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Fourier rebinning and consistency equations for time-of-flight PET planograms

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Abstract

Due to the unique geometry, dual-panel PET scanners have many advantages in dedicated breast imaging and on-board imaging applications since the compact scanners can be combined with other imaging and treatment modalities. The major challenges of dual-panel PET imaging are the limited-angle problem and data truncation, which can cause artifacts due to incomplete data sampling. The time-of-flight (TOF) information can be a promising solution to reduce these artifacts. The TOF planogram is the native data format for dual-panel TOF PET scanners, and the non-TOF planogram is the 3D extension of linogram. The TOF planograms is five-dimensional while the objects are three-dimensional, and there are two degrees of redundancy. In this paper, we derive consistency equations and Fourier-based rebinning algorithms to provide a complete understanding of the rich structure of the fully 3D TOF planograms. We first derive two consistency equations and John's equation for 3D TOF planograms. By taking the Fourier transforms, we obtain two Fourier consistency equations and the Fourier-John equation, which are the duals of the consistency equations and John's equation, respectively. We then solve the Fourier consistency equations and Fourier-John equation using the method of characteristics. The two degrees of entangled redundancy of the 3D TOF data can be explicitly elicited and exploited by the solutions along the characteristic curves. As the special cases of the general solutions, we obtain Fourier rebinning and consistency equations (FORCEs), and thus we obtain a complete scheme to convert among different types of PET planograms: 3D TOF, 3D non-TOF, 2D TOF and 2D non-TOF planograms. The FORCEs can be used as Fourier-based rebinning algorithms for TOF-PET data reduction, inverse rebinnings for designing fast projectors, or consistency conditions for estimating missing data. As a byproduct, we show the two consistency equations are *necessary* and *sufficient* for 3D TOF planograms. Finally, we give numerical examples of implementation of a fast 2D TOF planogram projector and Fourier-based rebinning for a 2D TOF planograms using the FORCEs to show the efficacy of the Fourier-based solutions.

Keywords

Fourier rebinning; consistency equations; Fourier consistency equations; John's equation; planogram; time-of-flight (TOF); positron emission mammography (PEM); positron emission tomography (PET)

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1. Introduction

A current trend in tomographic imaging is the adaptation of general cylindrical PET scanners to specialized PET scanners for dedicated applications, with the intent of improved spatial resolution and increased detection sensitivity. Most of these efforts go to the developments of dedicated breast scanners and on-board imaging for image-guided proton therapy. Breast cancer remains a leading disease among women in many parts of the world [31], and early detection of breast abnormality is important to achieve a high cure rate of breast cancer [52]. Positron emission mammography/tomography (PEM/PET) is an important imaging tool for breast cancer detection, characterization, treatment planning, and assessment of response to therapy. Due to their unique geometry, dual-panel PET scanners have many advantages in these dedicated applications since the compact scanners can be flexibly positioned around imaging objects and combined with other imaging modalities, e.g., X-ray, and treatment modalities, e.g., proton therapy. Many dedicated breast scanners using dual-panel detectors have been investigated and developed [61, 23, 20, 48, 1, 55, 56, 64, 3, 43, 13]. The dual-panel scanners are also applied to image-guided proton therapy to provide on-board imaging, which is crucial to verify and monitor that the proton irradiation is concentrated on a tumor [51, 62, 59, 35].

For 3D cylindrical scanners, the acquired data satisfy Orlov's condition [53], and TOF information is redundant in the sense that exact reconstructions can be performed after summing all TOF bins [39]. The redundant TOF information helps to improve the contrast-to-noise ratio (CNR) of the reconstructed images. For 3D dual-panel scanners, the acquired data violate Orlov's condition, and exact image reconstruction is not possible. However, the TOF information can reduce the artifacts due to incomplete data sampling [58, 63, 36, 60]. The major challenges of stationary dual-panel PET imaging are the limited-angle problem, data truncation and depth-of-interaction (DOI) effects, which can cause artifacts due to incomplete data sampling and anisotropic blurring.

The goal of the paper is to investigate the planogram data parametrization for TOF-PET scanners. The TOF planogram is the native data format for dual-panel TOF-PET scanners. The non-TOF planogram is a 3D extension of the 2D linogram [22, 21], all the LORs passing through a fixed point inside the field of view (FOV) lie on a two-dimensional plane in the four-dimensional space of the non-TOF planograms [5]. For 3D non-TOF planograms, the Fourier rebinning using the frequency-distance relationship was developed by Champley *et al* [6, 7]. The Fourier rebinning based on John's equation in the local detector coordinates was developed by Defrise and Liu [17], Kao *et al* [32]. Brasse *et al* [5] developed a filtered-backprojection (FBP) algorithm for 3D non-TOF planograms, which is a 3D extension of the FBP algorithm for linograms [22, 21]. Fourier rebinning algorithms were developed for 3D non-TOF and TOF cylindrical PET scanners [16, 41, 17, 15, 8, 9]. The consistency equations were also developed for 3D cylindrical TOF-PET scanners [18, 38, 39]. The existing Fourier-based rebinning algorithms and consistency equations developed for cylindrical TOF-PET geometry are not directly applicable to 3D TOF dual-panel scanners. One can convert 3D TOF planograms to 3D TOF sinograms, however, the conversion needs interpolation and produces sub-optimal results. The TOF planograms is five-dimensional while the objects are three-dimensional, and there are two degrees of redundancy. In this

paper, we derive Fourier-based rebinning and consistency equations to provide a complete understanding of the rich structure of the fully 3D TOF planograms. We give the shadow zones (the regions in 3D Fourier space where the Fourier transform of the imaging object cannot be measured) for 3D non-TOF and TOF limited-angle planograms, and show that the TOF measurement provides both redundant and new information. We also show that 3D non-TOF planograms provide more information than the corresponding 2D non-TOF planograms when the PET data violates Orlov's condition. The unified theory of TOF planograms developed in this paper can be useful to develop fast image reconstructions algorithms using Fourier-based rebinning or inverse Fourier rebinning, and it can also be useful for dual-panel system designs and optimizations.

2. TOF-PET data formation

2.1. A unified framework for 3D TOF data

First, we propose a general parametrization scheme for the 3D TOF-PET data of an unknown activity distribution $f \in C_0(\mathbb{R}^3)$ with an arbitrary geometry, which can accommodate the native data parametrization for a large class of scanner geometries. In general, an LOR with TOF measurement can be parameterized by $(\vec{r}, \hat{n})$ with $\vec{r} \in \mathbb{R}^3$ and $\hat{n} \in \mathcal{S}^2$. Here, the vector $\vec{r}$ includes two projection variables and one TOF variable, and $\hat{n}$ defines the TOF direction of the LOR. The relation between $\vec{r}$ and the object space coordinates is defined by a non-singular 3×3 matrix $A(\hat{n})$, which characterizes the selected data parametrization scheme. The general 3D TOF data can be formulated as

$$p(\vec{r}, \hat{n}) = \int_{-\infty}^{\infty} dl h(-l) f(A(\hat{n})\vec{r} + l\hat{n}), \quad (1)$$

where h is usually modeled as a Gaussian TOF profile with standard deviation σ ,

$$h(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{t^2}{2\sigma^2}\right). \quad (2)$$

We introduce the Fourier transform of p with respect to $\vec{r}$

$$\hat{p}(\vec{\omega}, \hat{n}) = \int \int \int_{\mathbb{R}^3} d^3\vec{r} p(\vec{r}, \hat{n}) e^{-j\vec{\omega} \cdot \vec{r}}, \quad (3)$$

where $\vec{\omega}$ is the frequency vector conjugate to $\vec{r}$. Putting (1) into (3), changing integration order and applying the Fourier stretch theorem [4], we obtain

$$\hat{p}(\vec{\omega}, \hat{n}) = \frac{1}{|\det \mathbf{A}(\hat{n})|} \hat{f}(\mathbf{A}^{-\top}(\hat{n}) \vec{\omega}) \int_{-\infty}^{\infty} dl h(l) e^{-jl \vec{\omega} \cdot \mathbf{A}^{-1}(\hat{n}) \hat{n}}, = \frac{1}{|\det \mathbf{A}(\hat{n})|} \hat{h}(\langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \vec{\omega} \rangle) \hat{f}(\mathbf{A}^{-\top}(\hat{n}) \vec{\omega}), \quad (4)$$

where $\hat{h}(\omega_t) = e^{-\sigma^2 \omega_t^2 / 2}$ is the Fourier transform of $h(t)$. Equation (4) is the generalized Fourier-slice theorem for 3D TOF data. For 3D TOF sinogram parametrization, the corresponding Fourier-slice theorem is given by (2) in Cho *et al* [9] and (7) in Li *et al* [39]. For histo-image parametrization [46], the transfer matrix is an identity matrix, i.e., $\mathbf{A}(\hat{n}) = \mathbf{I}$, an equivalent form is given by (A.4) in Li *et al* [37]. For the 3D parallel projection data format used for the 3DRP algorithm, we refer the corresponding data format with TOF measurement as TOF Kinahanograms⁴. And the transfer matrix for a TOF Kinahanogram is a unitary matrix, i.e., $\mathbf{A}^{-\top}(\hat{n}) = \mathbf{A}(\hat{n})$ [34, 33]. For non-TOF PET data, $\sigma \rightarrow \infty$, $\sigma \hat{h}(\omega_t) / \sqrt{2\pi}$ becomes the Dirac delta function, and (4) becomes the classical Fourier-slice theorem [50]. The generalized consistency equations for 3D TOF data with an arbitrary matrix $\mathbf{A}(\hat{n})$ are derived in appendix A. We focus in the rest of the article on the case of dual-panel detectors.

2.2. TOF-PET planogram

As illustrated in figure 1, the planogram is the native data acquisition format for dual-panel PET scanners with the two detectors located at the planes $y = \pm R$. Two detector elements located at $A = (x_a, -R, z_a)$ and $B = (x_b, R, z_b)$ determine an LOR, and we can parameterize

the LOR by the coordinates (r_1, r_2, u, v) where $r_1 = \frac{1}{2}(x_a + x_b)$, $r_2 = \frac{1}{2}(z_a + z_b)$, $u = \frac{1}{2R}(x_b - x_a)$ and $v = \frac{1}{2R}(z_b - z_a)$. The orientation of the LOR is $\hat{n} = \frac{1}{\sqrt{1+u^2+v^2}}[u, 1, v]^{\top}$ with slope parameters u and v along which the TOF t is measured. The TOF data in planogram format are

$$p(r_1, r_2, t; u, v) = \int_{-\infty}^{\infty} dl f(r_1 \hat{d}_1 + r_2 \hat{d}_2 + l \hat{n}) h(t - l), \quad (5)$$

where h is given in (2) and the three unit vectors are given by

$$\hat{d}_1 = [1, 0, 0]^{\top}, \hat{d}_2 = [0, 0, 1]^{\top}, \hat{n} = \frac{1}{\sqrt{1+u^2+v^2}}[u, 1, v]^{\top}. \quad (6)$$

⁴We introduce the term ‘Kinahanogram’ after an email communication with Dr Paul E Kinahan on October 27, 2015.

Equation (5) is the 3D TOF planogram formulation, which is a special case of (1) with transfer matrix $A(\hat{n}) = [\hat{d}_1, \hat{d}_2, \hat{n}]$. Similar to the 2D TOF sinograms (subset of the 3D PET data that have zero ring difference), we have the 2D TOF planograms with zero axial slope $v = 0$, i.e., both photons are received by the same axial detector slice. By changing variable $l = \sqrt{1+u^2+v^2}y$, we can rewrite (5) as

$$p(r_1, r_2, t; u, v) = \sqrt{1+u^2+v^2} \int_{-\infty}^{\infty} dy f(r_1+uy, y, r_2+vy) h\left(t - y \sqrt{1+u^2+v^2}\right). \quad (7)$$

When the TOF profile $h = 1$, (7) becomes the non-TOF planogram formulation [5, 6, 7]. Similar to (4), we can use $\hat{f}$ to represent the Fourier transform $\hat{p}$ in (5) as

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, v) = \sqrt{1+u^2+v^2} \hat{h}(\omega_t) \hat{f}\left(\omega_1 \hat{d}_1^* + \omega_2 \hat{d}_2^* + \omega_t \hat{n}^*\right), \quad (8)$$

where

$$\hat{d}_1^* = [1, -u, 0]^\top, \hat{d}_2^* = [0, -v, 1]^\top, \hat{n}^* = [0, \sqrt{1+u^2+v^2}, 0]^\top. \quad (9)$$

Here, $\hat{d}_1^*$, $\hat{d}_2^*$ and $\hat{n}^*$ are the three dual basis vectors of $\hat{d}_1$, $\hat{d}_2$ and $\hat{n}$, and they are the three column vectors of $A^*(\hat{n}) = A^{-\top}(\hat{n})$, i.e., $A^*(\hat{n}) = [\hat{d}_1^*, \hat{d}_2^*, \hat{n}^*]$. The two sets basis vectors denoted by $A(\hat{n})$ and $A^*(\hat{n})$ form a biorthogonal system. We can also rewrite (8) into the following non-vector form as

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, v) = \sqrt{1+u^2+v^2} \hat{h}(\omega_t) \hat{f}\left(\omega_1, -u\omega_1 - v\omega_2 + \sqrt{1+u^2+v^2}\omega_t, \omega_2\right). \quad (10)$$

Equation (8) and (10) are the generalized Fourier-slice theorem for 3D TOF planograms. Here, the TOF profile h can be an arbitrary one-dimensional function, though a Gaussian TOF profile is usually used. A non-TOF version of (10) with $\omega_t = 0$ is given by (23) in Brasse *et al* [5], and a similar version is given by (11) in Mariano-Goulart and Crouzet [44]. Here we use (5) and (8) as the two basis parametrizations for TOF planograms in respectively spatial and Fourier domains to develop Fourier rebinning and consistency equations.

To achieve full angular sampling for a dual-panel scanner, TOF data may be collected for a second position with the scanner rotated around the z-axis by 90° counter clockwise. The TOF planogram from this second scanner orientation is then formulated as

$$p_2(r_1, r_2, t; u, v) = \int_{-\infty}^{\infty} dl f \left(r_1 \hat{d}_1^{(\text{rot})} + r_2 \hat{d}_2^{(\text{rot})} + l \hat{n}^{(\text{rot})} \right) h(t - l), \quad (11)$$

where

$$\hat{d}_1^{(\text{rot})} = [0, 1, 0]^T, \hat{d}_2^{(\text{rot})} = [0, 0, 1]^T, \hat{n}^{(\text{rot})} = \frac{1}{\sqrt{1+u^2+v^2}} [1, -u, -v]^T. \quad (12)$$

The Fourier transform of (11) is the rotated version of the Fourier-slice theorem, which is given by

$$\hat{p}_2(\omega_1, \omega_2, \omega_t; u, v) = \sqrt{1+u^2+v^2} \hat{h}(\omega_t) \hat{f} \left(u\omega_1 + v\omega_2 - \sqrt{1+u^2+v^2} \omega_t, \omega_1, \omega_2 \right). \quad (13)$$

By comparing (11) with (5), we can obtain the relation between p_2 and p as

$$p_2(r_1, r_2, t; u, v) = p \left(\frac{r_1}{u}, \text{sign}(u)t + \frac{\sqrt{1+u^2+v^2}}{|u|} r_1, r_2 - \frac{vr_1}{u}; -\frac{1}{u}, \frac{v}{u} \right). \quad (14)$$

One can verify that (14) is equivalent to (E.4) with $\psi = -\frac{\pi}{2}$ since rotating the scanner by 90° is equivalent to rotating the object by -90° around the z-axis. By taking the Fourier transform of (14), we obtain the Fourier domain relation between $\hat{p}_2$ and $\hat{p}$ are

$$\hat{p}_2(\omega_1, \omega_2, \omega_t; u, v) = |u| \hat{p} \left(u\omega_1 + v\omega_2 - \omega_t \sqrt{1+u^2+v^2}, \omega_2, \text{sign}(u)\omega_t; -\frac{1}{u}, \frac{v}{u} \right), \quad (15)$$

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, v) = |u| \hat{p}_2 \left(-u\omega_1 - v\omega_2 + \omega_t \sqrt{1+u^2+v^2}, \omega_2, -\text{sign}(u)\omega_t; \frac{-1}{u}, \frac{-v}{u} \right). \quad (16)$$

2.3. Planogram data truncation with dual-panel detectors

To measure the complete planograms over all azimuthal angles with only one detector orientation, i.e., to satisfy Orlov's condition [53], infinitely large dual-panel detectors are required. Infinite large detectors are impractical, in addition large detectors can introduce large depth-of-interaction (DOI) parallax errors. In practice, finite-sized planar detectors are used, and there is a tradeoff between angular coverage and proximity. We consider the two detectors having width $2L$ and height $2H$. As shown in figure 2, the measured TOF

planogram domain is the direct product of the two diamond regions and the TOF region given by

$$M = \{(r_1, u) \in \mathbb{R}^2: |r_1| + R|u| \leq L\} \times \{(r_2, v) \in \mathbb{R}^2: |r_2| + R|v| \leq H\} \times \{t \in \mathbb{R}\}. \quad (17)$$

For real TOF data, t is measured in a finite region determined by the object support and the essential support of the TOF kernel h . In real imaging applications, the object f always has finite support. We follow Champley *et al* [6] and assume that the support of f is inside a cylindrical set S_f :

$$\text{supp}(f) \subset S_f \equiv \left\{ (x, y, z) \in \mathbb{R}^3: \sqrt{x^2 + y^2} \leq a, |z| \leq c \right\}, \quad (18)$$

where $a < \min(R, L)$ and $c < H$. In this case, the support of the TOF planogram is contained in the direct product of two hourglass-shaped regions (the boundaries of the first one are hyperbolas) given by

$$\text{supp}(p) \subset S_p \equiv \left\{ (r_1, u) \in \mathbb{R}^2: \frac{r_1^2}{a^2} - u^2 \leq 1 \right\} \times \{(r_2, v) \in \mathbb{R}^2: |r_2| - a|v| \leq c\} \times \{t \in \mathbb{R}\}. \quad (19)$$

The two diamond regions and the two hourglass-shaped regions are shown in figure 2. After observing the intersection between sets S_p and M , we see that there is no data truncation when

$$|u| \leq u_m = \frac{RL - a\sqrt{R^2 + L^2 - a^2}}{R^2 - a^2}, |v| \leq v_m = \frac{H - c}{R + a}. \quad (20)$$

In other words, all of the LORs passing through the cylindrical set S_f with $|u| \leq u_m$ and $|v| \leq v_m$ are measured by the finite-sized dual-panel detectors. When $u_m < |u| \leq L/R$ or $v_m < |v| \leq H/R$, there exist LORs passing through S_f that cannot be measured, i.e., the measured planograms with orientation (u, v) are truncated.

When the detector gantry is capable of rotation around the z-axis to acquire data at different azimuthal angles, the number of rotations/orientations N needs to be larger than or equal to

$\frac{\pi}{2 \tan^{-1} u_m}$ to satisfy Orlov's condition. For fixed N , we need the detector length $2L \geq 2R \tan \pi / (2N) + 2a \sqrt{1 + \tan^2 \pi / 2N}$. For instance, the detector length should be at least $2(R + 2a)$ and $2(R + 2a)/3$ for two and three rotations, respectively. We also show in figure 2(a) the measured region by the 90°-rotated detectors, i.e., the two back-to-back fan

regions, $|u| \leq (|r_1| + R)/L$, which can be obtained from (14) and (17). One can easily verify that the support of the planogram is contained in the union of the measured diamond region and the fan regions when $u_m = 1$, i.e., the combination of the two data sets with azimuthal angles of 0 and 90° satisfies Orlov's condition.

2.4. Shadow zone in Fourier domain of limited-angle TOF planograms

The measured planograms with dual-panel PET scanners with finite detectors have both limited-angle effects and data truncation; however, the planograms have no truncation when $u = u_m$ and $v = v_m$ for objects inside cylindrical set S_F . Here, we only consider the planograms with $u = u_m$ and $v = v_m$, which allows us to focus on the limited-angle effects of the TOF planograms. Since there is no data truncation, the Fourier transform of TOF planograms with respect to r_1 , r_2 , and t can be applied. The Fourier-slice theorem yields the relation between the Fourier transform of an object and the Fourier transform of the TOF planogram. The cut-off frequency or bandwidth for a Gaussian profile is difficult to determine. To overcome this difficulty, Slepian [57] introduced the *essential bandwidth*, i.e., the portion of the spectrum $\hat{h}(\omega)$ that contains most of the energy. We use $\omega_{t,m}$ to denote the essential bandwidth of the TOF profile. From (10) we can obtain the shadow zones (or missing regions) in which the frequency contents of the data are essentially zero and thus the corresponding frequency contents of the object cannot be obtained from the limited-angle TOF planograms. The shadow zone $S_{z,3D}$ for 3D TOF planograms can be given by

$$S_{z,3D} = \left\{ (\xi_x, \xi_y, \xi_z) \in \mathbb{R}^3 \mid |\xi_y| > u_m |\xi_x| + v_m |\xi_z| + \sqrt{1 + u_m^2 + v_m^2} |\omega_{t,m}| \right\}. \quad (21)$$

When $v_m = 0$, we have the shadow zone $S_{z,2D}$ for 2D TOF planograms

$$S_{z,2D} = \left\{ (\xi_x, \xi_y, \xi_z) \in \mathbb{R}^3 \mid |\xi_y| > u_m |\xi_x| + \sqrt{1 + u_m^2} |\omega_{t,m}| \right\}. \quad (22)$$

Since the spectra of imaging objects tend toward zero at high frequencies, it is natural to idealize the concept mathematically and assume the object spectrum is bandlimited to, or essentially bandlimited to, e.g., a spectral cube. The shadow zones for TOF and non-TOF planograms, as well as the spectral cube, are shown in figure 3. The shadow zone for TOF planograms in (21) can be compared to the shadow zone in cone beam computed tomography with a circular trajectory [24]. The difference regions between the non-TOF shadow zones and TOF shadow zones (3D and 2D) are associated to the TOF information. It is obvious that the volumes of the shadow zone of TOF planograms inside the spectral cube (3D and 2D) are smaller than those for the corresponding non-TOF planograms; the new information, i.e., the frequency content in the difference regions, is gained by TOF measurement. It is worth noting from figure 3 that the shadow zones of 3D non-TOF planograms are smaller than that of the corresponding 2D non-TOF planograms inside the spectral cube, which implies that 3D non-TOF planograms contain more information than

the corresponding 2D non-TOF planograms. The shadow zones can induce artifacts in the reconstructed images. When the scanner is capable of gantry rotation, shadow zones can be completely eliminated. Using large planar detectors, or improving TOF time resolution, can increase u_m , v_m and $\omega_{t,m}$, thus shrink the shadow zones. TOF measurement with currently achievable time resolution can reduce, but cannot eliminate, these artifacts; it may be necessary to fill the shadow zones using sophisticated extrapolation methods on these shadow zones.

3. Consistency equations for 3D TOF planograms

The space of 3D TOF planogram is five-dimensional, while the object space is three-dimensional. The two degrees of redundancy can be characterized by two independent partial differential equations (PDEs) known as consistency equations [18, 39]. In this section, we present the consistency equations for 3D TOF planograms assuming a Gaussian TOF profile. The derivation of consistency equations with arbitrary h is beyond the scope of the paper.

3.1. Planogram consistency equations

In appendix A, we derive the generalized consistency equations for 3D TOF data with an arbitrary geometry. Applying (A.4) and (A.5) with the transfer matrix $A(\hat{n}) = [\hat{d}_1, \hat{d}_2, \hat{n}]$ for the TOF planograms, we obtain the two basis spatial-domain consistency equations for 3D TOF planograms:

$$\frac{\partial p}{\partial u} - \frac{t}{\sqrt{1+u^2+v^2}} \frac{\partial p}{\partial r_1} + \frac{ut}{1+u^2+v^2} \frac{\partial p}{\partial t} = \frac{\sigma^2}{\sqrt{1+u^2+v^2}} \frac{\partial^2 p}{\partial t \partial r_1} - \frac{\sigma^2 u}{1+u^2+v^2} \frac{\partial^2 p}{\partial t^2}, \quad (23)$$

$$\frac{\partial p}{\partial v} - \frac{t}{\sqrt{1+u^2+v^2}} \frac{\partial p}{\partial r_2} + \frac{vt}{1+u^2+v^2} \frac{\partial p}{\partial t} = \frac{\sigma^2}{\sqrt{1+u^2+v^2}} \frac{\partial^2 p}{\partial t \partial r_2} - \frac{\sigma^2 v}{1+u^2+v^2} \frac{\partial^2 p}{\partial t^2}. \quad (24)$$

One can also directly verify that (23) and (24) are necessary by applying (7).

Other interesting consistency equations can be obtained by combining these two equations.

First, applying $\frac{\partial}{\partial r_2} - \frac{v}{\sqrt{1+u^2+v^2}} \frac{\partial}{\partial t}$ to (23) and $\frac{\partial}{\partial r_1} - \frac{v}{\sqrt{1+u^2+v^2}} \frac{\partial}{\partial t}$ to (24), and then adding the results, we obtain John's equation for 3D TOF planograms

$$\frac{v}{1+u^2+v^2} \frac{\partial p}{\partial r_1} + \frac{\partial^2 p}{\partial r_2 \partial u} - \frac{v}{\sqrt{1+u^2+v^2}} \frac{\partial^2 p}{\partial t \partial u} = \frac{u}{1+u^2+v^2} \frac{\partial p}{\partial r_2} + \frac{\partial^2 p}{\partial r_1 \partial v} - \frac{u}{\sqrt{1+u^2+v^2}} \frac{\partial^2 p}{\partial t \partial v}. \quad (25)$$

After removing the two terms with a t -derivative in (25), we obtain John's equation for 3D non-TOF planograms

$$\frac{v}{1+u^2+v^2} \frac{\partial p}{\partial r_1} + \frac{\partial^2 p}{\partial r_2 \partial u} = \frac{u}{1+u^2+v^2} \frac{\partial p}{\partial r_2} + \frac{\partial^2 p}{\partial r_1 \partial v}, \quad (26)$$

Equation (26) for 3D non-TOF planograms is essentially the same as John's equation [29, 17, 54]. The two first-order differential terms are due to the normalization factor

$$\sqrt{1+u^2+v^2}, \text{ and (26) can be simplified as } \left(\frac{\partial^2}{\partial r_2 \partial u} - \frac{\partial^2}{\partial r_1 \partial v} \right) \frac{p}{\sqrt{1+u^2+v^2}} = 0.$$

3.2. Fourier consistency equations

In appendix B, we derive the generalized Fourier consistency equations (FCEs) for 3D TOF data with an arbitrary geometry. Applying $A(\hat{n}) = [\hat{d}_1, \hat{d}_2, \hat{n}]$ to (B.2) and (B.3), we obtain the two Fourier-space consistency equations for 3D TOF planograms:

$$\frac{\partial \hat{p}}{\partial u} + \left(\frac{\omega_1}{\sqrt{1+u^2+v^2}} - \frac{u\omega_t}{1+u^2+v^2} \right) \frac{\partial \hat{p}}{\partial \omega_t} = \frac{u}{1+u^2+v^2} \hat{p} - \sigma^2 \left(\frac{\omega_1}{\sqrt{1+u^2+v^2}} - \frac{u\omega_t}{1+u^2+v^2} \right) \omega_t \hat{p}, \quad (27)$$

$$\frac{\partial \hat{p}}{\partial v} + \left(\frac{\omega_2}{\sqrt{1+u^2+v^2}} - \frac{v\omega_t}{1+u^2+v^2} \right) \frac{\partial \hat{p}}{\partial \omega_t} = \frac{v}{1+u^2+v^2} \hat{p} - \sigma^2 \left(\frac{\omega_2}{\sqrt{1+u^2+v^2}} - \frac{v\omega_t}{1+u^2+v^2} \right) \omega_t \hat{p}. \quad (28)$$

Equations (27) and (28) are the Fourier transforms (dual equations) of (23) and (24), and we refer them as the first and second FCEs for TOF planograms. One can also verify the above two Fourier consistency equations using (10).

Applying $\omega_2 - \frac{v\omega_t}{\sqrt{1+u^2+v^2}}$ to (27) and $\omega_1 - \frac{u\omega_t}{\sqrt{1+u^2+v^2}}$ to (28) and adding the results, we obtain

$$\left(\omega_2 - \frac{v\omega_t}{\sqrt{1+u^2+v^2}} \right) \frac{\partial \hat{p}}{\partial u} - \left(\omega_1 - \frac{u\omega_t}{\sqrt{1+u^2+v^2}} \right) \frac{\partial \hat{p}}{\partial v} = \frac{u\omega_2 - v\omega_1}{1+u^2+v^2} \hat{p}. \quad (29)$$

The above equation is the dual equation of the John's equation (25), and we refer this equation as the Fourier-John equation (FJE) for TOF planograms. Applying $\omega_t = 0$ in (29), we obtain Fourier-John equation for 3D non-TOF planograms

$$\omega_2 \frac{\partial \hat{p}}{\partial u} - \omega_1 \frac{\partial \hat{p}}{\partial v} = \frac{u\omega_2 - v\omega_1}{1+u^2+v^2} \hat{p}. \quad (30)$$

After considering the normalization factor $\sqrt{1+u^2+v^2}$, (30) can be simplified as

$$(\omega_2 \frac{\partial}{\partial u} - \omega_1 \frac{\partial}{\partial v}) \frac{\hat{p}}{\sqrt{1+u^2+v^2}} = 0.$$

4. Fourier-based rebinning equations among different data types

The generalized Fourier-slice theorem (10) is the solution to the Fourier consistency equations (27) and (28) and the Fourier-John equation (29). We can easily see from (7) that $\hat{p}/(\hat{h} \sqrt{1+u^2+v^2})$ is invariant if the argument of $\hat{f}$ is invariant under some coordinate transformations. This invariance/redundancy in Fourier space can be fully characterized by (27)–(29). By solving these equations, we can separate the two degrees of redundancy in 3D TOF data, which are entangled in (10). In appendices C and D, we give respectively the solution to the first consistency equation and the Fourier-John equation. These solutions can also be derived by applying the coordinate transformations to the Fourier-slice theorem (10), which in turn gives the geometric interpretation of the Fourier consistency equations [39]. The geometric interpretation of the redundancy in TOF-PET data is more general than the consistency equations since it does not require a Gaussian TOF profile. The equations developed in this section are valid independent of the TOF profile. Here we use the word ‘rebinning’ to refer to the process of generating a small data set, e.g. 2D non-TOF data, by a sophisticated data averaging from a large data set, e.g., 3D TOF data, using the consistency equations; we also use ‘inverse rebinning’ to refer to the reverse process.

4.1. Solutions to the Fourier consistency equations

Applying the method of characteristics [30, 12], the general solution to the first FCE (27) is (see appendix C for the detailed derivation)

$$\begin{aligned} \hat{p}(\omega_1, \omega_2, \omega_t; u, v) &= \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u_0^2+v_0^2}} \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \hat{p}(\omega_1, \omega_2, \omega_{t,0}; u_0, v), \\ \omega_t &= \frac{\sqrt{1+u_0^2+v_0^2}}{\sqrt{1+u^2+v^2}} \omega_{t,0} + \frac{(u-u_0)\omega_1}{\sqrt{1+u^2+v^2}}, \end{aligned} \quad (31)$$

where $\omega_{t,0}$ and u_0 are initial arguments. The characteristic curves are determined by the second equation in (31). For each initial point $(\omega_1, \omega_2, \omega_{t,0}; u_0, v)$, the values are propagated according to the first equation in (31) along the characteristic curve passing this point. The solution is obtained from the union of the family of the characteristic curves passing through different initial points. One can easily see that the general solution (31)

characterizes one degree of redundancy in the 3D TOF data. Alternatively, the planogram rebinning equation (31) can be derived from the planogram Fourier-slice theorem (10), which is similar to the derivation for the sinogram case [8, 9].

Similarly, by exchanging the role of the two slopes u and v , the general solution to the second FCE (28) is

$$\begin{aligned}\hat{p}(\omega_1, \omega_2, \omega_t; u, v) &= \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u^2+v_0^2}} \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \hat{p}(\omega_1, \omega_2, \omega_{t,0}; u, v_0), \\ \omega_t &= \frac{\sqrt{1+u^2+v_0^2}}{\sqrt{1+u^2+v^2}} \omega_{t,0} + \frac{\omega_2(v-v_0)}{\sqrt{1+u^2+v^2}}.\end{aligned}\quad (32)$$

We have $\omega_t = 0$ for non-TOF data, $v = 0$ for 2D PET data, so we can apply the special cases of (31) and (32) to obtain Fourier-based rebinning methods or mapping equations among different types of PET data.

In the rest of this section, we examine a few special cases and their applications. When $\omega_{t,0} = 0$, (31) becomes

$$\begin{aligned}\hat{p}(\omega_1, \omega_2, \omega_t; u, v) &= \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u_0^2+v^2}} \hat{h}(\omega_t) \hat{p}(\omega_1, \omega_2, 0; u_0, v), \\ \omega_t &= \frac{(u-u_0)\omega_1}{\sqrt{1+u^2+v^2}},\end{aligned}\quad (33)$$

One can use (33) to convert 3D non-TOF planograms to 3D TOF planograms, or exactly rebin 3D TOF planograms to 3D non-TOF planograms to reduce the redundancy.

When $\omega_{t,0} = 0$, (32) becomes

$$\begin{aligned}\hat{p}(\omega_1, \omega_2, \omega_t; u, v) &= \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u^2+v_0^2}} \hat{h}(\omega_t) \hat{p}(\omega_1, \omega_2, 0; u, v_0), \\ \omega_t &= \frac{(v-v_0)\omega_2}{\sqrt{1+u^2+v^2}}.\end{aligned}\quad (34)$$

Similar to (33), (34) can be used to convert 3D non-TOF planograms to 3D TOF planograms, or rebin 3D TOF planograms to 3D non-TOF planograms.

There is symmetry between (31) and (32) due to the symmetry of the two slopes u and v . By selecting v as the axial slope, we choose (32) as the primary consistency equation and only consider the special cases of (32) in the rest of this section. When $v_0 = 0$, (32) becomes

$$\begin{aligned}\hat{p}(\omega_1, \omega_2, \omega_t; u, v) &= \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u^2}} \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \hat{p}(\omega_1, \omega_2, \omega_{t,0}; u, 0), \\ \omega_t &= \frac{\sqrt{1+u^2}}{\sqrt{1+u^2+v^2}} \omega_{t,0} + \frac{v\omega_2}{\sqrt{1+u^2+v^2}}.\end{aligned}\quad (35)$$

Equation (35) can be used to convert 2D TOF planograms to 3D TOF planograms, or exactly rebin 3D TOF planograms to 2D TOF planograms for each axial panel defined by r_1 and u . It is worth noting that (35) becomes the single slice rebinning for TOF data (TOF-SSRB) when $\sigma \rightarrow 0$ [49, 47, 15]. Applying $\lim_{\sigma \rightarrow 0} e^{-\sigma^2 \omega_t^2 / 2} / e^{-\sigma^2 \omega_{t,0}^2 / 2} = 1$ and taking the inverse Fourier transform of (35) with respect to ω_1 , ω_2 and ω_t , we obtain the approximate rebinning formula

$$p(r_1, r_2, t; u, v) \approx p\left(r_1, r_2 + \frac{vt}{\sqrt{1+u^2+v^2}}, \frac{\sqrt{1+u^2}t}{\sqrt{1+u^2+v^2}}; u, 0\right). \quad (36)$$

When $\omega_{t,0} = 0$ and $v_0 = 0$, (32) becomes

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, v) = \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u^2}} \hat{h}(\omega_t) \hat{p}(\omega_1, \omega_2, 0; u, 0),$$

$$\omega_t = \frac{v\omega_2}{\sqrt{1+u^2+v^2}}. \quad (37)$$

Equation (37) can be used to exactly rebin 3D TOF planograms to 2D non-TOF planograms; however, TOF planograms may not be fully utilized since (37) only includes one degree of redundancy.

When $v = 0$, (31) becomes

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, 0) = \frac{\sqrt{1+u^2}}{\sqrt{1+u_0^2}} \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \hat{p}(\omega_1, \omega_2, \omega_{t,0}; u_0, 0),$$

$$\omega_t = \frac{\sqrt{1+u_0^2}}{\sqrt{1+u^2}} \omega_{t,0} + \frac{(u-u_0)\omega_1}{\sqrt{1+u^2}}. \quad (38)$$

Equation (38) exploits the redundancy in the 2D TOF planograms. The corresponding partial differential equation (23) with $v = 0$ can be used, e.g., to estimate the attenuation from 2D TOF emission data [19, 37].

When $\omega_{t,0} = 0$, (38) becomes

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, 0) = \frac{\sqrt{1+u^2}}{\sqrt{1+u_0^2}} \hat{h}(\omega_t) \hat{p}(\omega_1, \omega_2, 0; u_0, 0),$$

$$\omega_t = \frac{(u-u_0)\omega_1}{\sqrt{1+u^2}}. \quad (39)$$

This equation can be used to convert 2D TOF planograms to 2D non-TOF planograms, or rebin 2D TOF planograms to 2D non-TOF planograms to reduce the redundancy.

When $u_0 = 0$, (38) becomes

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, 0) = \sqrt{1+u^2} \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \hat{p}(\omega_1, \omega_2, \omega_{t,0}; 0, 0),$$

$$\omega_t = \frac{1}{\sqrt{1+u^2}} \omega_{t,0} + \frac{u\omega_1}{\sqrt{1+u^2}}. \quad (40)$$

In some cases, the 3D TOF data can be partitioned into many subsets of 2D TOF data, e.g., LORs in each subset lie on a plane that is parallel to the axial axis of the scanner. And 1D TOF data for each subset, e.g., the TOF data with axial slope $v = 0$, can be sufficient for the image reconstruction from all the subsets. The 1D TOF data can also be considered as one-view of histo-projection [46]. Equation (40) can be used to convert 2D TOF data to 1D TOF data, which has an embarrassingly parallel structure and can be easily implemented using massively parallel processors.

4.2. Solution to Fourier-John equation

Similar to (31), we can write the general solution to (29) as (see appendix D for the detailed derivation)

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, v) = \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u_0^2+v_0^2}} \hat{p}(\omega_1, \omega_2, \omega_t; u_0, v_0), (u-u_0)\omega_1 + (v-v_0)\omega_2 - \left(\sqrt{1+u^2+v^2} - \sqrt{1+u_0^2+v_0^2} \right) \omega_t = 0, \quad (41)$$

where u_0 and v_0 are initial arguments. Equation (41) can be considered as a combination of (31) and (32), and it is another useful equation to convert among different types of PET planograms.

When $\omega_t = 0$, (41) becomes

$$\hat{p}(\omega_1, \omega_2, 0; u, v) = \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u_0^2+v_0^2}} \hat{p}(\omega_1, \omega_2, 0; u_0, v_0), (u-u_0)\omega_1 + (v-v_0)\omega_2 = 0. \quad (42)$$

Equation (42) exploits the redundancy in 3D non-TOF planograms, and it can be applied, e.g., to estimate the missing data in 3D non-TOF planograms.

When $v_0 = 0$, (41) becomes

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, v) = \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u_0^2}} \hat{p}(\omega_1, \omega_2, \omega_t; u_0, 0), (u-u_0)\omega_1 + v\omega_2 - \left(\sqrt{1+u^2+v^2} - \sqrt{1+u_0^2} \right) \omega_t = 0. \quad (43)$$

Similar to (35), (43) can be used to rebin 3D TOF planograms to 2D TOF planograms, or to convert 2D TOF planograms to 3D TOF planograms.

When $\omega_t = 0$ and $u_0 = 0$, (41) becomes

$$\hat{p}(\omega_1, \omega_2, 0; u, v) = \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u_0^2}} \hat{p}(\omega_1, \omega_2, 0; u_0, 0), (u-u_0)\omega_1 + v\omega_2 = 0. \quad (44)$$

Equation (44) is the Fourier rebinning for 3D non-TOF planograms, which is equivalent to the exact Fourier rebinning for 3D non-TOF sinograms [16, 41, 17].

4.3. A unified picture for Fourier-based rebinning of TOF planograms

Based on the solutions to the two Fourier consistency equations and Fourier-John equation described in sections 4.1 and 4.2, we obtain a unified picture for converting different types of planograms as shown in figure 4. There are two degrees of redundancy in 3D TOF planograms, and one degree of redundancy in 3D non-TOF planograms and 2D TOF planograms. We have two operations for converting the 3D TOF planograms either to 2D TOF planograms or to 3D non-TOF planograms due to the two degrees of redundancy in 3D TOF planograms. Figure 4 shows the basis conversion scheme among different types of PET planograms. A similar mapping scheme was obtained for cylindrical scanners [39], and a simple mapping scheme was given in [9]. A combination of these conversions can also be applied. For instance, one can convert 3D TOF planograms to 2D non-TOF planograms by first converting to 2D TOF planograms using (35) or (43) and then converting to 2D non-TOF planograms using (39), or by first converting to 3D non-TOF planograms using (33) or (34) and then converting to 2D non-TOF planograms using (44). One can verify that the four combinations lead to the same equation as follows:

$$\hat{p}(\omega_1, \omega_2, \omega_t; u, v) = \hat{h}(\omega_t) \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+u_0^2}} \hat{p}(\omega_1, \omega_2, 0; u_0, 0), \quad \omega_t = \frac{(u-u_0)\omega_1 + v\omega_2}{\sqrt{1+u^2+v^2}}. \quad (45)$$

Similar to (37), (45) can be used to exactly rebin 3D TOF data to 2D non-TOF data, and the two degrees of redundancy are utilized since (45) is a composition of two Fourier-based

rebinning equations. Since $\hat{p}(\omega_1, \omega_2, 0; u_0, 0)$ depends only on three variables, we can write it as

$$\hat{p}(\omega_1, \omega_2, 0; u_0, 0) = \sqrt{1+u_0^2} \hat{f}(\omega_1, -u_0\omega_1, \omega_2). \quad (46)$$

Equation (46) is just the projection-slice theorem for 2D non-TOF planograms (linograms) [22, 5], but we consider it as the initial value for the two consistency equations for the 3D TOF planograms. This initial condition is also related to the Helgason-Ludwig consistency condition, which is a global condition for the Radon transform [25, 42, 11]. One can see

from (46) that the scaled 2D non-TOF Fourier data $\hat{p}(\omega_1, \omega_2, 0; u_0, 0) / \sqrt{1+u_0^2}$ is independent of u_0 when $\omega_1 = 0$. Taking the inverse Fourier transform of (46) with respect to ω_1 and ω_2 , we obtain

$$p_{2D, \text{non-TOF}}(r_1, r_2; u) \equiv \int_{-\infty}^{\infty} dt p(r_1, r_2, t; u, 0) = \sqrt{1+u_0^2} \int_{-\infty}^{\infty} dy f(r_1 + uy, y, r_2). \quad (47)$$

Equation (47) is the formulation of 2D non-TOF planograms, which can be considered as an initial condition in the formulation of 3D TOF planograms using the consistency equations. Putting the initial condition (46) into (45), we return to the generalized Fourier slice theorem (10). Finally, we have the following theorems for the sufficiency of the consistency equations for 3D planogram transform with TOF measurement.

Theorem 1 (Sufficiency of the two Fourier consistency equations). Let $\hat{p}(\omega_1, \omega_2, \omega_3; u, v) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^2)$ satisfy the two Fourier consistency equations (27) and (28), and $\hat{p}(\omega_1, \omega_2, 0; u, 0)$ satisfy the initial condition (46) with $\hat{f} \in C^\infty(\mathbb{R}^3)$, then $\hat{p}$ can be determined in the form of (10).

Proof. By solving the two consistency equations (27) and (28), as shown in appendix C, we give the general solutions in (31) and (32). By combining the two special cases (35) and (39) of the general solutions, we obtain (45) after applying the initial condition (46), which is equivalent to (10). We state in the theorem that $(u, v) \in \mathbb{R}^2$; however, one can apply the theorem only in the domain where $\hat{p}(\omega_1, \omega_2, \omega_3; u, v)$ and $\hat{p}(\omega_1, \omega_2, 0; u, v)$ are defined, e.g., the domain $|u| \leq u_m$ and $|v| \leq v_m$.

Theorem 2 (Sufficiency of the two consistency equations). Let a rapidly decreasing or compactly supported function $p(r_1, r_2, t; u, v) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^2)$ satisfy the two consistency equations (23) and (24), and the initial condition (47) with a rapidly decreasing or compactly supported function f on $\mathbb{R}^3$, then p can be determined in the form of (5), which is the 3D X-ray transform of the 3D object f with TOF measurement.

Proof. Equations (23) and (24) are the inverse Fourier transforms of (27) and (28); (5) is the inverse Fourier transforms of the Fourier-slice theorem (10). Applying **Theorem 1**, we prove that (23) and (24) are sufficient for 3D planogram transform with TOF measurement with

initial condition (47). The proof is based on Fourier transform; a mathematically rigorous proof of the theorem is beyond the scope of this paper.

It is obvious that (23) and (24) are necessary for (5), and (27) and (28) are necessary for (10) as well after applying Fourier transforms. So the two consistency equations are *necessary* and *sufficient* for 3D planogram transform with TOF measurement. The sufficiency of the two consistency equations in the sinogram format was proven for 3D X-ray transform with TOF measurement in Li *et al* [39]. For the non-TOF case, the sufficiency was proven by [29]. From **Theorem 2**, we know that there are two and only two independent consistency equations (23) and (24) for 3D TOF planograms. Any additional linear consistency equation must be a linear combination of the two consistency equations, e.g. John's equation (25).

5. Numerical examples

We developed a unified picture for converting different types of PET planograms using the FORCES. The full implementations of all the FORCES are beyond the scope of the paper. For 3D non-TOF planograms, a different rebinning based on the frequency-distance relationship was implemented in [6, 7]. Here, we perform numerical simulations with a generic 2D dual-panel PET system ($r_2 = 0$ and $v = 0$) with timing resolution of 300 ps FWHM and crystal size of 1.2 mm. We use a 120 mm diameter hot-rod phantom, which is discretized in 160×160 with 1 mm pixel size with an oversampling factor of 2^2 for each pixel, see figure 7 for the reconstructed images. There are nine hot spheres of 6 mm diameter with contrast ratio of 4 : 1 and six cold rods of 8 mm in the top-right half of the phantom. The distance between the centers of adjacent hot and cold rods is 16 mm. There are 35 TOF bins spaced by 7.5 mm which corresponds to 50 ps in time. We generated noise-free TOF planograms using a strip-integral model with a TOF profile of 45 mm FWHM and strip width of 1.2 mm. Scatter, randoms, attenuation, detector efficiency difference and depth-of-interaction (DOI) in detectors were not modeled.

5.1. 2D TOF planogram projector using Fourier-slice theorem

We implemented a fast 2D TOF planogram projector using Fourier-slice theorem (10). When complete sampling is satisfied, one can also perform the inverse Fourier rebinning to implement the fast TOF planogram projector from non-TOF planograms, similar to inverse Fourier rebinning from 2D non-TOF PET to 3D non-TOF PET [10]. The TOF planogram projector using the Fourier-slice theorem, similar to 3D-FRP algorithm [45], can be used in the implementation of fast iterative image reconstructions.

We first obtained the Fourier transform $\hat{f}$, and then performed gridded data interpolation to compute 2D TOF Fourier data $\hat{p}$ from $\hat{f}$ using (10). Since Fourier data $\hat{p}$ is Hermitian symmetric with respect to ω_1 , ω_t and ω_2 ; only the domain $\omega_1 \geq 0$ is needed. We applied B-spline interpolation method in each respective dimension of frequency variables (ξ_x, ξ_y) since it outperforms the bilinear method with similar computing time. We also generated reference TOF planograms using the strip-integral model with a TOF profile of 45 mm FWHM. The TOF planograms are $160 \times 121 \times 35$ along r_1 , u and t with $r_2 = 0$ and $v = 0$. The 121 samples along u are uniformly spaced over $(-1, 1)$, which corresponds to an angular coverage of 90° without data truncation. The comparison of the computed TOF planograms

with the corresponding reference planograms is shown in figure 5. The normalized root-mean-square errors (NRMSEs) of the four TOF planograms with TOF bins 0, 3, 6 and 9 are respectively 1.84%, 1.73%, 1.51% and 1.27%; the average NRMSE over 35 TOF bins is 1.23%.

5.2. Fourier-based rebinning of 2D TOF planograms and image reconstructions

We also performed Fourier-based rebinning from 2D TOF planograms to 2D non-TOF planograms. Two sets of planograms are generated with and without 90° rotation. Each set of planograms is $160 \times 121 \times 35$ along r_1 , u and t . The two combined data sets, having a complete data sampling, are concatenated along u . The concatenated data have an angular coverage of 180° but with a different angular sampling. We generated 60 noise realizations of noisy 2D TOF planograms with 1×10^6 total counts in each realization. We then performed Fourier-based rebinning to compute the 2D non-TOF planograms for each noise realization.

As shown in section 2.4, the TOF planogram of each data set contains more information than the corresponding non-TOF planogram. The rebinning of the first set of TOF planogram also contributes to the second set of non-TOF planogram, and vice versa. So we applied the relations between 90°-rotated Fourier data $\hat{p}_2$ and $\hat{p}$ in (15) and (16). Specifically, we computed the non-TOF Fourier data $\hat{p}(\omega_1, \omega_2, 0; u_0, 0)$ for the first data set of TOF planogram using the following weighted average

$$\begin{aligned} \hat{p}(\omega_1, \omega_2, 0; u_0, 0) &= \frac{1}{\int_{\omega_t \leq \sqrt{1+u_0^2}|\omega_1|} d\omega_t w_{\omega_1, u_0}(\omega_t)} \times \int_{\omega_t \leq \sqrt{1+u_0^2}|\omega_1|} d\omega_t \frac{w_{\omega_1, u_0}(\omega_t)}{\hat{h}(\omega_t)} \frac{\sqrt{1+u_0^2}}{\sqrt{1+u^2}} \hat{p}(\omega_1, \omega_2, \omega_t; u, 0), \omega_t \\ &= \frac{(u - u_0)\omega_1}{\sqrt{1+u^2}}. \end{aligned} \quad (48)$$

where $w_{\omega_1, u_0}(\omega_t)$ is the TOF weight for fixed ω_1 and u_0 , and the weight is zero when the corresponding u is outside the sampled region $|u| \leq 1$. Equation (48) is obtained from (39). In the 2D implementation, we only considered $\omega_2 = 0$ and $r_2 = 0$. We obtained u by solving $\omega_t = (u - u_0)\omega_1 / \sqrt{1+u^2}$, and we selected the root with smaller absolute value when there are two roots. The first data set of TOF planogram also contributes to the second data set of non-TOF planogram as

$$\begin{aligned}\hat{p}_2(\omega_1, \omega_2, 0; u_0, 0) &= \frac{1}{\int_{\omega_t \leq \sqrt{1+u_0^2}|\omega_1|} d\omega_t w'_{\omega_1, u_0}(\omega_t)} \times \int_{\omega_t \leq \sqrt{1+u_0^2}|\omega_1|} d\omega_t \frac{w'_{\omega_1, u_0}(\omega_t)}{\hat{h}(\omega_t)} \frac{\sqrt{1+u_0^2}}{\sqrt{1+u^2}} \hat{p}(u_0 \omega_1, \omega_2, \omega_t; u, 0), \omega_t \\ &= \frac{(u u_0 + 1) \omega_1}{\sqrt{1+u^2}}.\end{aligned}\quad (49)$$

where $w'_{\omega_1, u_0}(\omega_t)$ is the TOF weights for fixed ω_1 and u_0 . Equation (49) is obtained from (39) and (16) with $\omega_t = 0$ and $v = 0$. In the implementation of (49), u was obtained by solving $\omega_t = (u u_0 + 1) \omega_1 / \sqrt{1+u^2}$, and the root with smaller absolute value was selected when there are two roots. Similar to (48) and (49), we can also perform the Fourier-based rebinning for the second data set of TOF planogram. Again, $\hat{p}(\omega_1, \omega_2, 0; u_0, 0)$ and $\hat{p}_2(\omega_1, \omega_2, 0; u_0, 0)$ are Hermitian symmetric with respect to ω_1 and ω_2 , only the domain $\omega_1 = 0$ is needed. For optimal rebinning, we selected the TOF weights $\hat{h}^2(\omega_t) = \exp(-\sigma^2 \omega_t^2)$ for each pair of (ω_1, u_0) when the corresponding u is in the sampled region $|u| \leq 1$ [2]. Again, we applied the gridded B-Spline interpolation in respectively (ω_1, u) and $(u_0 \omega_1, u)$ after computing u from the second equations of (48) and (49) for each ω_t due to the relatively coarse sampling in ω_t . It is worth noting that the implementation of (48) only requires 1D interpolation in the frequency space, which is a benefit of TOF planograms compared to TOF sinograms.

The mean and variance of the rebinned non-TOF planograms were calculated and compared. The comparison of the rebinned planograms and the non-TOF planograms is shown in figure 6(a); the horizontal profiles of the mean and variance sinograms through $u = 0$ of the first data set are shown in figure 6(b).

We then performed image reconstructions from the noisy rebinned planograms using OSEM with 8 subsets and pixel size of 1 mm; these are called FORCE reconstructions. We also performed reconstructions from the non-TOF and TOF sinograms using the same OSEM for comparison. We ran up to 128 iterations to ensure that convergence can be achieved for all three types of reconstruction. The contrast recovery coefficients (CRCs) for the nine hot rods were also calculated [14, 38]. We selected six circle regions of 8 mm diameter in the bottom-left half of the phantom to calculate the background values. The six background regions are the mirrored regions of the six cold rods with respect to the diagonal line. Figure 7 shows a comparison of the sample, mean and variance of the three types of reconstructions (non-TOF, FORCE and TOF). The three types of reconstructed images in figures 7 have approximately matched CRC of 0.83 (shown as the dashed line in figure 9(a)) for the nine 6mm hot rods in the mean reconstructed images with respectively iteration numbers 21, 18 and 10. The central horizontal profiles of the mean and variance images through the three hot rods are shown in figure 8 for a quantitative comparison. The mean profiles are very similar; however, the variance profiles of FORCE and TOF reconstructions are much smaller

than the non-TOF reconstructions. The FORCE reconstructions from the rebinned planograms have similar variance (or marginally higher variance) compared to the corresponding TOF reconstructions.

For a more quantitative comparison, we calculated CRC and the standard deviation as image quality metric to compare the three types of reconstructions. The standard deviations for the hot rods were calculated as the square root of the mean variance obtained from the 60 noise realizations. The calculated CRC and standard deviation for the three types of reconstructions are shown in figure 9. We see the FORCE reconstructions from the rebinned non-TOF sinogram have similar performance compared to the TOF reconstructions based on the calculated CRC versus standard deviation; and both reconstructions show better performance than the non-TOF reconstructions.

The reconstruction code was implemented using MATLAB (MathWorks, Natick, MA); faster implementation can be achieved using C++ or parallel programming. We ran the reconstructions on a MacBook Pro (Mid 2014) with 2.5 GHz Intel Core i7-4870HQ and 16GB RAM. The reconstruction time for non-TOF and TOF with 128 iterations and 8 subsets were 6.6 s and 138 s, respectively. The Fourier rebinning using (48) and (49) took 1.5s for the two sets of TOF planograms including the overhead time, and most of the time was spent on the interpolation implemented using MATLAB function `griddedInterpolant`. The computational time for interpolation can be reduced by half after considering the Hermitian symmetric property. The FORCE reconstruction, combination of Fourier rebinning and non-TOF reconstruction, took about 8.1 s.

6. Discussion

In 3D time-of-flight PET imaging using dual-panel scanners, the data space of the TOF planograms are five-dimensional while the object is three-dimensional. We derived two independent consistency equations (23) and (24) to characterize the two degrees of redundancy. We also derived two Fourier consistency equations (FCEs) and the Fourier-John equation (FJE) for 3D TOF PET based on the generalized Fourier-slice theorem, which are the duals of the spatial-domain consistency equations and John's equation, respectively [18, 39]. We then solved the three equations using the method of characteristics and obtained the Fourier rebinning and consistency equations (FORCES). The two degrees of entangled redundancy of 3D TOF planograms can be explicitly elicited and exploited by the solutions of the FCEs and FJE along the characteristic curves, which give a complete understanding of the rich structure of 3D TOF planograms. Finally, we showed that the two consistency equations are necessary and *sufficient* for 3D TOF planograms. For non-TOF case, the sufficiency of John's equation for X-ray transform was proven by John [29]. For cylindrical TOF PET scanners, the sufficiency of the two consistency equations for 3D X-ray transform with TOF measurement was proved by Li *et al* [39].

For infinite large detectors, or finite dual-panel detectors with gantry rotations, the TOF planograms have complete sampling, i.e., satisfy Orlov's condition, which allows an exact tomographic reconstruction. In this case, TOF measurement provides redundant information about the object, and the redundant TOF information can reduce variance of the images

reconstructed for noisy measured TOF data, e.g., the TOF measurement in cylindrical TOF-PET systems [15, 39]. For finite stationary dual-panel detectors, the measured TOF planograms have both data truncations and limited-angle effects. The TOF planograms violate Orlov's condition, and an exact tomographic reconstruction is not possible. The reconstructed images always have artifacts caused by the incomplete data sampling. However, the TOF measurement provides both redundant and new information and reduces these artifacts. We showed in figure 3 the TOF measurement reduces the shadow zones in the Fourier domain, and thus provides new information about the objects. As we showed in section 2.4, TOF planograms contain more information than the corresponding non-TOF planograms. For the two sets of TOF planograms in the example, the combination of the two corresponding non-TOF planograms have complete data sampling due to the 180° angle coverage. So the TOF information in the first set of TOF planograms can actually contribute the 90° -rotated non-TOF planograms, and the contribution can be obtained using the FORCEs and relations in (15) and (16). We show in figure 10 the normalization factors in the Fourier domain, i.e., the rebinning weights integrated over ω_k used in the Fourier-based rebinning example. The limited-angle TOF planograms contains more information than the corresponding non-TOF planograms. Simply rebinning the limited-angle TOF planograms to non-TOF planograms causes information loss, which reduces the value of TOF measurement. We show in figure 11 the reconstructed image from the limited-angle planograms with 90° angle coverage (only one data set of the two sets). We see from figure 11 that the FORCE reconstruction has reduced noise. However, FORCE reconstruction has similar artifacts compared to the non-TOF reconstruction, and TOF reconstruction has reduced artifacts due to the new information contributed by TOF measurement.

In the numerical examples, we only considered limited-angle effects and generated non-truncated planograms. As shown in section 2.3, the acquired data using finite dual-panel detectors also have data truncation along r_1 and r_2 with $|u| = u_m$ or $|v| = v_m$. The FORCEs cannot be directly applied to the truncated data. When the scanner is capable of gantry rotation and acquired data satisfy Orlov's condition, the truncation problem is similar to the axial truncation in a cylindrical PET scanner. The 3DRP and 3D-FRP algorithms overcome this axial truncation by initially estimating the unmeasured oblique sinograms, which is done by forward projecting an initial image reconstructed from only the direct sinograms in spatial and Fourier domains, respectively [34, 45]. This approach was also adopted by the FOREX algorithm [16, 41]. Instead of projecting an initial image reconstructed from the direct sinograms, inverse Fourier rebinning can be applied to estimate the unmeasured planograms. When the acquired data violate Orlov's condition, the missing data can be estimated by integrating the PDEs (23) and (24) to complete the TOF planograms.

We obtained the general solutions (31), (32) and (41) to the two FCEs and FJE, from which we obtained the Fourier-based rebinning and consistency equations (FORCEs) in section 4 as the special cases of the general solutions. These FORCEs are in the planogram parametrization, and the corresponding FORCEs in sinogram parametrization are given in Li *et al* [39]. The FORCEs are generalized equations compared to the Fourier rebinning equation [16, 41] and the mapping equations [9]. We showed a clear connection between the mapping equations (including Fourier rebinning equation) and consistency equations—the mapping equations are the special cases of the general solutions to the Fourier consistency

equations. The FORCES can be used as Fourier-based rebinning algorithms for TOF-PET data reduction, or inverse rebinnings for designing fast projectors, or consistency conditions for estimating missing data. From these FORCES, we obtained a complete scheme to convert among different types of PET planograms: 3D TOF, 3D non-TOF, 2D TOF and 2D non-TOF, as shown in figure 4.

7. Conclusion

Fully 3D TOF planograms have both redundancy and new information compared to 3D or 2D non-TOF planograms, and the rich structure and redundancy are governed by the two consistency equations. In this paper, we first gave general solutions to the Fourier consistency equations and the Fourier-John equation, and then presented the Fourier-based rebinning and consistency equations (FORCES) as the special cases of the general solutions. We have obtained a unified and complete picture to convert between different types of PET planograms (3D TOF, 3D non-TOF, 2D TOF and 2D non-TOF) using the FORCES. As a byproduct, we have proven that the consistency equations are *necessary* and *sufficient* for 3D TOF planograms. The FORCES can be very useful in many TOF-PET imaging applications with dual-panel scanners, e.g., developing fast TOF image reconstructions using the Fourier-based rebinning or inverse Fourier rebinning, estimation of missing data and simultaneous emission and attenuation reconstructions from TOF-PET data without transmission scan. Finally, we presented numerical examples of implementation of Fourier-slice theorem and Fourier-based rebinning using FORCES based on a 2D dual-panel TOF PET, and we showed that the FORCE reconstructions from the rebinned non-TOF planograms can fully utilize TOF information and produce comparable results compared to the TOF OSEM reconstructions. The potential applications of the FORCES can only be partially demonstrated by the numerical examples.

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Appendix A. Generalized consistency equations for 3D TOF data

In this appendix, we derive the generalized consistency equations for 3D TOF data with a Gaussian TOF profile from an arbitrary scanner geometry defined by a non-singular matrix $\mathbf{A}$. By taking the total angular gradient of (1) with respect to $\hat{n}$ with fixed most likely annihilation point $\vec{x} = \mathbf{A}(\hat{n})\vec{r}$, we respectively obtain from left and right sides

$$\mathcal{D}_{\hat{n}}p = \nabla_{\hat{n}}p + \langle (\nabla_{\hat{n}}\mathbf{A}^{-1}(\hat{n}))\mathbf{A}(\hat{n})\vec{r}, \nabla p \rangle \quad (\text{A.1})$$

and

$$\begin{aligned}
\mathcal{D}_{\hat{n}}p &= \int_{-\infty}^{\infty} dl h(l) \mathbf{A}^{-T}(\hat{n}) \nabla f(\mathbf{A}(\hat{n}) \vec{r} + l\hat{n}), \\
&= -\sigma^2 \int_{-\infty}^{\infty} dl h'(l) \mathbf{A}^{-T}(\hat{n}) \nabla f(\mathbf{A}(\hat{n}) \vec{r} + l\hat{n}), \\
&= \sigma^2 \int_{-\infty}^{\infty} dl h(l) \frac{\partial}{\partial l} \left[\mathbf{A}^{-T}(\hat{n}) \nabla f(\mathbf{A}(\hat{n}) \vec{r} + l\hat{n}) \right], \\
&= \sigma^2 \int_{-\infty}^{\infty} dl h(l) \mathbf{A}^{-T}(\hat{n}) \nabla \langle \hat{n}, \mathbf{A}^{-T}(\hat{n}) \nabla \rangle f(\mathbf{A}(\hat{n}) \vec{r} + l\hat{n}), \\
&= \sigma^2 \mathbf{A}^{-T}(\hat{n}) \nabla \langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \nabla \rangle p,
\end{aligned} \tag{A.2}$$

where ∇ denotes the gradient with respect to $\vec{r}$ at fixed $\hat{n}$. Here we dropped the arguments $(\vec{r}, \hat{n})$ of p for conciseness; we used $\nabla_{\vec{x}} = \mathbf{A}^{-T}(\hat{n}) \nabla$ and $h'(l) = -lh(l)/\sigma^2$ for a Gaussian TOF profile given in (2). Putting (A.1) and (A.2) together, we obtain the generalized consistency equation (in vector calculus form)

$$\nabla_{\hat{n}} p + \langle (\nabla_{\hat{n}} \mathbf{A}^{-1}(\hat{n})) \mathbf{A}(\hat{n}) \vec{r}, \nabla p \rangle = \sigma^2 \mathbf{A}^{-T}(\hat{n}) \nabla \langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \nabla \rangle p. \tag{A.3}$$

Since the TOF direction $\hat{n}$ is normalized, i.e., $\|\hat{n}\| = 1$, $\nabla_{\hat{n}}$ is only meaningful in the directions perpendicular to $\hat{n}$. Without loss of generality, we assume that $\hat{n}$ depends on two variables η and ζ , i.e., $\hat{n} = \hat{n}(\eta, \zeta)$. The two variables can be azimuthal angle ϕ and elevation angle θ for TOF sinograms, or two slopes u and v for TOF planograms. We can then find two vectors,

$\frac{\partial \hat{n}}{\partial \eta}$ and $\frac{\partial \hat{n}}{\partial \zeta}$, that are perpendicular to $\hat{n}$ due to, e.g., $\hat{n} \cdot \frac{\partial \hat{n}}{\partial \eta} = \frac{1}{2} \frac{\partial \|\hat{n}\|^2}{\partial \eta} = 0$. All directions

perpendicular to $\hat{n}$ can be represented as a combination of $\frac{\partial \hat{n}}{\partial \eta}$ and $\frac{\partial \hat{n}}{\partial \zeta}$. By taking the inner

product with respectively $\frac{\partial \hat{n}}{\partial \eta}$ and $\frac{\partial \hat{n}}{\partial \zeta}$, we have the following two basis consistency equations

$$\frac{\partial p}{\partial \eta} + \left\langle \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \eta} \mathbf{A}(\hat{n}) \vec{r}, \nabla p \right\rangle = \sigma^2 \left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \eta}, \nabla \right\rangle \langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \nabla \rangle p, \tag{A.4}$$

$$\frac{\partial p}{\partial \zeta} + \left\langle \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \zeta} \mathbf{A}(\hat{n}) \vec{r}, \nabla p \right\rangle = \sigma^2 \left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \zeta}, \nabla \right\rangle \langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \nabla \rangle p. \tag{A.5}$$

Applying $\left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \zeta}, \nabla \right\rangle$ to (A.4) and $\left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \eta}, \nabla \right\rangle$ to (A.5) and comparing the results, we obtain the generalized John's equation for 3D TOF data

$$\left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \zeta}, \nabla \right\rangle \left(\frac{\partial p}{\partial \eta} + \left\langle \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \eta} \mathbf{A}(\hat{n}) \vec{r}, \nabla p \right\rangle \right) = \left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \eta}, \nabla \right\rangle \left(\frac{\partial p}{\partial \zeta} + \left\langle \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \zeta} \mathbf{A}(\hat{n}) \vec{r}, \nabla p \right\rangle \right). \quad (\text{A.6})$$

Appendix B. Generalized Fourier consistency equations for 3D TOF data

By taking the total angular gradient of (4) with respect to $\hat{n}$ with fixed $\vec{\xi} = \mathbf{A}^{-T}(\hat{n})\vec{\omega}$, we obtain the generalized Fourier consistency equation (in vector calculus form)

$$\nabla_{\hat{n}} \hat{p} + \left\langle \left(\nabla_{\hat{n}} \mathbf{A}^T(\hat{n}) \right) \mathbf{A}^{-T}(\hat{n}) \vec{\omega}, \nabla \hat{p} \right\rangle + \frac{\nabla_{\hat{n}} |\det \mathbf{A}(\hat{n})|}{|\det \mathbf{A}(\hat{n})|} \hat{p} = -\sigma^2 \langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \vec{\omega} \rangle \mathbf{A}^{-T}(\hat{n}) \vec{\omega} \hat{p}, \quad (\text{B.1})$$

where ∇ denotes the gradient with respect to $\vec{\omega}$. We used that $\hat{h}'(\omega_\rho) = -\sigma^2 \omega_\rho \hat{h}(\omega_\rho)$ and $\nabla_{\hat{n}} \langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \vec{\omega} \rangle |_{\vec{\xi}=\mathbf{A}^{-1}(\hat{n})\vec{\omega}} = \mathbf{A}^{-T}(\hat{n})\vec{\omega}$. Similar to (A.4) and (A.5), we can obtain the two basis Fourier consistency equations

$$\frac{\partial \hat{p}}{\partial \eta} + \left\langle \frac{\partial \mathbf{A}^T(\hat{n})}{\partial \eta} \mathbf{A}^{-T}(\hat{n}) \vec{\omega}, \nabla \hat{p} \right\rangle + \frac{1}{|\det \mathbf{A}(\hat{n})|} \frac{\partial |\det \mathbf{A}(\hat{n})|}{\partial \eta} \hat{p} = -\sigma^2 \langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \vec{\omega} \rangle \left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \eta}, \vec{\omega} \right\rangle \hat{p}, \quad (\text{B.2})$$

$$\frac{\partial \hat{p}}{\partial \zeta} + \left\langle \frac{\partial \mathbf{A}^T}{\partial \zeta} \mathbf{A}^{-T}(\hat{n}) \vec{\omega}, \nabla \hat{p} \right\rangle + \frac{1}{|\det \mathbf{A}(\hat{n})|} \frac{\partial |\det \mathbf{A}(\hat{n})|}{\partial \zeta} \hat{p} = -\sigma^2 \langle \mathbf{A}^{-1}(\hat{n}) \hat{n}, \vec{\omega} \rangle \left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \zeta}, \vec{\omega} \right\rangle \hat{p}. \quad (\text{B.3})$$

Applying $\left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \zeta}, \vec{\omega} \right\rangle$ to (A.4) and $\left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \eta}, \vec{\omega} \right\rangle$ to (A.5) and comparing the results, we obtain the generalized Fourier-John equation for 3D TOF data

$$\begin{aligned}
& \left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \zeta}, \vec{\omega} \right\rangle \left[\frac{\partial \hat{p}}{\partial \eta} \right. \\
& + \left\langle \frac{\partial \mathbf{A}^T(\hat{n})}{\partial \eta} \mathbf{A}^{-T}(\hat{n}) \vec{\omega}, \nabla \hat{p} \right\rangle + \frac{1}{|\det \mathbf{A}(\hat{n})|} \frac{\partial |\det \mathbf{A}(\hat{n})|}{\partial \eta} \hat{p} = \left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \hat{n}}{\partial \eta}, \vec{\omega} \right\rangle \left[\frac{\partial \hat{p}}{\partial \zeta} \right. \\
& + \left\langle \frac{\partial \mathbf{A}^T(\hat{n})}{\partial \zeta} \mathbf{A}^{-T}(\hat{n}) \vec{\omega}, \nabla \hat{p} \right\rangle \\
& \left. + \frac{1}{|\det \mathbf{A}(\hat{n})|} \frac{\partial |\det \mathbf{A}(\hat{n})|}{\partial \zeta} \hat{p} \right]. \tag{B.4}
\end{aligned}$$

One can verify that (B.1–B.4) are respectively Fourier dual equations of (A.3–A.6) by applying the following derivative properties of Fourier transform

$$\nabla \xleftrightarrow{\mathcal{F}} j \vec{\omega}, \quad \vec{r} \xleftrightarrow{\mathcal{F}} j \nabla, \tag{B.5}$$

where we used symbol ∇ to denote the gradient with respect to $\vec{r}$ and $\vec{\omega}$ in spatial domain and Fourier domain, respectively. It is worth noting that the order of the operators should be kept when taking the Fourier transform because two operators may not be commutative. For

instance, the Fourier transform of $\left\langle \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \eta} \mathbf{A}(\hat{n}) \vec{r}, \nabla p \right\rangle$ is

$$\begin{aligned}
& \mathcal{F} \left[\left\langle \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \eta} \mathbf{A}(\hat{n}) \vec{r}, \nabla p \right\rangle \right] = \left\langle \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \eta} \mathbf{A}(\hat{n}) j \nabla, j \vec{\omega} \hat{p} \right\rangle, \\
& = - \left\langle \vec{\omega}, \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \eta} \mathbf{A}(\hat{n}) \nabla \hat{p} \right\rangle - \left\langle \frac{\partial \mathbf{A}^{-1}(\hat{n})}{\partial \eta} \mathbf{A}(\hat{n}) \nabla, \vec{\omega} \right\rangle \hat{p}, \\
& = - \left\langle \mathbf{A}^T(\hat{n}) \frac{\partial \mathbf{A}^{-T}(\hat{n})}{\partial \eta} \vec{\omega}, \nabla \hat{p} \right\rangle + \left\langle \mathbf{A}^{-1}(\hat{n}) \frac{\partial \mathbf{A}(\hat{n})}{\partial \eta} \nabla, \vec{\omega} \right\rangle \hat{p}, \\
& = \left\langle \frac{\partial \mathbf{A}^T(\hat{n})}{\partial \eta} \mathbf{A}^{-T}(\hat{n}) \vec{\omega}, \nabla \hat{p} \right\rangle + \text{tr} \left(\mathbf{A}^{-1}(\hat{n}) \frac{\partial \mathbf{A}(\hat{n})}{\partial \eta} \right) \hat{p}, \\
& = \left\langle \frac{\partial \mathbf{A}^T(\hat{n})}{\partial \eta} \mathbf{A}^{-T}(\hat{n}) \vec{\omega}, \nabla \hat{p} \right\rangle + \frac{1}{|\det \mathbf{A}(\hat{n})|} \frac{\partial |\det \mathbf{A}(\hat{n})|}{\partial \eta} \hat{p}. \tag{B.6}
\end{aligned}$$

In the last step of (B.6), Jacobi's formula for the derivative of the determinant of a matrix was applied, see formula (0.8.10.1) of Horn and Johnson [28].

Appendix C. The solution to the first consistency equation

Applying the method of characteristics [30, 12], we can obtain from (27) the following characteristic equations:

$$du = \frac{d\omega_t}{\frac{\omega_1}{\sqrt{1+u^2+v^2}} - \frac{u\omega_t}{1+u^2+v^2}} = \frac{d\hat{p}}{\frac{u}{1+u^2+v^2}\hat{p} - \sigma^2 \left(\frac{\omega_1}{\sqrt{1+u^2+v^2}} - \frac{u\omega_t}{1+u^2+v^2} \right) \omega_t \hat{p}}, d\omega_1=0, d\omega_2=0, \text{ and } dv=0. \quad (\text{C.1})$$

Since ω_1 , ω_2 and v are invariant along the characteristic curves, we can determine the characteristic curves by solving the following two ordinary differential equations (ODEs)

$$du = \frac{d\omega_t}{\frac{\omega_1}{\sqrt{1+u^2+v^2}} - \frac{u\omega_t}{1+u^2+v^2}}, du = \frac{d\log\hat{p} + \sigma^2\omega_t d\omega_t}{\frac{u}{1+u^2+v^2}}. \quad (\text{C.2})$$

The second ODE is obtained by combining the second and the third terms in (C.1). The solutions to the above two ODEs are

$$\omega_t = \frac{\omega_1 u + C_1}{\sqrt{1+u^2+v^2}}, \hat{p} = C_2 \sqrt{1+u^2+v^2} e^{-\sigma^2\omega_t^2/2}. \quad (\text{C.3})$$

The constants C_1 and C_2 can be determined from initial conditions. The first solution (with constant-valued ω_1 , ω_2 and v) determines the parameterized characteristic curves and the second one determines the planogram along the characteristic curves. The solution to PDE (27) (also called the integral surface) consists of the union of the characteristic curves with different values of C_1 [30, 12]. By choosing the initial argument $(\omega_1, \omega_2, \omega_{t,0}; u_0, v)$ for $\hat{p}$, we obtain the two constants:

$$C_1 = \sqrt{1+u_0^2+v^2}\omega_{t,0} - \omega_1 u_0, C_2 = \frac{\hat{p}(\omega_1, \omega_2, \omega_{t,0}; u_0, v)}{\sqrt{1+u_0^2+v^2}} e^{\sigma^2\omega_{t,0}^2/2}. \quad (\text{C.4})$$

Putting (C.4) into (C.3), we obtain (31).

Appendix D. The solution to the Fourier-John equation

Again, applying the method of characteristics [30, 12], we can obtain from (29) the following characteristic equations:

$$\frac{du}{\omega_2 - \frac{v\omega_t}{\sqrt{1+u^2+v^2}}} = \frac{dv}{-\omega_1 + \frac{u\omega_t}{\sqrt{1+u^2+v^2}}} = \frac{d\hat{p}}{\frac{u\omega_2 - v\omega_1}{1+u^2+v^2}\hat{p}}, d\omega_1=0, d\omega_t=0, \text{ and } d\omega_2=0. \quad (\text{D.1})$$

The variables ω_1 , ω_t and ω_2 are invariant along the characteristic curves, and we can rewrite (D.1) as the following two ODEs:

$$\frac{du}{\omega_2 - \frac{v\omega_t}{\sqrt{1+u^2+v^2}}} = \frac{dv}{-\omega_1 + \frac{u\omega_t}{\sqrt{1+u^2+v^2}}}, \quad \frac{u du + v dv}{u\omega_2 - v\omega_1} = \frac{d \log \hat{p}}{\frac{u\omega_2 - v\omega_1}{1+u^2+v^2}}. \quad (\text{D.2})$$

The left-hand side of the second ODE is obtained by combining the first two terms in (D.1). The solutions to the two first-order ODEs are

$$u\omega_1 + v\omega_2 - \sqrt{1+u^2+v^2}\omega_t = C_1, \quad \hat{p} = C_2 \sqrt{1+u^2+v^2}. \quad (\text{D.3})$$

By choosing the initial argument $(\omega_1, \omega_t, \omega_2; u_0, v_0)$, we obtain the two constants:

$$C_1 = u_0\omega_1 + v_0\omega_2 - \sqrt{1+u_0^2+v_0^2}\omega_t, \quad C_2 = \frac{\hat{p}(\omega_1, \omega_2, \omega_t; u_0, v_0)}{\sqrt{1+u_0^2+v_0^2}}. \quad (\text{D.4})$$

Putting (D.4) into (D.3), we obtain (41).

Appendix E. The shifting and rotation properties of TOF planograms

Sometimes, we need to create an image to represent an object based on some basis functions, e.g., voxels- or blobs-based images, or to create an analytic phantom to test reconstruction algorithms. The basis functions can be obtained by shifting a generating function, e.g. a box function for voxel images, or a generalized Kaiser-Bessel function for blob images [27]. The analytic phantom can be constructed by superimposing a number of elemental objects, e.g., rectangles, ellipses, placed at desired positions, at desired orientations and of desired sizes and activities [26]. We give in this appendix the shifting and rotation properties of 3D TOF planograms, which can be used to generate the projections from shifted or rotated objects.

We denote a shifted object as

$$f_{\text{shift}}(x, y, z) = f(x - x_0, y - y_0, z - z_0) \quad (\text{E.1})$$

where (x_0, y_0, z_0) is the offset. Putting (E.1) into (7), we obtain the 3D TOF planograms for the shifted object

$$p_{\text{shift}}(r_1, r_2, t; u, v) = p\left(r_1 + uy_0 - x_0, r_2 + vy_0 - z_0, t - y_0 \sqrt{1+u^2+v^2}; u, v\right). \quad (\text{E.2})$$

We denote a rotated object as

$$f_{\text{rot}}(x, y) = f(x \cos \psi + y \sin \psi, -x \sin \psi + y \cos \psi, z) \quad (\text{E.3})$$

where ψ is the rotation angle. Putting (E.3) into (7), we obtain the TOF planograms for the rotated object

$$\begin{aligned} p_{\text{rot}}(r_1, r_2, t; u, v) &= p\left(\frac{r_1}{\cos \psi - u \sin \psi}, r_2 + \frac{r_1 v \sin \psi}{\cos \psi - u \sin \psi}, t'; u', v'\right), \\ t' &= \text{sign}(\cos \psi - u \sin \psi) t - r_1 \sin \psi \sqrt{1+u^2+v^2} / |\cos \psi - u \sin \psi|, \\ u' &= \frac{u \cos \psi + \sin \psi}{\cos \psi - u \sin \psi}, v' = \frac{v}{\cos \psi - u \sin \psi}. \end{aligned} \quad (\text{E.4})$$

Here we applied $\sqrt{1+u^2+v^2} / |\cos \psi - u \sin \psi| = \sqrt{1+u'^2+v'^2}$.

Appendix F. Analytic TOF planograms for rectangle and ellipse objects

In this appendix, we give the analytic 2D TOF planogram formulas for rectangle and ellipse objects, which can be used in combination with the results in appendix E for fast generating the 2D TOF planograms from analytic phantoms or pixel-based images.

Appendix F.1. 2D TOF planograms for rectangle objects

We formulate a rectangle object with half of the two sides, a and b , as

$$f(x, y) = \Pi\left(\frac{x}{2a}\right) \Pi\left(\frac{y}{2b}\right), \quad (\text{F.1})$$

where $\Pi(x)$ is the rectangular function, which is one for $|x| < 1/2$ [40]. The Fourier transform of (F.1) is

$$\hat{f}(\xi_x, \xi_y) = \frac{4 \sin a \xi_x \sin b \xi_y}{\xi_x \xi_y}. \quad (\text{F.2})$$

Putting (F.1) into (7) with $r_2 = 0$ and $v = 0$, we can formulate the 2D TOF planogram as

$$p_{\text{TOF}}(r_1, t; u) = [F_h(\ell_+) - F_h(\ell_-)] \Pi\left(\frac{r_1}{2a+2b|u|}\right),$$

$$\ell_{\pm} = \begin{cases} t \pm b\sqrt{1+u^2} & u=0, \\ t \pm \min\left(b, \frac{a \pm \text{sign}(u)r_1}{|u|}\right) \sqrt{1+u^2} & u \neq 0. \end{cases} \quad (\text{F.3})$$

Here F_h is the cumulative distribution function (CDF) of Gaussian TOF profile h , and we dropped r_2 and v for conciseness. After integrating with respect to TOF variable t , we can formulate the corresponding non-TOF planogram as

$$p_{\text{non-TOF}}(r_1, u) = \sqrt{1+u^2} \begin{cases} 2b \Pi\left(\frac{r_1}{2a}\right), & u=0, \\ \frac{a+b|u|}{|u|} \Lambda\left(\frac{r_1}{a+b|u|}\right) - \frac{|a-b|u|}{|u|} \Lambda\left(\frac{r_1}{|a-b|u|}\right), & u \neq 0. \end{cases} \quad (\text{F.4})$$

Here $\Lambda(x) = (1 - |x|)\Pi(x/2)$ is the triangular function [40].

After some tedious calculations, we can give the Fourier transform of (F.3) as

$$\hat{p}_{\text{TOF}}(\omega_1, \omega_t; u) = \sqrt{1+u^2} \hat{h}(\omega_t) \frac{4 \sin a \omega_1 \sin b(-u\omega_1 + \sqrt{1+u^2}\omega_t)}{\omega_1(-u\omega_1 + \sqrt{1+u^2}\omega_t)}. \quad (\text{F.5})$$

When $\omega_t = 0$, (F.5) becomes the non-TOF case

$$\hat{p}_{\text{non-TOF}}(\omega_1, u) = \sqrt{1+u^2} \frac{4 \sin a \omega_1 \sin b u \omega_1}{u \omega_1^2}. \quad (\text{F.6})$$

Appendix F.2. 2D TOF planograms for ellipse objects

We formulate an ellipse object with semi-major and semi-minor axes a and b as

$$f(x, y) = \Pi\left(\frac{1}{2} \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\right). \quad (\text{F.7})$$

The Fourier transform of (F.7) is

$$\hat{f}(\xi_x, \xi_y) = 2\pi ab \frac{J_1 \left(\sqrt{a^2 \xi_x^2 + b^2 \xi_y^2} \right)}{\sqrt{a^2 \xi_x^2 + b^2 \xi_y^2}}, \quad (\text{F.8})$$

where J_1 denotes the Bessel function of the first kind of order 1. Similar to (F.3), we can formulate the 2D TOF planogram as

$$p_{\text{TOF}}(r_1, t; u) = \frac{1}{2} [F_h(\ell_+) - F_h(\ell_-)] \prod \left(\frac{r_1}{2\sqrt{a^2 + b^2 u^2}} \right),$$

$$\ell_{\pm} = t \pm \frac{b^2 u r_1 \pm ab \sqrt{a^2 + b^2 u^2} - r_1^2}{a^2 + b^2 u^2} \sqrt{1 + u^2}. \quad (\text{F.9})$$

Here F_h is the CDF of Gaussian TOF profile h . After integrating with respect to TOF variable t , we can formulate the corresponding non-TOF planogram as

$$p_{\text{non-TOF}}(r_1, u) = \sqrt{1 + u^2} \frac{2ab \sqrt{a^2 + b^2 u^2} - r_1^2}{a^2 + b^2 u^2} \prod \left(\frac{r_1}{2\sqrt{a^2 + b^2 u^2}} \right) \quad (\text{F.10})$$

Similar to (F.5) we can give the Fourier transform of (F.9) as

$$\hat{p}_{\text{TOF}}(\omega_1, \omega_t; u) = \sqrt{1 + u^2} \hat{h}(\omega_t) 2\pi ab \frac{J_1 \left(\sqrt{a^2 \omega_1^2 + b^2 (-u\omega_1 + \omega_t \sqrt{1 + u^2})^2} \right)}{\sqrt{a^2 \omega_1^2 + b^2 (-u\omega_1 + \omega_t \sqrt{1 + u^2})^2}}, \quad (\text{F.11})$$

When $\omega_t = 0$, (F.11) becomes the non-TOF case

$$\hat{p}_{\text{TOF}}(\omega_1, \omega_t; u) = \sqrt{1 + u^2} \hat{h}(\omega_t) 2\pi ab \frac{J_1 \left(\sqrt{a^2 \omega_1^2 + b^2 u^2 \omega_1^2} \right)}{\sqrt{a^2 \omega_1^2 + b^2 u^2 \omega_1^2}}. \quad (\text{F.12})$$

References

1. Abreu MC, et al. Design and evaluation of the Clear-PEM scanner for positron emission mammography. IEEE Trans Nucl Sci. 2006; 53:71–77.
2. Ahn S, Cho S, Li Q, Lin Y, Leahy RM. Optimal rebinning of time-of-flight PET data. IEEE Trans Med Imag. 2011; 30:1808–1818.
3. Bowen SL, et al. Initial characterization of a dedicated breast PET/CT scanner during human imaging. J Nucl Med. 2009; 50:1401–1408. [PubMed: 19690029]
4. Bracewell, RN. Fourier Analysis and Imaging. Springer; New York, NY: 2006.

5. Brasse D, Kinahan PE, Clackdoyle R, Defrise M, Comtat C, Townsend DW. Fast fully 3-D image reconstruction in PET using planograms. *IEEE Trans Med Imag.* 2004; 23:413–425.
6. Champley K, Defrise M, Clackdoyle R, Raylman RR, Kinahan PE. Planogram rebinning with the frequency-distance relationship. *IEEE Trans Med Imag.* 2008; 27:925–933.
7. Champley KM, Raylman RR, Kinahan PE. Advancements to the planogram frequency-distance rebinning algorithm. *Inverse Problems.* 2010; 26:045008.
8. Cho S, Ahn S, Li Q, Leahy RM. Analytical properties of time-of-flight PET data. *Phys Med Biol.* 2008; 53:2809–2821. [PubMed: 18460746]
9. Cho S, Ahn S, Li Q, Leahy RM. Exact and approximate Fourier rebinning of PET data from time-of-flight to non time-of-flight. *Phys Med Biol.* 2009; 54:467–484. [PubMed: 19124956]
10. Cho S, Li Q, Ahn S, Bai B, Leahy RM. Iterative image reconstruction using inverse Fourier rebinning for fully 3-D PET. *IEEE Trans Med Imag.* 2007; 26:745–756.
11. Clackdoyle R. Necessary and sufficient consistency conditions for fanbeam projections along a line. *IEEE Trans Nucl Sci.* 2013; 60:1560–1569.
12. Courant, R., Hilbert, D. *Methods of Mathematical Physics.* 2nd. Vol. 2. Wiley-VCH; New York, NY: 1989.
13. Courant R, et al. Development of ClearPEM-Sonic, a multimodal mammography system for PET and Ultrasound. *J Instrum.* 2014; 9:C03008.
14. Daube-Witherspoon ME, Karp JS, Casey ME, et al. PET performance measurements using the NEMA NU 2-2001 standard. *J Nucl Med.* 2002; 43:1398–1409. [PubMed: 12368380]
15. Defrise M, Casey M, Michel C, Conti M. Fourier rebinning of time-of-flight PET data. *Phys Med Biol.* 2005; 50:2749–2763. [PubMed: 15930600]
16. Defrise M, Kinahan PE, Townsend DW, Michel C, Sibomana M, Newport DF. Exact and approximate rebinning algorithms for 3-D PET data. *IEEE Trans Med Imag.* 1997; 16:145–158.
17. Defrise M, Liu X. A fast rebinning algorithm for 3D positron emission tomography using John's equation. *Inverse Problems.* 1999; 15:1047–1065.
18. Defrise M, Panin VY, Casey ME. New consistency equation for time-of-flight PET. *IEEE Trans Nucl Sci.* 2013; 60:124–133.
19. Defrise M, Rezaei A, Nuyts J. Time-of-flight PET data determine the attenuation sinogram up to a constant. *Phys Med Biol.* 2012; 57:885–899. [PubMed: 22290428]
20. Doshi NK, Silverman RW, Shao Y, Cherry SR. maxPET: A dedicated mammary and axillary region PET imaging system for breast cancer. *IEEE Trans Nucl Sci.* 2001; 48:811–815.
21. Edholm P, Herman GT, Roberts DA. Image reconstruction from linograms: implementation and evaluation. *IEEE Trans Med Imag.* 1988; 7:239–246.
22. Edholm PR, Herman GT. Linograms in image reconstruction from projections. *IEEE Trans Med Imag.* 1987; 6:301–307.
23. Freifelder R, Karp JS. Dedicated PET scanners for breast imaging. *Phys Med Biol.* 1997; 42:2463–2480. [PubMed: 9434301]
24. Grangeat, P. Mathematical framework of cone beam 3D reconstruction via the first derivative of the Radon transform. In: Herman, GT, Louis, AK., Natterer, F., editors. *Mathematical Methods in Tomography Lecture Notes in Mathematics.* Vol. 1497. Springer; Berlin: 1991. p. 66-97.
25. Helgason S. The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds. *Acta Math.* 1965; 113:153–80.
26. Herman, GT. *Fundamentals of Computerized Tomography: Image Reconstruction from Projections.* 2nd. Springer; London: 2009.
27. Herman GT. Basis functions in image reconstruction from projections: A tutorial introduction. *Sens Imaging.* 2015; 16
28. Horn, RA., Johnson, CR. *Matrix Analysis.* Cambridge University Press; New York, NY: 2013.
29. John F. The ultrahyperbolic differential equation with four independent variables. *Duke Math J.* 1938; 4:300–322.
30. John, F. *Partial Differential Equations.* 4th. Springer-Verlag; New York NY: 1982.

31. Kamangar F, Dores GM, Anderson WF. Patterns of cancer incidence, mortality, and prevalence across five continents: defining priorities to reduce cancer disparities in different geographic regions of the world. *J Clin Oncol*. 2006; 24:2137–2150. [PubMed: 16682732]
32. Kao CM, Pan XC, Chen CT. An exact Fourier rebinning algorithm for 3D PET imaging using panel detectors. *Phys Med Biol*. 2004; 49:2407–2423. [PubMed: 15248586]
33. Kinahan, PE., Defrise, M., Clackdoyle, R. Analytic Image Reconstruction Methods. In: Wernick, MN., Aarsvold, JN., editors. *Emission Tomography: The Fundamentals of PET and SPECT*. Academic Press; London, UK: 2004.
34. Kinahan PE, Rogers JG. Analytic three-dimensional image reconstruction using all detected events. *IEEE Trans Nucl Sci*. 1989; 36:964–968.
35. Kraan AC, et al. Online monitoring for proton therapy: A real-time procedure using a planar PET system. *Nucl Instrum Methods Phys Res A*. 2015; 786:120–126.
36. Lee E, Werner ME, Karp JS, Surti S. Design optimization of a time-of-flight, breast PET scanner. *IEEE Trans Nucl Sci*. 2013; 60:1645–1652. [PubMed: 24078744]
37. Li Y, Defrise M, Metzler SD, Matej S. Transmission-less attenuation estimation from time-of-flight PET histo-images using consistency equations. *Phys Med Biol*. 2015a; 60:6563–6583. [PubMed: 26267223]
38. Li Y, Matej S, Karp JS, Metzler SD. LOR-interleaving image reconstruction for PET imaging with fractional-crystal collimation. *Phys Med Biol*. 2015b; 60:647–670. [PubMed: 25555160]
39. Li Y, Matej S, Metzler SD. A unified Fourier theory for time-of-flight PET data. *Phys Med Biol*. 2016; 61:601–624. [PubMed: 26689836]
40. Li Y, Oldendick J, Ordonez CE, Chang W. The geometric response function for convergent slit-slat collimators. *Phys Med Biol*. 2009; 54:1469–1482. [PubMed: 19218734]
41. Liu X, Defrise M, Michel C, Sibomana M, Comtat C, Kinahan P, Townsend D. Exact rebinning methods for three-dimensional PET. *IEEE Trans Med Imag*. 1999; 18:657–664.
42. Ludwig D. The Radon transform on Euclidean space. *Commun Pure Appl Math*. 1966; 19:49–81.
43. MacDonald L, Edwards J, Lewellen T, Haseley D, Rogers J, Kinahan P. Clinical imaging characteristics of the positron emission mammography camera: PEM Flex Solo II. *J Nucl Med*. 2009; 50:1666–1675. [PubMed: 19759118]
44. Mariano-Goulart D, Crouzet JF. A new reconstruction method in the Fourier domain for 3D positron emission tomography. *C R Physique*. 2005; 6:133–137.
45. Matej S, Lewitt RM. 3D-FRP: Direct fourier reconstruction with Fourier reprojection for fully 3-D PET. *IEEE Trans Nucl Sci*. 2001; 48:1378–85.
46. Matej S, Surti S, Jayanthi S, Daube-Witherspoon ME, Lewitt RM, Karp JS. Efficient 3-D TOF PET reconstruction using view-grouped histo-images: DIRECT–direct image reconstruction for TOF. *IEEE Trans Med Imag*. 2009; 28:739–751.
47. Moses WW. Time of flight in PET revisited. *IEEE Trans Nucl Sci*. 2003; 50:1325–1330.
48. Motta A, Del Guerra A, Belcari N, Moehrs S, Panetta D, Righi S, Valentini D. Fast 3D-EM reconstruction using Planograms for stationary planar positron emission mammography camera. *Computerized Medical Imaging and Graphics*. 2005; 29:587–596. [PubMed: 16290284]
49. Mullani N, Wong W, Hartz R, Philippe E, Yerian K. Sensitivity improvement of TOFPET by the utilization of the inter-slice coincidences. *IEEE Trans Nucl Sci*. 1982; 29:479–483.
50. Natterer, F. *The Mathematics of Computerized Tomography*. Wiley; New York, NY: 1986.
51. Nishio T, Ogino T, Nomura K, Uchida H. Dose-volume delivery guided proton therapy using beam on-line PET system. *Med Phys*. 2006; 33:4190–4197. [PubMed: 17153398]
52. Nover AB, Jagtap S, Anjum W, Yegingil H, Shih WY, Shih WH, Brooks AD. Modern breast cancer detection: a technological review. *Int J Biomed Imaging*. 2009; 2009:902326. [PubMed: 20069109]
53. Orlov SS. Theory of three-dimensional image reconstruction: I. Conditions for a complete set of projections. *Sov Phys Crystallogr*. 1975; 20:312–314.
54. Patch SK. Consistency conditions upon 3D CT data and the wave equation. *Phys Med Biol*. 2002; 47:2637–2650. [PubMed: 12200929]

55. Raylman RR, Majewski S, Kross B, Popov V, Proffitt J, Smith MF, Weisenberger AG, Wojcik R. Development of a dedicated positron emission tomography system for the detection and biopsy of breast cancer. *Nucl Instrum Methods Phys Res A*. 2006; 569:291–295.
56. Raylman R, et al. The positron emission mammography/tomography breast imaging and biopsy system (PEM/PET): design, construction and phantom-based measurements. *Phys Med Biol*. 2008; 53:637–653. [PubMed: 18199907]
57. Slepian D. On bandwidth. *Proc IEEE*. 1976; 64:292–300.
58. Surti S, Karp JS. Design considerations for a limited angle, dedicated breast, TOF PET scanner. *Phys Med Biol*. 2008; 53:2911–2921. [PubMed: 18460745]
59. Surti S, Zou W, Daube-Witherspoon ME, McDonough J, Karp JS. Design study of an in situ PET scanner for use in proton beam therapy. *Phys Med Biol*. 2011; 56:2667–2685. [PubMed: 21464528]
60. Tashima H, Yamaya T, Kinahan PE. An OpenPET scanner with bridged detectors to compensate for incomplete data. *Phys Med Biol*. 2014; 59:6175–6193. [PubMed: 25255296]
61. Thompson CJ, Murthy K, Weinberg IN, Mako F. Feasibility study for positron emission mammography. *Med Phys*. 1994; 21:529–538. [PubMed: 8058019]
62. Vecchio, Set. A PET prototype for “in-beam” monitoring of proton therapy. *IEEE Trans Nucl Sci*. 2009; 56:51–56.
63. Xiao P, Cao X, Wan L, Li Y, Xie Q. TOF capability evaluation on a panel PET for human body imaging. *J Instrum*. 2013; 8:P04001.
64. Zhang J, Olcott PD, Chinn G, Foudray AMK, Levin CS. Study of the performance of a novel 1 mm resolution dual-panel PET camera design dedicated to breast cancer imaging using Monte Carlo simulation. *Med Phys*. 2007; 34:689–702. [PubMed: 17388187]

Figure 1.

Data parametrization for 3D dual-panel TOF-PET planogram. The LOR between detectors A and B is parameterized in planogram format by the variables r_1 , r_2 , t , u and v . The TOF profile is centered at the most likely position $\vec{x} = r_1 \hat{d}_1 + t \hat{n} + r_2 \hat{d}_2$ along the TOF direction $\hat{n}$.

Figure 2.

The two diamond regions are the domains of measured planogram with finite dual-panel detectors without rotation. The two hourglass-shaped regions in (a) and (b) are the supports of planograms for an object inside a cylindrical set. The cross-hatched regions have no data truncation, while the single-hatched regions have data truncation. After a 90° rotation around z -axis, the measured diamond region in (a) becomes the two back-to-back fan regions (in gray), and the hourglass-shaped region is contained in the union of the diamond region and the two fan regions when $u_m = 1$. The case $u_m = 1$ is shown in (a).

Figure 3.

Shadow zones in the Fourier domain associated to the 3D and 2D limited-angle TOF planograms. The frequency contents of an object inside the shadow zones cannot be obtained from the limited-angle TOF planograms. Inside the spectral cube, the new information in the difference regions of the TOF and non-TOF shadow zones is gained by TOF measurement.

Figure 4.

A unified picture for converting different types of PET planogram: 3D TOF, 2D TOF, 3D non-TOF and 2D non-TOF data. (In the colored version, the red, blue, and green colors represent the equations related to the first FCE, the second FCE, and FJE, respectively.)

Figure 5.

Comparison of 2D TOF planograms using the Fourier-slice theorem with the corresponding directly computed 2D TOF planograms using the strip-integral model. The horizontal and vertical axes represent variable r_1 and slope u , respectively. The columns show the TOF bin indices 0, 3, 6 and 9; the rows show the inverse rebinned TOF planograms (FORCE), the directly computed planograms (Reference) and their difference (Difference).

(a) Rebinned sinogram and the non-TOF sinogram reference

(b) Horizontal profiles of mean and variance through $\phi = 45^\circ$ **Figure 6.**

Comparison the rebinned planograms and the corresponding reference non-TOF planograms. The columns show the sample, mean and variance planograms; the rows show the non-TOF and FORCE rebinned planograms. The horizontal and vertical axes in (a) represent variable r_1 and slope u , respectively. The horizontal profiles of the mean and variance sinograms through $u = 0$ of the first date set are shown in (b).

Figure 7.

The reconstructed images of non-TOF, FORCE and TOF (top). The mean and variance reconstructions calculated from 60 noise realizations are shown in the middle and bottom rows. The number of iterations for the 3 types of reconstructed images are respectively 21, 18 and 10, which are selected to yield approximately the same CRC of 0.83 for the nine hot rods in the mean images.

Figure 8.
Central horizontal profiles of mean and variance reconstructions.

Figure 9.

Calculated CRC vs iteration number (a), standard deviation vs iteration number (b) and CRC vs standard deviation for the non-TOF, FORCE and TOF reconstructions of the nine hot rods. Each marker represents one iteration. The dashed horizontal line in (a) indicates the CRC level of 0.83, which is used to show the images in figure 7.

Figure 10.

The normalization factors in the Fourier domain (the rebinning weights integrated over ω_l) used in the Fourier-based rebinning example shown in figure 6. The horizontal and vertical axes represent variable ω_1 and slope u , respectively. The direct weights (left) are the rebinning weights contributed by the two sets of TOF planograms to the corresponding non-TOF data sets. The cross weights (middle) are the rebinning weights contributed by the first (second) set of TOF planograms to the second (first) set of non-TOF planograms. The total rebinning weights (right) are the summation of the two weights.

Figure 11.
The reconstructed images of non-TOF (left), FORCE (middle) and TOF (right) using the limited-angle planograms (one position only) with 90° angle coverage.