Dubbelman (2005), indicating that intrinsic astigmatism of the isolated human lens is nearly independent of accommodation both in theory and in real lenses studied *ex-vivo* with collimated light.

Although the domain of tangentially oriented astigmatism is very small in Fig. 6A, and quite remote from the locus of points inferred from human lens data, it doesn't necessarily follow that compensation of corneal astigmatism by the lens is unlikely. As noted above in connection with Fig. 4, oblique astigmatism is strongly influenced by the vergence of incident light. This influence is demonstrated again in Fig. 6B which shows the effect of surface curvature on astigmatism of the isolated lens for converging light (vergence = +42.1D leaving the corneal plane). The domain of tangentially-oriented astigmatism is larger in Fig. 6B than in Fig. 6A, and now includes the Dubbelman (2005) data [19].

For a full account of compensation we computed internal astigmatism, which takes into account the effects of refraction on converging light by the corneal posterior surface, wavefront propagation through the anterior chamber, and refraction by the crystalline lens. The effects of surface curvatures on the results are shown in Fig. 6C. Thus the difference between Figs. 6B and 6C reveals the importance of factors other than the astigmatism of the isolated lens in determining internal astigmatism.

The compensatory region of tangential astigmatism ($J_0 > 0$) is larger in Fig. 6C than in Fig. 6B and includes all of the black diamonds (Dubbelman's 2005 data)[19] and several of the black circles (Rosales' 2006 data)[18] representing the more relaxed states of accommodation. We conclude from these results that partial compensation of corneal astigmatism by internal astigmatism is to be expected in human eyes, despite the fact that corneal astigmatism and the intrinsic oblique astigmatism of the isolated lens for collimated light are of the same sign. Compensation is greatest for the relaxed eye but should diminish as the eye accommodates. Although the numerical values of this compensation are small ($< 0.1D$) at 13.5° eccentricity, compensation is expected to grow as the square of eccentricity.

To further explore the importance of corneal power on internal astigmatism, we replaced the cornea of the Navarro eye by an aberration-free surface with positive power in the range 0–60 D. ZEMAX treats this surface as an ideal thin lens that alters the vergence of light incident on the anterior surface of the lens, taking into account wavefront propagation from cornea to lens but omitting the oblique astigmatism of the cornea. Astigmatism computed by this method is therefore equivalent to internal astigmatism. The results are summarised in Fig. 7. When corneal power is zero, internal oblique astigmatism ($-0.1 D$) equals that of the isolated lens reported in Fig. 6A, which confirms that the Navarro eye model exhibits conventional oblique astigmatism (radial axes) when the cornea is neutralised, as demonstrated experimentally on human eyes.[6] As corneal power increases, the internal oblique astigmatism becomes less negative and eventually changes sign from negative to positive when corneal power equals +43 D. When corneal power equals the Navarro value (+48 D, for the anterior corneal surface), internal optics offsets corneal astigmatism by 0.013 D, which is close to the 0.03 value reported in Fig. 2C. Thus we conclude that compensation of corneal astigmatism by internal optics is made possible by the highly converging nature of wavefronts incident upon the lens resulting from corneal refraction. In fact, those incident wavefronts will be slightly astigmatic (Fig. 2B), which will make a significant contribution to ocular astigmatism, but is of minor importance for compensation in the central visual field ($< 15^\circ$ eccentricity).

Discussion

Our theoretical study of oblique astigmatism in a schematic eye was motivated by a need to understand our recent empirical finding that oblique astigmatism of the whole eye is less than that of the cornea by itself, even when the eye accommodates.[10] An example of such behaviour by human eyes is shown in Fig. 8. This particular eye (subject DK from our empirical study) is selected for display here because of its absence of axial astigmatism, thereby avoiding complex interactions that might obscure our view of corneal oblique astigmatism and its compensation by internal astigmatism.[14]

Ocular oblique astigmatism and the compensation of corneal oblique astigmatism by internal oblique astigmatism were typically invariant with accommodation over the central 30° field of view in our experiments. Therefore we averaged the measurements taken at 8 levels of accommodation to produce visual field maps for ocular (Fig.8 A) and internal oblique astigmatism (Fig. 8 C) along with corneal astigmatism shown in Fig. 8 B. All three maps have a centre of symmetry (red diamond) displaced from the line-of-sight (coordinate origin), which was typical of our study population. As predicted by theoretical modelling (Fig. 2), empirically measured ocular and corneal astigmatism maps exhibit radially oriented axes, yet the internal astigmatism map demonstrates tangentially oriented axes. This 90° rotation of internal astigmatism axis is necessary to achieve partial compensation of corneal astigmatism by the internal optics, which results in ocular oblique astigmatism that is smaller than oblique astigmatism of the cornea alone. Deeper analysis revealed this non-classical pattern of tangential axes for internal astigmatism is due to the influence of corneal power on the angles of incidence of rays striking the internal lens as shown analytically in Appendix and by numerical ray-tracing (Fig. 7). From these results we conclude that vergence of light incident on the crystalline lens is the critical factor determining whether internal astigmatism will compensate or augment the oblique astigmatism of the corneal first surface.

The effect of incident vergence on oblique astigmatism highlighted in Results has implications for experimental studies of the isolated crystalline lens. If off-axis aberrations of an isolated lens are measured using collimated light (e.g. an excised lens measured with a laser beam[20] or an intact eye with neutralised cornea[6]), then the axis and magnitude of oblique astigmatism will be different from that present *in-vivo* and might lead to misleading conclusions about compensatory mechanisms and the quality of the retinal image in peripheral vision.

One way to understand why the cornea normally produces classical (radial) oblique astigmatism but the lens *in situ* produces non-classical (tangential) oblique astigmatism is to note that the cornea normally refracts diverging wavefronts from real objects whereas the lens images refracts converging wavefronts from virtual objects. This change in sign of object vergence reverses the sign of oblique astigmatism even for the simple case of a plane surface.[21] For diverging light in the meridional plane produced by a real point source, a plane surface produces a virtual image with negative vergence (see Mahajan 's Fig. 5–27). However, for converging light, the same surface produces a real image with positive vergence (see Mahajan 's Fig. 5–26). This sign reversal for image vergence reverses the sign

of all the Seidel aberration coefficients, including oblique astigmatism (see Mahajan's eqn. 5–254), which explains why the axis of oblique astigmatism changes from radial to tangential.

Numerical evaluation of the Navarro eye model using optical design software revealed a range of parameter values of the eye model for which partial compensation would be expected. This parametric range includes reported values obtained from two empirical studies of human eyes, which suggests that internal compensation of corneal aberrations is true in general for oblique astigmatism just as it is for spherical aberration and coma (see Introduction for literature citations). Our study therefore supports the claim that the human eye has a robust optical design that produces relatively constant optical quality despite large variations in the structural properties of the system. Our evidence extends this claim beyond the domain of foveal vision in the relaxed eye[7] to include the central (and probably the peripheral) field of view even when the surface curvatures of the crystalline lens undergo major changes as the eye accommodates.

Theoretical analysis of schematic eye models based on population data provide useful insights into expected optical behaviour of individual eyes. Thus one of the important outcomes of our study is the articulation of a testable hypothesis that corneal oblique astigmatism is partially compensated by the crystalline lens. Factors not included in our schematic eye, such as the gradient-index profile of the human lens, the variation of lens thickness with accommodation, and the variation of asphericity of the surfaces of the lens during accommodation will surely influence the outcome of experimental tests of our hypothesis. For example, oblique astigmatism of a reduced eye model depends on the axial position of the entrance pupil[14, 22] and therefore pupil location may also influence the compensation of corneal astigmatism by internal astigmatism. We tested this possibility by varying the anterior chamber depth of the Navarro model eye from 2mm to 4mm. As expected, internal and ocular oblique astigmatism both decline as anterior chamber depth increases because the iris shifts away from cornea and towards the nodal point. Nevertheless, compensation of corneal astigmatism by internal optics persists and in fact increases with anterior chamber depth because of the larger propagation distance increases the vergence of light incident on the lens, which pushes internal astigmatism further into the non-classical zone of tangentially oriented axes (Fig. 7).

Many topics in clinical and visual science may benefit from improved understanding of oblique astigmatism. Pattern detection[23, 24] and motion perception[25, 26] of stimuli located in the peripheral visual field both improve when oblique astigmatism is reduced. Quality of life for low-vision patients deprived of foveal vision may also be improved by correcting oblique astigmatism at peripheral retina loci with residual vision.[27–30] Foveal vision may also be hampered by oblique astigmatism if the eye's optical axis departs significantly from the foveal line-of-sight.[14] Deeper understanding of the role of the crystalline lens in compensating for corneal oblique astigmatism may also help improve the design of intraocular lenses. Retinal imaging with adaptive optics for basic science and diagnostic purposes may also be improved by correcting oblique astigmatism. Even small amounts of unintended instrument astigmatism ($<0.16\text{D}$ over a central 3.6 degree field of view) compromises the acquisition of high resolution retinal images, which suggests that

correcting small, yet critical, levels of ocular oblique astigmatism could significantly improve the quality of fundus images.[31, 32]

Perhaps the most important result of our analysis is that refraction by the cornea, if strong enough, has the ability to convert internal oblique astigmatism from the conventional, textbook type (“spokes focus on the saucer, tire focuses on the teacup”) to the unconventional type (“spokes focus on the teacup, tire focuses on the saucer”). The underlying reason may be found in Coddington’s equations, which generalise the classical Gaussian conjugate equation of paraxial imaging to describe the refraction of rays striking a surface obliquely. As shown in the Appendix, the effective vergence of incident sagittal rays depends only on object distance but the effective vergence of tangential rays depends also on the angle of obliquity. In other words, effective vergence of incident light can be astigmatic even for a spherical wavefront. If that spherical wavefront is strongly converging, as normally occurs at the anterior surface of the crystalline lens, it is possible for the sign of oblique astigmatism to reverse and thus the axis switch from radial to tangential, just as we have observed in human eyes (Fig. 8) and confirmed by tracing rays through an eye model (Fig. 7).

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Appendix: Power crosses for Coddington’s equations

For paraxial imaging, power crosses are a convenient, clinic-friendly depiction in graphical format of the familiar Gaussian conjugate equation $L' = L + F$, where L is vergence of incident rays, F is surface (or thin lens) power, and L' is vergence of emerging rays. This equation is written twice when the optical system has axial astigmatism, once for each principal meridian. In general the incident wavefront can also be astigmatic, so we distinguish between incident vergence in the two principal meridians when writing the equations governing refraction in these meridians,

$$L'_1 = L_1 + F_1 \quad (1^{st} \text{ principal meridian}) \quad (\text{A1a})$$

$$L'_2 = L_2 + F_2 \quad (2^{nd} \text{ principal meridian}) \quad (\text{A1b})$$

where L_1 and L_2 are the incident vergences, F_1 and F_2 are the refracting powers of the surface, and L'_1 and L'_2 are the emerging vergences of the wavefront in the first and second principal meridians, respectively. The corresponding power cross diagram of Fig. A1 embodies both of these Gaussian conjugate equations:

Exit vergence = Incident vergence + Refractive power

Appendix Figure A1.

Power cross depiction of the Gaussian conjugate equation of paraxial imaging by a refracting surface with axial astigmatism.

The difference $L'_2 - L'_1$ between emerging vergences in the two principal planes represents the dioptric length of Sturm's interval, which quantifies the amount of astigmatism in image space. If the input wavefront is rotationally symmetric (i.e. $L_1=L_2$), then $L'_2 - L'_1 = F_2 - F_1$ = intrinsic astigmatism of the refracting surface. Therefore we conclude that, for spherical incident wavefronts, axial astigmatism of the emerging wavefront does not depend on incident vergence.

If instead the astigmatism is due to oblique incidence, then the Gaussian conjugate equations are replaced by the more general Coddington equations that govern refraction in the sagittal and tangential principle meridians of conic surfaces,[17, 22]

$$\begin{aligned} \frac{n'}{s'} - \frac{n}{s} &= \frac{n' \cos i' - n \cos i}{r_S} \quad (\text{sagittal}) \\ \frac{n' \cos^2 i'}{t'} - \frac{n \cos^2 i}{t} &= \frac{n' \cos i' - n \cos i}{r_T} \quad (\text{tangential}) \end{aligned} \quad (\text{A2})$$

To aid interpretation, eqn. A2 can be converted to vergence notation for display as a power cross diagram. The first step is to determine the refracting powers F_S and F_T of the sagittal and tangential meridians. This step is accomplished by setting object distances $s=t=\infty$ in eqn. A2 to get

$$F_S = \frac{n'}{s'} = \frac{n' \cos i' - n \cos i}{r_S} \quad (\text{sagittal power}) \quad (\text{A3a})$$

$$F_T = \frac{n'}{t'} = \frac{n' \cos i' - n \cos i}{r_T \cos^2 i'} \quad (\text{tangential power}) \quad (\text{A3b})$$

Substituting equations A3 into A2 gives Coddington's equations in the optometric vergence format

$$S' = S + F_S \quad (\text{refraction in sagittal plane}) \quad (\text{A4a})$$

$$T' = T \frac{\cos^2 i}{\cos^2 i'} + F_T \quad (\text{refraction in tangential plane}) \quad (\text{A4a})$$

where S, T are incident vergences and S', T' are exit vergences in the two principal planes. Comparing eqn. A4 with A1 reveals a key feature of oblique astigmatism: the effective vergence of incident sagittal rays depends only on object distance but the effective vergence of incident tangential rays depends also on the angle of obliquity. Thus, unlike the simpler case of imaging in the presence of axial astigmatism, effective vergence for incident oblique rays can be astigmatic even for a spherical wavefront.

The corresponding power-cross diagram for Coddington's eqn. A4 (Fig. A2) includes the scaling of incident vergence in the tangential meridian needed to take account of the angle of obliquity,

Exit vergence = Effective incident + Refractive power vergence

$$T' = T \frac{\cos^2(i)}{\cos^2(i')} + F_T$$

$$S' = S + F_S$$

$$T \frac{\cos^2(i)}{\cos^2(i')}$$

$$S$$

$$F_T$$

$$F_S$$

$$F_T = \frac{n'}{t'} = \frac{n' \cos i' - n \cos i}{r_T \cos^2 i'}$$

$$F_S = \frac{n'}{s'} = \frac{n' \cos i' - n \cos i}{r_S}$$

Appendix Figure A2.

Power cross depiction of Coddington's equations for imaging by a refracting surface with oblique astigmatism.

Again we compute the difference between exit vergences in the principal meridians to find the amount of astigmatism in image space,

$$T' - S' = T \frac{\cos^2 i}{\cos^2 i'} + F_T - S - F_S \quad (\text{A5})$$

If the incident wavefront is spherical, then $T=S=L$ and (A5) becomes

$$T' - S' = L \left[\frac{\cos^2 i}{\cos^2 i'} - 1 \right] + (F_T - F_S) \quad (\text{A6})$$

The term $(F_T - F_S)$ quantifies the astigmatism of the refracting surface, which is positive for the textbook example of “teacup and saucer” image shells produced when light leaves a rare medium (e.g. the aqueous humour) to enter a dense medium (e.g. the crystalline lens) separated by a convex surface. In this case $i > i'$ and $\cos(i) < \cos(i')$ so the quantity in square brackets, which quantifies the effective astigmatism of a spherical incident wavefront, is negative. Thus for a converging incident wavefront ($L > 0$), astigmatism in the emerging wavefront is the sum of two terms of opposite signs, only one of which depends on incident vergence L . Therefore, if L grows large enough, the sign of astigmatism of the emerging wavefront will reverse and the axis of astigmatism will change from radial to tangential.

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Figure 1.

Schematic drawing to of the Navarro eye model implemented in ZEMAX. Lens parameters were varied parametrically to simulate accommodation and to explore the influence of surface asphericity and lens shape on oblique astigmatism.

Figure 2.

Visual field maps of oblique astigmatism for the whole eye (A), the isolated cornea (B), the isolated lens (D), and for internal optics (C) computed by vector subtraction of map B from map A. Colour indicates magnitude of astigmatism interpolated from sample points shown by symbols. Lines drawn through symbols indicate axis of oblique astigmatism. The magnitude of oblique astigmatism J (dioptries) as a function of eccentricity e is fit with the quadratic equation $J = \sigma e^2$ where σ (in $10^{-3} D/deg^2$) is the regression coefficient shown in the lower left corner of each map. Other modelling parameters are given in Methods.

Figure 3.

Cross sections of field maps in Fig. 2 along the superior vertical meridian. For this meridian, positive J_0 values indicate a tangential axis, whereas negative values indicate a radial axis. The quadratic coefficient σ is in $10^{-3} D/deg^2$.

Figure 4.

Effect of surface asphericity on oblique astigmatism of the isolated lens for collimated light (vergence = 0D) produced by a real object located at infinity (A) and for converging light (vergence = +42.1D at the corneal plane) produced by a virtual object (B). Abscissa shows conic constant for anterior surface and ordinate shows conic constant for posterior surface. Colour indicates the signed value of oblique astigmatism at 13.5° eccentricity in the superior visual field.

Figure 5.
Range of lens shapes examined by altering surface curvatures of the model lens.

Figure 6.

Nomogram showing the effect of surface curvature on oblique astigmatism of (A) the isolated lens for collimated light (vergence = 0D), (B) for the isolated lens with converging light (vergence = +42.1D at the corneal plane), and (C) for internal optics. Abscissa shows radius of curvature of the anterior surface and ordinate shows radius of curvature of the posterior surface of the model lens. Colour indicates the signed value of oblique astigmatism at 13.5° eccentricity in the superior visual field. The conic constants for anterior and posterior lens are -2.13 and 0 respectively. Black cross symbol indicates the Navarro eye

model, black circles indicate Rosales data[18], and black diamonds indicate Dubbelman data[19]. See text for further description of contour lines and symbols.

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Figure 7.

Effect of corneal power (assuming zero corneal astigmatism) on the signed value of internal oblique astigmatism at 13.5° eccentricity in the superior visual field. The conic constants for anterior and posterior lens are -2.13 and 0 respectively.

Figure 8.

Visual field maps of a human eye demonstrating tangentially-oriented internal oblique-astigmatism compensating for corneal oblique-astigmatism. Maps of oblique astigmatism for the whole eye (A), the isolated cornea (B), and for internal optics (C) computed by vector subtraction of map B from map A are shown for an eye with negligible axial astigmatism. Colour indicates magnitude of astigmatism interpolated from sample points shown by symbols. The magnitude of astigmatism for object location 13.5° in the superior

visual field is labelled. Lines drawn through symbols indicate axis of oblique astigmatism. Red diamonds indicates optical axis, along which oblique astigmatism vanishes.