ON GENERALIZED STANDARD ALGEBRAS*

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Among finite-dimensional nonassociative algebras, alternative algebras and (commutative) Jordan algebras are outstanding for the depth of the results which have been achieved. There are striking parallels between the theories for these two classes of algebras that lead to the expectation that a unified theory exists, embracing both classes. Noncommutative Jordan algebras are too inclusive, since as basic a theorem as the Wedderburn principal theorem fails to hold for these algebras.¹ A significant step toward a unified theory was taken by A. A. Albert in defining *standard algebras*,² which include all associative algebras and all (commutative) Jordan algebras.

In this note we define *generalized standard algebras* by identities (1)–(5), which are satisfied in all alternative algebras and standard algebras (hence in commutative Jordan algebras). We assume characteristic $\neq 2$ throughout. We state the fundamental theorems in a structure theory for finite-dimensional generalized standard algebras. Proofs of these results will appear elsewhere.

We shall call a nonassociative algebra $\mathfrak{A}$ over F (of characteristic $\neq 2$) a *generalized standard algebra* in case the following five conditions are satisfied:

$$\mathfrak{A} \text{ is flexible; that is, } (xy)x = x(yx) \text{ for all } x, y \text{ in } \mathfrak{A}; \quad (1)$$

$$H(x, y, z)x = H(x, y, xz) \text{ for all } x, y, z \text{ in } \mathfrak{A}, \text{ where } H(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) \text{ and } (x, y, z) = (xy)z - x(yz); \quad (2)$$

$$(x, y, wz) + (w, y, xz) + (z, y, xw) = [x, (w, z, y)] + (x, w, [y, z]) \text{ for all } w, x, y, z \text{ in } \mathfrak{A}, \text{ where } [x, y] = xy - yx; \quad (3)$$

$$\text{if } F \text{ has characteristic } 3, \text{ then } (x, y, x^2) = 0 \text{ for all } x, y \text{ in } \mathfrak{A}; \quad (4)$$

$$[L_x, L_y] + [L_x, R_y] + [R_x, R_y] \text{ is a derivation of } \mathfrak{A} \text{ for all } x, y \text{ in } \mathfrak{A}; \text{ that is, } y(x(wz)) - x(y(wz)) + (x, wz, y) + ((wz)x)y - ((wz)y)x = [y(xw) - x(yw) + (x, w, y) + (wx)y - (wy)x]z + w[y(xz) - x(yz) + (x, z, y) + (zx)y - (zy)x] \text{ for all } w, x, y, z \text{ in } \mathfrak{A}. \quad (5)$$

Any generalized standard algebra $\mathfrak{A}$ is a noncommutative Jordan algebra (hence power-associative).

THEOREM 1. *Any finite-dimensional generalized standard nilalgebra is nilpotent.*

Thus, in any finite-dimensional generalized standard algebra $\mathfrak{A}$, there exists a unique maximal nilpotent (= solvable = nil) ideal $\mathfrak{N}$ of $\mathfrak{A}$, the *radical* of $\mathfrak{A}$, and $\mathfrak{A}$ is called *semisimple* in case $\mathfrak{N} = 0$. In order to obtain the structure of semisimple algebras (Theorem 3), we determine first the simple algebras (Theorem 2).

Relative to any idempotent element e in a generalized standard algebra $\mathfrak{A}$, we define *Peirce spaces* $\mathfrak{A}_i$ ($i = 0, 1$) and $\mathfrak{A}_{ij}$ ($i + j = 1$; $i, j = 0, 1/2, 1$) by

$$\mathfrak{A}_i = \{x \in \mathfrak{A} | ex = xe = ix\}, \quad i = 0, 1, \quad (6)$$

and

$$\mathfrak{A}_{i,j} = \{x \in \mathfrak{A} | ex = ix, xe = jx\}, \quad i+j = 1; \quad i, j = 0, 1/2, 1. \quad (7)$$

Then $\mathfrak{A}$ is the vector space direct sum of the five Peirce spaces (6) and (7). Multiplicative relationships in this Peirce decomposition combine the features of both alternative algebras and commutative Jordan algebras. We use the Peirce decomposition in the proof of

THEOREM 2. *Any finite-dimensional simple generalized standard algebra is either alternative or a commutative Jordan algebra.*

These finite-dimensional simple algebras are completely known. In the light of Kleinfeld's theorems on simple standard and accessible rings,³ we conjecture that any simple generalized standard ring (of characteristic $\neq 2$) is either alternative or a commutative Jordan ring.

THEOREM 3. *Any finite-dimensional semisimple generalized standard algebra $\mathfrak{A}$ is uniquely expressible as a direct sum $\mathfrak{A} = \mathfrak{A}_1 \oplus \dots \oplus \mathfrak{A}_r$ of simple ideals $\mathfrak{A}_i$.*

Let $\mathfrak{B}$ be any ideal in a generalized standard algebra $\mathfrak{A}$. Then $\mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}^2 = \mathfrak{B}^2\mathfrak{A} + \mathfrak{B}^2$ is an ideal of $\mathfrak{A}$. Hence any ideal $\mathfrak{B}$ leads to a descending chain

$$\mathfrak{B}^{(0)} \supseteq \mathfrak{B}^{(1)} \supseteq \dots \supseteq \mathfrak{B}^{(k)} \dots$$

of ideals $\mathfrak{B}^{(i)}$ of $\mathfrak{A}$ defined by $\mathfrak{B}^{(0)} = \mathfrak{B}$, $\mathfrak{B}^{(i+1)} = \mathfrak{A}(\mathfrak{B}^{(i)})^2 + (\mathfrak{B}^{(i)})^2$. We call $\mathfrak{B}$ *Penico solvable* in case there is some k for which $\mathfrak{B}^{(k)} = 0$. As an important tool in the proof of the Wedderburn principal theorem, we have

THEOREM 4. *Any solvable ideal of a finite-dimensional generalized standard algebra is Penico solvable.*

THEOREM 5 (Wedderburn principal theorem). *Let $\mathfrak{A}$ be a finite-dimensional generalized standard algebra, and let $\mathfrak{N}$ be the radical of $\mathfrak{A}$. If $\mathfrak{A}/\mathfrak{N}$ is separable, then $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$ where $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$, $\mathfrak{S} \cong \mathfrak{A}/\mathfrak{N}$.*

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¹ Schafer, R. D., *An Introduction to Nonassociative Algebra* (New York and London: Academic Press, 1966), p. 147. Also see this book for general terminology concerning nonassociative algebras.

² Albert, A. A., "Power-associative rings," *Trans. Am. Math. Soc.*, **64**, 552-593 (1948). For other papers on standard algebras, see Kleinfeld, E., "Standard and accessible rings," *Can. J. Math.*, **8**, 335-340 (1956); Penico, A. J., "On the structure of standard algebras," in *Proceedings International Congress of Mathematicians, 6th, Cambridge, Mass., 1950* (Providence, R.I.: Am. Math. Soc., 1952), abstr., vol. 1, p. 320; Schafer, R. D., "Standard algebras," to appear in *Pacific J. Math.*

³ See Kleinfeld's paper cited in ref. 2.