A novel investigation of a micropolar fluid characterized by nonlinear constitutive diffusion model in boundary layer flow and heat transfer

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The rheological and heat-conduction constitutive models of micropolar fluids (MFs), which are important non-Newtonian fluids, have been, until now, characterized by simple linear expressions, and as a consequence, the non-Newtonian performance of such fluids could not be effectively captured. Here, we establish the novel nonlinear constitutive models of a micropolar fluid and apply them to boundary layer flow and heat transfer problems. The nonlinear power law function of angular velocity is represented in the new models by employing generalized “ n -diffusion theory,” which has successfully described the characteristics of non-Newtonian fluids, such as shear-thinning and shear-thickening fluids. These novel models may offer a new approach to the theoretical understanding of shear-thinning behavior and anomalous heat transfer caused by the collective micro-rotation effects in a MF with shear flow according to recent experiments. The nonlinear similarity equations with a power law form are derived and the approximate analytical solutions are obtained by the homotopy analysis method, which is in good agreement with the numerical solutions. The results indicate that non-Newtonian behaviors involving a MF depend substantially on the power exponent n and the modified material parameter K_0 introduced by us. Furthermore, the relations of the engineering interest parameters, including local boundary layer thickness, local skin friction, and Nusselt number are found to be fitted by a quadratic polynomial to n with high precision, which enables the extraction of the rapid predictions from a complex nonlinear boundary-layer transport system. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4976642>]

I. INTRODUCTION

Complex non-Newtonian fluids, being ubiquitous in the natural world and engineering applications, such as paints, polymers, blood, body fluids, and colloidal suspensions, have been extensively scrutinized. It is generally accepted that these fluids usually exhibit intricate constitutive relationships (generally called non-Newtonian behaviors); for instance, the nonlinear variation between stress and strain in rheology, and the nonlinear variation between heat current and temperature gradient in thermodynamics, which are strikingly divergent from the Newtonian fluids with the simple linear constitutive equations. Macroscopically, the pseudo-plasticity (shear thinning) and dilatant (shear thickening) and thixotropy and yield stress, in particular the heat transfer of a non-Newtonian fluid, are all rather sensitive to the shear effects.^{1–3} All these alluring thermal and physicochemical performances of non-Newtonian fluids have fueled a rapid growth of interest in research due to their ubiquity in daily life.

Micropolar fluids (MFs) as a class of non-Newtonian fluids are considered to be a special kind of suspensions described by micropolar theories. Such theories were based on pioneering work by Eringen^{4,5} and Dahler, Scriven, and Condiff,^{6,7} in which the stress tensor is no longer symmetric but rather an anti-symmetric characteristic due to the oriented micro-rotation of particles. In the research monograph by Stokes,⁸ the microstructure in micropolar fluids with the form of the gyration tensor restricted was discussed in detail. It is worth mentioning that Stokes has earlier presented the effects of couple stresses in fluids⁹ with different flow conditions, for example, hydromagnetic channel flows,¹⁰ flow past a sphere,¹¹ and plane Poiseuille flow,¹² which were meaningful to the understanding of the couple stress structure in a MF. Furthermore, review papers and books reported the systematic applications of MFs theoretically and experimentally.^{13–16}

Recently, more investigations into the physical and engineering problems involving MFs have begun to emerge. The main research topics concern stagnation point flow,^{17–19} heat transfer by free/mixed convection (with inclined enclosure)^{20–22} and forced convection;^{23,24} the stretching/shrinking sheet problems;^{25–27} incorporating the boundary conditions such as suction/injection, velocity slip, heat radiation and generation, magneto hydrodynamics (MHDs);^{26–29} and even a wavy differentially heated cavity.³⁰ Also, the flow and heat

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transfer of a MF through a horizontal/vertical channel^{31,32} and applications in biological science³³ have been performed. These previous publications remind us that the micropolar fluids play a practical role in the chemical industry, food industry, manufacturing, and bio-medical science.

Nevertheless, and more importantly, these earlier studies oversimplify the non-Newtonian behaviors of MFs even though an additional equation with respect to the conservation of local angular momentum is used, and also the spin gradient is present in the momentum equation. The usual linear constitutive models can hardly describe the non-Newtonian viscous flow and heat transfer of MFs, which are constrained by the constant thermophysical parameters of micropolar fluids. As reported by Mitarai *et al.*,³⁴ a micropolar fluid can be understood in terms of granular flows because its model describes a fluid consisting of oriented spinning particles. The typical non-Newtonian behaviors of a MF in shear flow were still not elucidated. However, until 2010, when Peters *et al.*³⁵ explicitly showed that the collective rotation effects of the particles are responsible for a significant decrease of the effective viscosity of the suspension according to the measurement of the velocity profile. These results imply that new nonlinearity constitutive models of a MF are necessary.

Here, we propose novel constitutive models by taking into account the micro-rotation N dependence of variable viscosities (dynamic, vortex, and spin gradient viscosity) and thermal conductivity to characterize the shear-thinning behavior of a MF.³⁵ The generalized n -diffusion theory³⁶ is employed in our nonlinear constitutive models, since it has been used successfully to investigate the flow and heat transfer of non-Newtonian power law fluids with respect to shear-thinning and shear-thickening performances.

The new generalized constitutive model can be expressed as $\mathbf{Q} = k_0 |\mathbf{U}|^{n-1} \nabla \mathbf{B}$, where $\mathbf{Q}$ is the flux density vector, $\nabla \mathbf{B}$ is the gradient of a vector (such as velocity gradient, temperature gradient, and concentration gradient), $\mathbf{U}$ the physical quantities which affect the fluid properties such as viscosity, thermal conductivity and mass diffusion coefficients, k_0 a coefficient matching the dimension, n the important physical parameter named power exponent, and $k_0 |\mathbf{U}|^{n-1}$ is the so-called apparent coefficients of fluids. Taking the two dimensional viscous flow of power law fluids, for example, their rheological constitution model is written as $\tau_{xy} = k_0 |\partial u / \partial y|^{n-1} \partial u / \partial y$, where τ_{xy} is the shear stress, $\partial u / \partial y$ the shear rate (the velocity gradient), and $k_0 |\partial u / \partial y|^{n-1}$ the apparent viscosity. Therefore, the case $n < 1$ means the reduced apparent viscosity with shear rate, i.e., shear-thinning, the case $n > 1$ indicates the increased apparent viscosity with shear rate, i.e., shear-thickening, and $n = 1$ for Newtonian fluid with the constant viscosity. Obviously, the power exponent n determines the deviation level from the normal constitutive model (Newtonian fluids). The generalized power law diffusion model has been utilized to address some typical nonlinear diffusion processes such as shear flow and heat transfer of non-Newtonian power law fluids by Zheng³⁶ and Pascal *et al.*^{37,38}

The advantage that such theories can characterize the typical shear-thinning and shear-thickening performances of non-Newtonian fluids by modulating the power exponent n adequately, while the rheological relationships of some real

complex fluids such as carboxymethyl cellulose (CMC) solutions and polyvinyl alcohol (PVA) were used to verify the value range of n experimentally.³⁹⁻⁴² With such motivation, we introduce in this work a generalized nonlinear diffusion model to remodel the rheological and heat-conduction constitutive relations of a micropolar fluid. The model first shows the effects of shear-thinning on boundary layer flow and heat transfer theoretically by analyzing the salient parameters in detail. Moreover, the nonlinear similarity equations of boundary layer flow and heat transfer are derived and the approximate analytical solutions are obtained by the homotopy analysis method (HAM).⁴³⁻⁴⁷ At present, the approximate analytical approaches in terms of nonlinear boundary layer problems for non-Newtonian fluids still attract a great deal of research interest, in particular the transport problems with the nonlinear power-law criterion. Hence, the present investigation for the shear flow and heat transfer of a micropolar fluid by means of novel constitutive models is valuable.

II. BASIC GOVERNING EQUATIONS

Here, we consider the steady two-dimensional laminar boundary layer flow and heat transfer on the stretching sheet of an incompressible micropolar fluid that is described by novel nonlinear power-law constitutive models. The large velocity gradient in the boundary layer contributing to the noticeable shear flow is controlled by a stretching sheet with a certain velocity u_w . The steady rotation of spherical particles in a micropolar fluid is treated as co-rotating without interplay due to the dilute fluid. Moreover, the spheres are larger than the nano-particles (10–100 nm) so that the collective micro-rotation effects of particles can play a dominant role on the apparent viscosities and effective thermal conductivity of a micropolar fluid.

Fig. 1 depicts the schematic illustration of the velocity boundary layer (VBL) and temperature boundary layer (TBL). The fluid outside the VBL, containing the nonrotating suspended spherical particles, remains stationary. The surface temperature of the sheet T_w is larger than the ambient temperature T_∞ . Meanwhile, we postulate the stretching orientation as positive along the x -axis and a homogeneously collective anti-clockwise micro-rotation in the x - y plane where the y -axis is perpendicular to the x -axis. Taking into account the micropolar theory described by Eringen^{4,5,16} and new power-law constitutive relationship for heat transfer of a micropolar fluid in the laminar boundary layer flow with the stretching sheet condition, the basic governing equations can be written in Cartesian coordinates as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left(\frac{\mu_0 + \kappa_0}{\rho} \right) \frac{\partial}{\partial y} \left(|N|^{n-1} \frac{\partial u}{\partial y} \right) + \frac{\kappa_0}{\rho} \frac{\partial}{\partial y} (|N|^{n-1} N), \quad (2)$$

$$\rho j \left(u \frac{\partial N}{\partial x} + v \frac{\partial N}{\partial y} \right) = \frac{\partial}{\partial y} \left(\gamma \frac{\partial N}{\partial y} \right) - \kappa_0 |N|^{n-1} \left(2N + \frac{\partial u}{\partial y} \right), \quad (3)$$

FIG. 1. Schematic illustration of the physical problem within a two-dimensional Cartesian coordinate system. The oriented micro-rotation of suspended particles is driven by a large velocity gradient in the boundary layer, whereas the particles out of the boundary layer are assumed at rest. The anticlockwise micro-rotation and the rightward stretching flow are assumed in positive directions.

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\lambda_0}{\rho c_p} \frac{\partial}{\partial y} \left(|N|^{n-1} \frac{\partial T}{\partial y} \right), \quad (4)$$

where u and v are the velocity field components along the x -axis and the y -axis, respectively, N is the micro-rotation or angular velocity in the x - y plane, ρ the density, and c_p the specific heat capacity at constant pressure of a MF. By employing the generalized n -diffusion model,³⁶ the new couple shear stress of a MF in this problem is characterized as $-\tau = (\mu_0 + \kappa_0) |N|^{n-1} \partial u / \partial y + \kappa_0 |N|^{n-1} N$, where $\mu_0 |N|^{n-1}$ and $\kappa_0 |N|^{n-1}$ are the so-called apparent dynamic viscosity and vortex viscosity of a MF with the positive constants μ_0 and κ_0 , respectively. Fourier's heat conduction law is modified as $q = -\lambda_0 |N|^{n-1} \partial T / \partial y$, where $\lambda_0 |N|^{n-1}$ refers to the modified thermal conductivity with the constant λ_0 and T the temperature. The power exponent n denotes the magnitude of deviation from the ordinary model, which is set in the range $0 < n \leq 1$ to manifest a shear thinning behavior due to the collective effects of micro-rotation of spherical particles. Once $n = 1$, these models can be simplified to traditional ones.^{25–27} Furthermore, the definition of spin gradient viscosity γ is based on Ahmadi's reports⁴⁸ as

$$\gamma = (\mu_0 + \kappa_0/2) |N|^{n-1} = \mu_0 |N|^{n-1} (1 + K_0/2)j, \quad (5)$$

where $K_0 = \kappa_0/\mu_0$ denotes the dimensionless viscosity ratio and is named the material parameter and j is the micro-inertia per unit mass with the dimension L^2 (L means length dimension). The appropriate boundary layers of Eqs. (1)–(4) are

$$u = u_w, \quad v = 0, \quad N = -m \partial u / \partial y, \quad T = T_w \text{ at } y = 0, \quad (6)$$

$$u \rightarrow 0, \quad v \rightarrow 0, \quad N \rightarrow 0, \quad T = T_\infty, \text{ as } y \rightarrow \infty, \quad (7)$$

where the stretching velocity u_w is a constant independent of the horizontal size of the sheet, m is a constant which is located in $0 \leq m \leq 1$, the case $m = 0$ corresponds to a concentrated MF, namely, the micro-spheres (microelements) near the sheet retain little spin $N = 0$, and the alternative case $m = 1$ signifies the turbulent boundary layer flow model.^{23,25,26} In this paper, neither constants are used, instead $m = 1/2$ is employed to meet the requirements of a dilute MF, which means the vanishing of the anti-symmetrical part of the stress tensor.^{24–26}

We introduce the non-dimensional similarity variables for a transformation of the original physical governing equation

systems with the boundary conditions as follows:

$$\begin{aligned} \psi &= \left(u_w^{2n-1} x \mu_0 / \rho \right)^{\frac{1}{n+1}} F(\eta), \quad N = \left(u_w^3 \rho / \mu_0 x \right)^{\frac{1}{n+1}} R(\eta), \\ \eta &= \left(u_w^{2-n} \rho / \mu_0 x \right)^{\frac{1}{n+1}} y, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}, \end{aligned} \quad (8)$$

where ψ is the stream function to describe the velocity components as $u = \partial \psi / \partial y$ and $v = -\partial \psi / \partial x$, η is regarded as the dimensionless coordinate axis along the y -axis, and θ the normalized temperature. Substituting variables (8) into Eqs. (1)–(7), we obtain the nonlinear ordinary differential equations

$$(1 + K_0) \left(|R|^{n-1} F'' \right)' + K_0 \left(|R|^{n-1} R \right)' + \frac{1}{n+1} FF'' = 0, \quad (9)$$

$$\begin{aligned} \left(1 + \frac{K_0}{2} \right) \left(|R|^{n-1} R' \right)' + \frac{1}{n+1} [RF' + FR'] \\ - K_0 |R|^{n-1} [2R + F''] = 0, \end{aligned} \quad (10)$$

$$\left(|R|^{n-1} \theta' \right)' + \frac{\text{Pr}_N}{n+1} F \theta' = 0, \quad (11)$$

with the boundary conditions

$$F'(0) = 1, \quad F(0) = 0, \quad R(0) = -\frac{1}{2} F''(0), \quad \theta(0) = 1, \text{ at } \eta = 0, \quad (12)$$

$$F'(\infty) = 0, \quad R(\infty) = 0, \quad \theta(\infty) = 0, \text{ as } \eta \rightarrow \infty, \quad (13)$$

where $F' = u/u_w$ denotes the normalized velocity, $\text{Pr}_N = \mu_0 c_p / \lambda_0$ is defined as the generalized Prandtl number, which is reduced to the usual Prandtl number Pr when $n = 1$. $j = \left(U_w^{n-2} \mu_0 x / \rho \right)^{\frac{2}{n+1}}$ as a local reference length with the dimension L^2 (length square)^{18,21,24} and only such similarity expression can give rise to a full similarity transformation from Eqs. (3)–(10) by employing similarity variables Eq. (8).

Furthermore, the physical quantities concerned in engineering are the skin friction coefficient for shear flow and the Nusselt number for heat transfer, expressed as

$$C_f = \frac{2\tau_w}{\rho u_w^2}, \quad \text{Nu}_x = \frac{x q_w}{\lambda_w (T_w - T_\infty)}, \quad (14)$$

where λ_w is the thermal conductivity on the surface of the sheet with expression $\left[\lambda_0 |N|^{n-1} \right]_{y=0}$ and the skin friction τ_w

and the skin heat current q_w are given by

$$\begin{aligned}\tau_w &= [(\mu_0 + \kappa_0) |N|^{n-1} \partial u / \partial y + \kappa_0 |N|^{n-1} N]_{y=0}, \\ q_w &= -\lambda_0 [|N|^{n-1} \partial T / \partial y]_{y=0}.\end{aligned}\quad (15)$$

Considering the similarity variables Eq. (8) and the boundary condition $R(0) = -\frac{1}{2}F''(0)$ yields the local skin friction coefficient C_f^{local} and local Nusselt number Nu_x^{local} ,

$$\begin{aligned}C_f \text{Re}_N^{\frac{1}{n+1}} &= 2^{2-n} \left(1 + \frac{K_0}{2}\right) |F''(0)|^{n-1} F''(0), \\ Nu_x \text{Re}_N^{\frac{1}{n+1}} &= -\theta'(0),\end{aligned}\quad (16)$$

where $\text{Re}_N = x^n u_w^{2-n} \rho / \mu_0$ denotes the generalized Reynolds number for the present model, which can be reduced to the usual one when $n = 1$.

Furthermore, on the basis of the boundary layer (BL) theory,⁴⁹ the important physical quantities we focus on are the thickness of the boundary layers, for instance the momentum layer thickness (MLT) for shear flow, given as

$$\Delta_m = \int_0^\infty (1 - u/u_w) \rho u / \rho_w u_w dy \quad (17)$$

and the enthalpy layer thickness (ELT) for heat transfer whose explicit definition is

$$\Delta_e = \int_0^\infty \rho u (T - T_\infty) / \rho_w u_w (T_w - T_\infty) dy. \quad (18)$$

Here, we discuss the circumstance of a homogenous MF with the low flow rate, thus the dimensionless local MLT and ELT are derived by substituting the similarity variables, Eq. (8), into Eqs. (17) and (18) as

$$\frac{\Delta_m}{L} \text{Re}_N^{\frac{1}{n+1}} = \int_0^\infty (1 - F') F' d\eta, \quad \frac{\Delta_e}{L} \text{Re}_N^{\frac{1}{n+1}} = \int_0^\infty F' \theta d\eta, \quad (19)$$

where L is assumed as the primary length of the sheet before stretching, and we employ the notations δ_m and δ_e to represent the dimensionless local MLT and dimensionless local ELT, respectively.

III. APPLICATION OF HOMOTOPY ANALYSIS METHOD IN THE NEW MODEL

The approximation analytical solutions of the nonlinear differential equations with power exponent items Eqs. (9)–(13) and conditions (14) and (15) are usually difficult to be achieved but meaningful. Homotopy analysis method (HAM) is an effective approach in solving the nonlinear differential equations, which was proposed and developed by Liao.^{39–41} Considering the basic essentials of HAM, we present here only the main structure of the computations.

We choose the base functions to express the solutions $F(\eta)$, $R(\eta)$, and $\theta(\eta)$ as

$$\{\eta^i e^{-j\eta} | i \geq 0, j \geq 0\}. \quad (20)$$

The solutions can be written as $F(\eta) = \sum_{j=0}^\infty \sum_{i=0}^\infty a_{j,i} \eta^i e^{-j\eta}$, $R(\eta) = \sum_{j=0}^\infty \sum_{i=0}^\infty b_{j,i} \eta^i e^{-j\eta}$, and $\theta(\eta) = \sum_{j=0}^\infty \sum_{i=0}^\infty c_{j,i} \eta^i e^{-j\eta}$, where $a_{j,i}$, $b_{j,i}$, and $c_{j,i}$ are the coefficients to be determined. The initial approximation solutions and auxiliary linear operators

are chosen to meet the criterion of present conditions (14) and (15) as

$$F_0(\eta) = 1 - e^{-\eta}, \quad R_0(\eta) = \frac{1}{2}e^{-\eta}, \quad \theta_0(\eta) = e^{-\eta}, \quad (21)$$

$$\begin{aligned}L_F(\Phi) &= \frac{\partial^3 \Phi}{\partial \eta^3} + \frac{\partial^2 \Phi}{\partial \eta^2}, \quad L_R(\Theta) = \frac{\partial^2 \Theta}{\partial \eta^2} + \frac{\partial \Theta}{\partial \eta}, \\ L_\theta(\Theta) &= \frac{\partial^2 \Theta}{\partial \eta^2} + \frac{\partial \Theta}{\partial \eta},\end{aligned}\quad (22)$$

where these linear operators satisfy the conditions $L_F(c_1 e^{-\eta} + c_2 \eta + c_3) = 0$, $L_R(c_4 e^{-\eta} + c_5) = 0$, and $L_\theta(c_6 e^{-\eta} + c_7) = 0$ with c_{1-7} as constants. The nonlinear operators are chosen by

$$\begin{aligned}N_F &= (1 + K_0) \frac{\partial}{\partial \eta} \left(|R|^{n-1} \frac{\partial^2 F}{\partial \eta^2} \right) \\ &\quad + K_0 \frac{\partial}{\partial \eta} \left(|R|^{n-1} R \right) + \frac{1}{n+1} F \frac{\partial^2 F}{\partial \eta^2}, \\ N_R &= \left(1 + \frac{K_0}{2} \right) \frac{\partial}{\partial \eta} \left(|R|^{n-1} \frac{\partial R}{\partial \eta} \right) + \frac{1}{n+1} \left(R \frac{\partial F}{\partial \eta} + F \frac{\partial R}{\partial \eta} \right) \\ &\quad - K_0 |R|^{n-1} \left(2R + \frac{\partial^2 F}{\partial \eta^2} \right), \\ N_\theta &= \frac{\partial}{\partial \eta} \left(|R|^{n-1} \frac{\partial \theta}{\partial \eta} \right) + \frac{\text{Pr}_N}{n+1} F \frac{\partial \theta}{\partial \eta}.\end{aligned}\quad (23)$$

Notice that $p \in [0, 1]$ is an embedding parameter; h_F , h_R , and h_θ indicate non-zero auxiliary parameters for velocity, micro-rotation, and temperature fields, respectively. We then construct the relevant deformation equations as follows: 0th-order deformation equations:

$$\begin{aligned}(1-p)L_F[f(\eta, p) - F_0(\eta)] &= ph_F N_F[F(\eta, p), R(\eta, p), \theta(\eta, p)], \\ (1-p)L_R[r(\eta, p) - R_0(\eta)] &= ph_R N_R[F(\eta, p), R(\eta, p), \theta(\eta, p)], \\ (1-p)L_\theta[w(\eta, p) - \theta_0(\eta)] &= ph_\theta N_\theta[F(\eta, p), R(\eta, p), \theta(\eta, p)].\end{aligned}\quad (24)$$

Subjecting to the boundary conditions:

$$\begin{aligned}f(0, p) &= 0, \quad \left. \frac{\partial f(0, p)}{\partial \eta} \right|_{\eta=0} = 1, \quad \left. \frac{\partial f(\eta, p)}{\partial \eta} \right|_{\eta \rightarrow +\infty} = 0, \\ r(0, p) &= -\frac{1}{2} \left. \frac{\partial^2 f(0, p)}{\partial \eta^2} \right|_{\eta=0}, \quad r(\infty, p) = 0, \\ w(0, p) &= 1, \\ w(\infty, p) &= 0.\end{aligned}\quad (25)$$

Obviously, as p increases from 0 to 1, the functions $f(\eta, p)$, $r(\eta, p)$, and $w(\eta, p)$ vary from the initial approximation solutions $F_0(\eta)$, $R_0(\eta)$, and $\theta_0(\eta)$ to the exact solutions $F(\eta)$, $R(\eta)$, and $\theta(\eta)$, respectively. Furthermore, we rewrite the

functions mentioned earlier in the Taylor series at $p=0$ as

$$\begin{aligned} f(\eta, p) &= F_0(\eta) + \sum_{k=1}^{\infty} F_k(\eta) p^k, \\ r(\eta, p) &= R_0(\eta) + \sum_{k=1}^{\infty} R_k(\eta) p^k, \\ w(\eta, p) &= \theta_0(\eta) + \sum_{k=1}^{\infty} \theta_k(\eta) p^k, \end{aligned} \quad (26)$$

where $F_k(\eta) = \frac{1}{k!} \frac{\partial^k f(\eta, p)}{\partial p^k} \Big|_{p=0}$, $R_k(\eta) = \frac{1}{k!} \frac{\partial^k r(\eta, p)}{\partial p^k} \Big|_{p=0}$, and $\theta_k(\eta) = \frac{1}{k!} \frac{\partial^k w(\eta, p)}{\partial p^k} \Big|_{p=0}$. The series (26) are convergent at $p = 1$, so the solution series are obtained as

$$\begin{aligned} F(\eta) &= F_0(\eta) + \sum_{k=1}^{\infty} F_k(\eta), \quad R(\eta) = R_0(\eta) + \sum_{k=1}^{\infty} R_k(\eta), \\ \theta(\eta) &= \theta_0(\eta) + \sum_{k=1}^{\infty} \theta_k(\eta). \end{aligned} \quad (27)$$

k th-order deformation equations:

$$\begin{aligned} L_F [F_k(\eta) - \chi_k F_{k-1}(\eta)] &= h_F H(\eta) \Psi_k^F(\eta), \\ L_R [R_k(\eta) - \chi_k R_{k-1}(\eta)] &= h_R H(\eta) \Psi_k^R(\eta), \\ L_\theta [\theta_k(\eta) - \chi_k \theta_{k-1}(\eta)] &= h_\theta H(\eta) \Psi_k^\theta(\eta), \end{aligned} \quad (28)$$

satisfying the boundary conditions

$$\begin{aligned} F_k(0) &= 0, \quad F'_k(0) = 0, \quad F'_k(\infty) = 0, \quad R_k(0) + \frac{1}{2} F''_k(0) = 0, \\ R_k(\infty) &= 0, \quad \theta_k(0) = 0, \quad \theta_k(\infty) = 0, \end{aligned} \quad (29)$$

where $H(\eta)$ is the so-called auxiliary function which needs to be determined appropriately to promote the convergent rate of solutions,

$$\begin{aligned} \Psi_k^F(\eta) &= (1 + K_0) \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial p^{k-1}} \left[\frac{\partial}{\partial \eta} \left(|R|^{n-1} \frac{\partial^2 F}{\partial \eta^2} \right) \right]_{p=0} \\ &+ K_0 \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial p^{k-1}} \left[\frac{\partial}{\partial \eta} \left(|R|^{n-1} R \right) \right]_{p=0} \\ &+ \frac{1}{n+1} \sum_{j=0}^{k-1} F_j(\eta) F''_{k-j-1}(\eta), \end{aligned} \quad (30)$$

$$\begin{aligned} \Psi_k^R(\eta) &= \left(1 + \frac{K_0}{2} \right) \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial p^{k-1}} \left[\frac{\partial}{\partial \eta} \left(|R|^{n-1} \frac{\partial R}{\partial \eta} \right) \right]_{p=0} \\ &+ \frac{1}{n+1} \left(\sum_{j=0}^{k-1} R_j(\eta) F'_{k-j-1}(\eta) + \sum_{j=0}^{k-1} R'_j(\eta) F_{k-j-1}(\eta) \right) \\ &- K_0 \sum_{j=0}^{k-1} |R_j(\eta)| \left(2R_{k-j-1}(\eta) + F''_{k-j-1}(\eta) \right), \end{aligned} \quad (31)$$

$$\begin{aligned} \Psi_k^\theta(\eta) &= \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial p^{k-1}} \left[\frac{\partial}{\partial \eta} \left(|R|^{n-1} \frac{\partial \theta}{\partial \eta} \right) \right]_{p=0} \\ &+ \frac{\text{Pr}_N}{n+1} \sum_{j=0}^{k-1} F_j(\eta) \theta'_{k-j-1}(\eta), \end{aligned} \quad (32)$$

$$\chi_k = \begin{cases} 0 & k = 1 \\ 1 & k > 1 \end{cases}. \quad (33)$$

The convergences of series approximation solutions by HAM are strongly dependent on the auxiliary parameter, then as an example, Fig. 2 depicts the admissible values range of auxiliary parameters h_F , h_R , and h_θ located in the horizontal part (as noted by Liao) corresponding to the 9-th order HAM solutions of velocity and temperature fields with $n = 0.8$ and $\text{Pr}_N = 2$ for various material parameters K_0 . We can make the choices that $-0.7 \leq h_F \leq -0.4$, $-0.5 \leq h_R \leq -0.4$, and $-1 \leq h_\theta \leq -0.4$ are the acceptable options for convergent solutions as shown in Fig. 2. Sometimes, the conditions of the range of auxiliary parameters alone are not enough; therefore, the optimal values of the auxiliary parameters remain to be determined via minimum error calculation for approaching accurate enough solutions. The change of h_F , h_R , and h_θ will influence the analytical solutions significantly, in particular for the nonlinear boundary layer equations with power law form in the present problem. Theoretically, the average residual error at k th-order can be defined as in discrete format⁴¹

$$E_k = \frac{1}{m} \sum_{j=0}^m \left[N_{F,\theta} \left(\sum_{i=0}^k f_i(j\Delta x) \right) \right]^2, \quad (34)$$

where $\Delta x = 10/m$ and $m = 20$ for the boundary layer problems in general.^{45,50} We also find that Eq. (33) can greatly decrease the CPU time in the comparison of the so-called exact square residual error as $\int_0^{+\infty} (N_{F,\theta} [\sum_{i=0}^k f_i(\eta)])^2 d\eta$ for the present problem.^{45,50} Fig. 3 depicts the average residual error of homotopy-series solutions (26) of momentum Equation (9) in Fig. 3(a) and energy Equation (11) in Fig. 3(b), respectively.

FIG. 2. The h_F -curve of $F''(0)$, h_R -curve of $R'(0)$, and h_θ -curve of $\theta'(0)$ obtained by the 9-th order approximation of the HAM solutions with $n = 0.8$ ($\text{Pr}_N = 2$) for various K_0 for the new model.

FIG. 3. Average residual error versus auxiliary parameters h_F (a) and h_θ (b) at 9th-order HAM solutions with $n = 0.8$ ($Pr_N = 2$) for various K_0 for the new model.

The optimal values of h_F and h_θ are given at the corresponding minimum average residual errors. One can notice that these optimal values are obviously dependent on additional parameters, such as K_0 .

IV. RESULTS AND DISCUSSION

We first introduce the nonlinear model in the rheological and heat-conducting constitutive relations of a micropolar fluid by considering generalized n -diffusion theory. Such nonlinearity with the power law criterion can highlight the non-Newtonian characteristics of a MF in that the apparent viscosity and heat conductivity are susceptible by micro-rotation of dispersed particles. As pointed out by Peters,³⁵ oriented micro-rotation effects of particles dispersing in the suspension, e.g., a MF, will reduce the effective viscosity with shear flow. Theoretically, the generalized power law model can present a new perspective to understand a MF being largely associated with the shear-thinning non-Newtonian fluids.

As elucidated earlier, the oriented micro-rotation has positive directions in the boundary layer flow, i.e., $R(\eta) > 0$, then the absolute value notations in Eqs. (9)–(11) can be removed. Although we cannot find the accurate values of n for a MF with

a certain particle loading (little research has been reported), the range $0.5 \leq n \leq 1$ can probe the shear-thinning effects on boundary layer flow and heat transfer clearly. Fig. 4 depicts the appreciable effects of the power exponent n on the normalized velocity and temperature profiles of a MF, which manifests a key role of the present model on presenting the obvious shear-thinning behavior.

One can see in Fig. 4(a) that the normalized velocity profiles are increased as n decreases from 1. Such results illuminate that the larger divergent values of n from the ordinary model with $n = 1$, the more obvious shear-thinning effects, which is in agreement with the conclusions by Peters that an apparent decrease of the effective viscosity of the suspension (MFs) is attributed to the rotation of the particles.³⁵ With decreasing values of n , the shear-thinning behavior caused by the collective effects of micro-particle rotation results in better fluid migration layer-by-layer, consequently the velocity boundary layer is noticeably thickened as shown in the inset of Fig. 4(a), namely, the viscous diffusions with large n are enhanced by exerting the same shear effects. For the temperature field, the visible increased temperature profiles with

FIG. 4. (a) The normalized velocity profiles and (b) the temperature profiles for different n ($Pr_N = 2$, $K_0 = 0.5$). The numerical solution and the HAM solution with $n = 1$ are in good agreement. The δ_m inset (a) and δ_e inset (b) are both fitted by the quadratic polynomial of n with high precision and the red solid line insets are the fitting curves.

decrease of n are shown in Fig. 4(b), as a result the effects of n also make the thermal boundary layer thicken, which shows a similar trend with the velocity profiles, whereas, the influence of n on the temperature field is greater than the velocity field. The thickened temperature boundary layer with decreasing n (see the inset of Fig. 4(b)) not only depends on the heat conduction behavior of a MF modeled by modified thermal conductivity but also to a great extent on the shear-thinning behavior as a rheological characteristic in shear flow.

It is noteworthy that δ_m and δ_e are both nonlinearly dependent on n seeing from the insets of Figs. 4(a) and 4(b), respectively. Finding the accurate and simple relations between δ_m or δ_e and n are important in practical applications. One can imagine how difficult and inconvenient the processes would be if we calculate the original complex nonlinear equations repeatedly for every certain n . Therefore, we desire to show the useful prediction curves by means of several values of n . We are pleased to find that the polynomial functions can fit such dependent relations quite well in comparison with any other functions and the quadratic polynomial is adopted to fit the data for δ_m and δ_e ultimately due to the minimal fitting coefficients and high fitting precision. The prediction formulas for momentum layer thickness (MLT) and enthalpy layer thickness (ELT) are obtained as $\delta_m = 1.208 - 0.202n - 0.207n^2$ and $\delta_e = 1.508 - 0.672n - 0.139n^2$, respectively, with fitting precision $R^2 = 0.9999$, and the corresponding errors are listed in the inset of Figs. 4(a) and 4(b). In addition, the HAM solutions of normalized velocity and temperature are compared with the numerical solutions by Runge-Kutta method for the $n = 1$ case, which presents a good agreement between them in Fig. 4.

The distribution of particles angular velocity is another important factor in a micropolar fluid, which is susceptible to the nonlinear spin gradient viscosity by power exponent n . Fig. 5 illuminates the unique micro-rotation velocity distributions of the particles for different shear-thinning performance controlled by n .

We notice that the micro-rotation velocity profile reduces from a certain value on the sheet surface to zero at the edge of the boundary layer (BL). The velocity on the surface declined

FIG. 5. The non-dimensional micro-rotation (angular) velocity profiles for various n .

with n , whereas it increased rapidly across the boundary layer with the decrease of n , consequently they intersect each other. As shown in Fig. 5, the decreased rate of the granular velocity profile is small at first (close the surface) and then large (reaching the edge of the BL), which signifies a critical point, the so-called inflection point for every angular velocity profile. It is the point where the first derivative of a profile reaches the extreme value mathematically, which separates the concave and convex of a profile. The inset in Fig. 5 shows the inflection points where the R' reaches the extreme value for various n . It can be seen that the inflection points are increased from $\eta = 1.5$ to $\eta = 2.4$ as n decreases from 1 to 0.5 correspondingly. Such inflection points are helpful to understand the different responses of R to the parameter n , i.e., the increase or decrease with it, when reaching the critical position within the boundary layer. The results indicate that the micro-rotation velocity of particles declines slowly as the shear-thinning behavior strengthens (the smaller n).

As noted earlier, one can derive the dimensionless similarity shear stress and heat current density as $\tilde{\tau} = -(1 + K_0)|R|^{n-1}F'' - K_0R^n$ and $\tilde{q} = -|R|^{n-1}\theta'$, respectively. We further probe how the new models play the critical role in the transition of both internal shear stress and heat current density in a MF showing in Figs. 6(a) and 6(b), respectively. Moreover, the corresponding local skin friction coefficient and local Nusselt number are located in the insets. An increasing n always results in an enhanced shear stress transfer in the whole boundary layer, meanwhile the heat flux density through the fluid is strengthened prominently by n , which provides the evidence that the novel models can characterize how the shear-thinning behavior leads to these unusual viscous and thermal diffusions layer-by-layer.

As expected, the local skin friction coefficient C_f^{local} is increased, but the local Nusselt number Nu_x^{local} is reduced as the values of n decline, as shown in the insets of Figs. 6(a) and 6(b).

According to an analogous experience earlier, we also fit the data by the quadratic polynomial, as a result both of them meet the high fitting accuracy with $R^2 = 0.9999$ seeing the insets in Fig. 6. The fitting curves provide the rapid and accurate predictions of C_f^{local} and Nu_x^{local} in terms of n with the simple formulas, which are useful to achieve the velocity and energy transfer information at the sheet surface. All these conclusions are significant enough to reveal their intrinsic characteristics in which the pseudo-plasticity is predominant once the shear effects are employed, whereas such non-Newtonian behaviors of a MF are impossible to be achieved by still using the ordinary model.

The modified material parameter K_0 of a micropolar fluid by the novel model (a degradation to the parameter K in ordinary model with $n = 1$) denotes the ratio of the generalized apparent vortex viscosity coefficient to the generalized apparent dynamic viscosity coefficient, which determines some non-Newtonian behaviors of a MF. In the traditional model, a MF exhibits the pure Newtonian behavior when $K = 0$, in other words, the collective micro-rotation effects of particles, e.g., vortex viscosity, are vanishing. However, in the new model, it is meaningless for the case $K_0 = 0$ with $n \neq 1$ because

FIG. 6. The non-dimensional shear stress diffusions (a) and heat flux diffusions (b) for different n . The local skin friction coefficient C_f^{local} inset (a) and local Nusselt number Nu_x^{local} inset (b) are both fitted by the quadratic polynomial of n with high precision and the red solid lines in the insets are the fitting curves for guiding eyes.

the shear-thinning behavior ($n \neq 1$) is mainly caused by the presence of great effects of vortex viscosity.

We explore the effects of K_0 on the shear flow and heat transfer of a MF described by the new model with $n = 0.8$ as shown in Fig. 7. As you can see, the increasing K_0 leads to a large increase in the velocity boundary layer thickness but a moderate decrease in the thermal boundary layer thickness. The results indicate that the momentum transfer layer-by-layer is enhanced significantly due to the increase of viscosity caused by the collective micro-rotation of particles, i.e., large values of K_0 , on the other hand, the thermal diffusions are weakened slightly. What is more, the local skin friction coefficient C_f^{local} and local Nusselt number Nu_x^{local} are both increased with K_0 but in divergent ways. Herein, the rising rate of C_f^{local} is faster, while that of Nu_x^{local} is slower with the increase of K_0 . Fig. 7(c) illustrates the striking effects of K_0 on the micro-rotation velocity in the novel model with $n = 0.8$ that the micro-rotation velocity is decreased monotonously across the entire boundary layer and such velocity at the sheet surface is reduced with K_0 , whereas it is increased with K_0 above a certain position η_c in the boundary layer. There are also the

FIG. 7. (a) The normalized velocity profiles, (b) temperature profiles, and (c) granular velocity profiles are dependent on K_0 for the new model $n = 0.8$, $Pr_N = 2$. The local skin friction coefficient C_f^{local} inset in (a) and local Nusselt number Nu_x^{local} inset in (b) are increasing with K_0 in divergent avenues marked by the lines for guiding eyes.

inflection points for every micro-rotation velocity profile for different K_0 , which present the critical point of declining rate of R . Obviously, the inflection points are increased with K_0 as shown in the inset in Fig. 7(c), which is analogous to the cases in Fig. 5. In addition, we can draw the conclusion that the

certain position is $\eta_c = 3.0$, namely, the micro-rotation velocity increases with K_0 above $\eta_c = 3.0$, but decreases below this position.

V. CONCLUSION

In the present work, we have investigated theoretically the boundary layer flow and heat transfer of a micropolar fluid on the stretching sheet by introducing the novel constitutive models. Such models taking into consideration nonlinear generalized n -diffusion theory with power law criterion were first proposed to characterize the non-Newtonian behaviors in rheology and heat conduction of a MF caused by the collective micro-rotation effects of particles. The approximate analytical solutions of nonlinear similarity equations are obtained by utilizing HAM, and we can draw the conclusions from the results by some points.

- (1) The power law dependence of angular velocity N is modeled in rheological and heat-conducting constitution relations of a MF. The power exponent n which governs the magnitude of shear-thinning behavior plays an important role on the entire system. As n decreases from 1, the velocity and temperature profiles are both increased, specifically, their respective boundary layers will be thickened, which denotes these micropolar fluids as pseudo-plastic non-Newtonian fluids.
- (2) With the decrease of n , the micro-rotation velocity on the surface of the sheet declined, while it increased above the critical position $\eta_c = 2.4$ in the boundary layer. Consequently, the different intersections between the angular velocity profiles can be captured in Fig. 5. Moreover, the inflection points of granular velocity profiles are rising as n decreases.
- (3) The increase of modified material parameter K_0 for the new model $n = 0.8$ will result in the thickened velocity boundary layer obviously, but give rise to the thin thermal boundary layer. The n and K_0 are both characteristics of non-Newtonian behaviors of a MF in the new model. The C_f^{local} is increased, but the Nu_x^{local} is reduced with decreasing values of n , however, they are both increased with K_0 . The effects of K_0 on the granular velocity are analogous to that of n . The nonlinear dependences of the dimensionless engineering parameters including C_f^{local} , Nu_x^{local} , δ_m , and δ_e on n are all formulated by the quadratic polynomial accurately.

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