Published in final edited form as:

IEEE Trans Med Imaging. 2000 May ; 19(5): 548–555. doi:10.1109/42.870265.

Unmatched Projector/Backprojector Pairs in an Iterative Reconstruction Algorithm

Gengsheng L. Zeng* [Member, IEEE] and Grant T. Gullberg [Senior Member, IEEE]

Department of Radiology, University of Utah, Salt Lake City, UT 84108-1218 USA

Abstract

Computational burden is a major concern when an iterative algorithm is used to reconstruct a three-dimensional (3-D) image with attenuation, detector response, and scatter corrections. Most of the computation time is spent executing the projector and backprojector of an iterative algorithm. Usually, the projector and the backprojector are transposed operators of each other. The projector should model the imaging geometry and physics as accurately as possible. Some researchers have used backprojectors that are computationally less expensive than the projectors to reduce computation time. This paper points out that valid backprojectors should satisfy a condition that the projector/backprojector matrix must not contain negative eigenvalues. This paper also investigates the effects when unmatched projector/backprojector pairs are used.

Index Terms

Backprojection; image reconstruction; iterative algorithm

I. Introduction

In an iterative reconstruction algorithm, a projector/back-projector pair that models the imaging geometry and physics is an essential component. The projector in an iterative algorithm should model the imaging geometry and physics as accurately as possible, but the requirements on the backprojector may be more relaxed. If the projector is represented by a matrix C , the matched backprojector is the transposed matrix C^T . If another matrix B^T is used as the backprojector, C and B^T are then an unmatched projector/backprojector pair. One motivation for using unmatched projector/backprojector pairs is to speed up the iteration process and to reduce the total computational burden.

Unmatched projector/backprojector pairs have been investigated and used in image reconstruction [1]–[6]. For example, a ramp filter can be incorporated in the backprojector B^T to greatly increase the rate of convergence [1], [8] because an iterative algorithm converges faster when $B^T C$ is closer to the identity matrix I in the sense that the eigenvalues of $B^T C$ are clustered closer together. This approach is adopted in the iterative Chang algorithm [1]. Others have attempted to use unmatched projector/backprojector pairs to reduce computation cost by modeling attenuation and scatter in the projector but not in the

larry@doug.med.utah.edu.

backprojector [2]–[6]. In our iterative cone-beam reconstruction experiences [2], we have observed that a matched, line-length-weighted, projector/backprojector pair produces ring artifacts, whereas an unmatched pair, in which the projector is ray-driven with line-length-weighting but the backprojector is voxel-driven with bilinear interpolation on the projection plane, can effectively remove the ring artifacts. Therefore, it is important to investigate the unmatched projector/backprojector pairs further and to build a theoretical foundation for their application.

This paper will address fundamental issues, including whether it is legitimate to use an unmatched projector/backprojector pair in an iterative algorithm, whether one can use any backprojector, what restrictions are involved when choosing a projector/backprojector pair, and does the iterative algorithm converge to a correct answer if it converges at all? These questions will be answered in Section II in terms of the eigenvalues and singular values of the combined matrix $B^T C$. In Section III, the eigenvalue and singular value studies are performed for some matched and unmatched projector/backprojector pairs and images are reconstructed for some selected pairs to verify the results.

II. Theory

We will consider a generalized Landweber iterative algorithm [9] to solve $CX = P$ with unmatched projector C and backprojector B^T [9]

$$X^{(k+1)} = X^{(k)} + SB^T (P - CX^{(k)}) \quad (1)$$

where $S = \text{diag}\{s_1, s_2, \dots, s_n\}$ with positive diagonal elements controlling the step size of each iteration. We will develop a condition for B under which (1) will converge.

In $CX = P$, we assume that the number m of projection elements in P not less than the number n of image voxels in X . Let

$$T = I - SB^T C \quad (2)$$

then (1) becomes

$$\begin{aligned} X^{(k+1)} &= TX^{(k)} + SB^T P \\ &= T^2 X^{(k-1)} + (T + I)SB^T P \\ &\quad \dots \\ &= T^{k+1} X^{(0)} + (T^k + \dots + T + I)SB^T P. \end{aligned} \quad (3)$$

Therefore, (1) converges if and only if

$$\max_i \{|\lambda_i|\} < 1 \quad (4)$$

where λ_i 's are the eigenvalues of the square matrix T [7]. If inequality (4) holds, the algorithm converges to [9]

$$X^{(\infty)} = (B^T C)^{-1} B^T P \quad (5)$$

because $T^{(\infty)} = 0$ and $(I - T)^{-1} = I + T + \dots + T^k + \dots$

As a special case, if C and B are square and each has an inverse, then

$$X^{(\infty)} = (B^T C)^{-1} B^T P = C^{-1} (B^T)^{-1} B^T P = C^{-1} P. \quad (6)$$

Namely, if C^{-1} and B^{-1} exist, the unmatched backprojector B^T does not alter the true solution $C^{-1}P$ at all. However, the solution given in (5) generally depends on the choice of B .

In the following example, we use singular value decomposition (SVD) to investigate how the choice of the backprojector affects the solution in (5). Let

$$C = U_c \Lambda_c V_c^T \quad (7)$$

where $U_c U_c^T = I_{m \times m}$, $V_c V_c^T = I_{n \times n}$, and Λ_c is an $m \times n$ matrix with singular values $\{\lambda_i^c\}$ as its diagonal elements and zero elsewhere. Let B have a similar decomposition

$$B = U_b \Lambda_b V_b^T \quad (8)$$

then we have

$$(B^T C)^{-1} B^T = (V_c \Lambda_c^\dagger U_c^T) \left(U_b \begin{bmatrix} I_{n \times n} & 0_{n \times (m-n)} \\ 0_{(m-n) \times n} & 0_{(m-n) \times (m-n)} \end{bmatrix} U_b^T \right) = C^\dagger F_b \quad (9)$$

where $C^\dagger = V_c \Lambda_c^\dagger U_c^T$ is a pseudoinverse of C , and

$$F_b = U_b \begin{bmatrix} I_{n \times n} & 0_{n \times (m-n)} \\ 0_{(m-n) \times n} & 0_{(m-n) \times (m-n)} \end{bmatrix} U_b^T \quad (10)$$

is a semipositive definite, symmetric matrix with the property that $F_b \rightarrow I_{m \times n}$ as $n \rightarrow m$. Equation (5) can be rewritten as

$$X^{(\infty)} = C^\dagger F_b P. \quad (11)$$

This result implies that the projection data are first filtered by F_b , and then the pseudoinverse operator of C is applied to the result. When m and n are close in value, the prefiltering does not have a significant effect on the result.

Now, we give a requirement for a valid projector/backprojector pair. We call a projector/backprojector pair valid if algorithm (1) converges to (11), which is guaranteed by inequality (4).

In order to be certain that inequality (4) is satisfied, we require that the eigenvalues of $I - SB^TC$ be in the interval $(-1, 1)$, namely, the eigenvalues of SB^TC be in the interval $(0, 2)$. Because S in (2) contains positive scaling factors that can be selected as small as necessary, it then becomes a requirement that the eigenvalues of B^TC be positive. For example, if the square matrix B^TC is diagonally dominant with positive diagonal elements, we know that the eigenvalues of B^TC will be positive [7].

III. Methods and Results

In this section, the minimum eigenvalues are calculated with the power method [7] to investigate the validity of some projector/backprojector pairs. The singular value spectrum is also calculated to study the potential speed-up of the projector/back-projector pairs in an iterative algorithm. Finally, the Landweber and maximum likelihood expectation maximization (ML-EM) algorithms [10] are used with selected pairs to observe the convergent and divergent effects in the reconstruction of selected images.

A. Validity of a Projection/Backprojection Pair

In this section, we propose using the power method [7] to find the minimal eigenvalue of B^TC and to determine whether that value is negative. The power method is an iterative procedure used to find the *dominant eigenvalue* that has the *largest* absolute value. We first use the power method to find the dominant eigenvalue λ_1 of B^TC . If $\lambda_1 < 0$, the backprojector B is not valid. If $\lambda_1 > 0$, we choose a constant $\beta > \lambda_1$ and form a matrix A

$$A = B^TC - \beta I_{n \times n}. \quad (12)$$

We then apply the power method to obtain the dominant eigenvalue σ_1 of A . Therefore, $\lambda_n = \sigma_1 - \beta$ is the smallest eigenvalue of $B^T C$. If $\lambda_n < 0$, the backprojector B is not valid. If $\lambda_n \geq 0$, the backprojector B is valid.

The procedure for using the power method to find the largest (in the sense of absolute value) eigenvalue of matrix A is as follows:

Step 1) Pick an initial vector $Z^{(0)}$, and let $k = 1$.

Step 2) Compute $w^{(k)} = AZ^{(k-1)}$.

Step 3) Let β_k be the component of $w^{(k)}$ that is maximum in value. Define $z^{(k)} = w^{(k)}/\beta_k$.

Return to Step 2), and repeat until $z^{(k)}$ converges to the dominant eigenvalue of A .

B. Examples Verifying Valid Projector/Backprojector Pairs

Examples of projection/backprojection pairs are presented here to illustrate how the power method can be used to select valid projection/backprojection pairs. In these examples, more than 5000 iterations were usually needed to converge to the dominant eigenvalue. In the examples presented, the image size of $40 \times 40 \times 8$ was relatively small. The detector size was 40×8 , and the number of projection views over 360° was 120. These examples are for illustrative purposes, and the results may hold for other sizes of applications. It is suggested that if we intend to use a projector/backprojector pair, the validity for its use should be verified first. By validity, we mean that the smallest eigenvalue of $B^T C$ must be positive.

In these examples, parallel beam imaging geometry was used. The three-dimensional (3-D) detector geometric response was determined by using the following parameters: collimator hole length of 2.7 cm, hole radius of 0.1 cm, and distance from center of rotation to the detector of 26.8 cm. The geometric response was depth dependent. The point response function was modeled by a 7×7 inverse-cone (see [11] for details). A constant attenuation coefficient of $\mu = 0.1$ per pixel-length was used, the attenuator was an ellipsoid of semiaxes of 19.5, 19.5, and 3.5 pixels, and the pixel size was 0.67 cm. No scatter was modeled in the projector. The projector C remained unchanged for all examples and used line-length weighting with attenuation and 3-D detector response modeling [11]. The backprojection matrix was B^T . The results of the power method for eight of projectors/backprojector pairs are listed in Table I. In backprojectors 2, 4, 6, and 8, the ramp filtering was performed row-by-row one dimensionally.

It is known that the rate of convergence in an iterative algorithm depends on the distribution of the eigenvalues of $B^T C$. A faster rate of convergence is expected if the eigenvalues of $B^T C$ cluster closely together. Singular values of the matrix and eigenvalues of the matrix A are closely related in that the singular values of A are the square roots of the eigenvalues of $A^T A$. Because the singular values are easier to compute than are the eigenvalues, we computed the singular values of $B^T C$ for the examples in Table I. The singular value spectra were normalized and displayed using a log scale in Fig. 1. Note that the singular values are always nonnegative; therefore, the singular values cannot be used to verify whether a projector/backprojector pair is valid. These examples imply that it is not necessary to model

the attenuation in the backprojector and that a ramp filter has the potential to greatly improve the rate of convergence.

C. Landweber and ML-EM Reconstructions

In this section, we apply sets of valid and invalid projector/backprojector pairs in the Landweber iterative algorithm and the ML-EM algorithm. In the Landweber algorithm (1), we chose all diagonal elements in S to be one constant s . We chose s as 0.05 if a ramp filter was not in the backprojector, and s as 0.5 if a ramp filter was used in the backprojector. The parameter s was determined by a trial-and-error method, such that the algorithm does not diverge for a valid projector/back-projector pair.

We modified the ML-EM algorithm so that it allowed the use of an unmatched projector/backprojector pair

$$x_i^{(k+1)} = \frac{x_i^{(k)}}{\sum_j b_{ji}} \sum_j \frac{p_j b_{ji}}{\sum_l c_{jl} x_l^{(k)}}. \quad (13)$$

In fact, this algorithm can be written in an additive form [2]

$$X^{(k+1)} = X^{(k)} + S^{(k)} B^T \Lambda^{(k)} (P - C X^{(k)}), \quad (14)$$

where

$$S^{(k)} = \text{diag} \left\{ x_i^{(k)} / \sum_j b_{ji} \right\} \quad (15)$$

and

$$\Lambda^{(k)} = \text{diag} \left\{ 1 / \left(\sum_l c_{jl} x_l^{(k)} \right) \right\} \approx \text{diag} \{ 1/p_j \}. \quad (16)$$

Using this notation and combining the Λ matrix with B^T , the principles in Section II still apply. The necessary condition to make $I - S^{(k)} B^T \Lambda^{(k)} C$ to be a contracting operator, that is,

$\prod_{k=1}^N [I - S^{(k)} B^T \Lambda^{(k)} C] \rightarrow 0$ as $N \rightarrow \infty$, is that $B^T \Lambda^{(k)} C$ does not have negative eigenvalues when k is large enough. However, in general the converged solution $X^{(\infty)}$ of (13) will be different from that given in (5). In order to preserve the nonnegativity property

of the ML-EM algorithm, the negative values were rectified to zero in the backprojector whenever a ramp filter was used in the backprojector.

In all examples in this section, a computer-generated phantom was used. The phantom consisted of three ellipsoids. Ellipsoid number 1 had semiaxes of 18, 18, and 3 pixels with intensity 5 and was centered at (0, 0, 0). Ellipsoid number 2 had semiaxes of 5, 5, and 1.5 pixels with intensity 7.5 and was centered at (−7, 0, 0). Ellipsoid number 3 had semiaxes of 5, 5, and 1.5 pixels with intensity 2.5 and was centered at (−7, 0, 0). A uniform, ellipsoidal attenuator had semiaxes of 19.5, 19.5, 3.5 pixels, attenuation coefficient of $\mu = 0.1$ per pixel length, and was centered at (0, 0, 0). The imaging geometry was exactly the same as that in Section III-B.

The same projector was used in projection data generation and in the reconstruction algorithms. This approach should not be encouraged in developing new algorithms, because the robustness of the new algorithm needs to be tested for model-mismatch errors. In this paper, no model-mismatch errors are considered because we wanted to eliminate the deterministic noise. However, random noise was added in some studies. Two sets of projection data were used for reconstruction. One set was noise free. In the other set, the projection data were randomized using a Poisson noise model, with the mean and variance being the noiseless projection values.

Reconstruction results, in terms of the normalized mean square error as a function of iteration number, are shown in Figs. 2–5 using Landweber and ML-EM algorithms with noiseless and noisy projection data, respectively. The central slice of each reconstruction is also shown in those figures with a profile drawn horizontally across the center. The images displayed were chosen by using an unrealistic stopping rule so that the iteration was stopped when the normalized mean square error reached the minimum or N_{max} , whichever was smaller. Here, N_{max} was 250 for noisy data and 500 for noiseless data.

Fig. 1 predicts the SVD spectra for different projector/back-projector pairs. It is observed that using a ramp filter and not modeling the system point response can speed the iterative reconstruction. Adding a ramp filter does not invalidate a backprojector. However, not modeling the system point response makes the backprojector invalid.

Fig. 2 illustrates the results of Landweber reconstructions with noiseless data. It is seen that when using noiseless data, all projector/backprojector pairs, including invalid pairs, demonstrate a convergent trend. A ramp filter is shown to provide a faster convergent rate. When invalid pairs are used, we believe that a divergent trend will sooner or later appear when the iteration number is large enough.

Fig. 3 shows the behavior of the Landweber algorithm in a noisy environment. For all projector/backprojector pairs, valid and invalid alike, the algorithm first has a short convergent trend and then diverges. In the “convergent period,” a ramp filter gives a faster convergent rate. This is a usual phenomenon for an ill-conditioned problem, in which a small perturbation in the input data can cause a very large deviation in the outcome. The solution $X^{(\infty)} = (B^T C)^{-1} B^T P$ is very sensitive to the noise in projection data P . For valid projector/

backprojector pairs, the Landweber algorithm does not diverge but converges to a perturbed solution, which is far away from the true solution.

We can view an iterative algorithm as a summation procedure; it picks up a frequency component at each iteration. Usually, a lower frequency component is picked up sooner than a higher frequency component. In a lot of cases, the desired solution and the solution that an iterative algorithm converges to have many lower frequency components in common. Their higher frequency components differ. Therefore, an initial convergent trend and a divergent trend can be observed in a mean square error versus an iteration-number curve.

Fig. 4 displays the normalized mean square error curves when using the ML-EM algorithm with noiseless data. All of the curves with the ramp filter show a small “divergent” trend after some point, even for those valid projection/backprojection pairs. As implied in (11), the solution is backprojector and algorithm dependent. The small “divergent” trend may be a convergent trend to a perturbed solution (because of intrinsic small computation errors) that is different from the true one.

Fig. 5 presents a similar situation to Fig. 3, except that the Landweber algorithm is replaced by the ML-EM algorithm. It is interesting to observe that the curves do not change much with and without the modeling of attenuation.

IV. Discussion

In some cases, it is beneficial to use an unmatched projector/backprojector pair in an iterative reconstruction algorithm to speed an iterative algorithm. The composite backprojector–projector matrix B^TC should be as similar to the identity matrix as possible, in the sense that the eigenvalues of B^TC should be clustered as close as possible. An SVD spectrum is a useful tool for determining a backprojector that gives a fast convergent rate.

In order to guarantee the convergence, the backprojector cannot be arbitrarily chosen. A valid backprojector B^T should not have negative eigenvalues in B^TC . The power method can be used to find the smallest eigenvalue. Another method is to test if B^TC is a diagonally dominant matrix, because a diagonally dominant matrix is positive definite. A diagonally dominant B^TC can be achieved by meeting the following requirement: for every projection ray in C , there should be an identical ray in B so that a point in the image space can be backprojected to the same location after projection and back-projection, thereby creating a large value at the corresponding diagonal position in B^TC . If the iteration number goes to infinity, a valid backprojector can provide a finite solution, whereas an invalid backprojector makes the algorithm diverge to infinity.

A filter, for example a ramp filter, is useful when applied between the projector and backprojector to change the SVD spectrum, provided the filter does not introduce negative eigenvalues. In other words, the filter’s block Toeplitz form [2] should be positive definite. Other filters (not investigated in this paper) may outperform the ramp filter in speeding up an iterative algorithm.

Our examples also indicate that an invalid backprojector is sometimes useful in the early iterations. It is an open problem how the prefilter F_b in (10) affects the final reconstruction when projection data are noisy or corrupted with Poisson noise.

Because the ill-condition nature of the image reconstruction problem, when noise is present, both Landweber and ML-EM algorithms demonstrate first a short convergent trend then diverge from the desired solution, no matter if a valid or an invalid projector/backprojector pair is used. Therefore, choosing a valid backprojector may not be a very critical factor in a practical image reconstruction problem. A “converged” solution usually is very noisy and not desirable. In practice, we should pay attention to the initial “convergent” trend and choose a rapid projector/backprojector pair, which may be an invalid pair, and then use regularization methods to guide or stop the iteration process.

Acknowledgments

This work was supported in part by the National Institute of Health, under Grants R29 HL/CA51462 and HL39792, and by Marconi Medical Systems, Inc. The Associate Editors responsible for coordinating the review of this paper and recommending its publication was F. J. Beekman.

The authors would like to thank S. Webb for editing the manuscript. They would also like to thank Dr. F. Noo of University de Liege, Belgium, Dr. J.-S. Liow of the University of Minnesota, and Dr. F. Natterer of Westfaelische Wilhelms-Universitaet, Germany, for their helpful suggestions and discussion.

References

1. Chang LT. A method of attenuation correction in radionuclide computed tomography. *IEEE Trans Nucl Sci.* 1978; 25:638–643.
2. Zeng GL, Weng Y, Gullberg GT. Iterative reconstruction with attenuation compensation from cone-beam projections acquired via non-planar orbits. *IEEE Trans Nucl Sci.* 1997; 44:98–106.
3. Welch A, Gullberg GT. Implementation of a model-based nonuniform scatter correction scheme for SPECT. *IEEE Trans Med Imag.* 1997; 16:717–726.
4. Kamphuis C, Beekman FJ, Viergever MA, van Rijk PP. Dual matrix ordered subset reconstruction for accelerated 3D scatter correction in SPECT. *Eur J Nucl Med.* 1998; 25:8–18. [PubMed: 9396869]
5. Kadrmas DJ, Frey EC, Karimi SS, Tsui BMW. Fast implementation of reconstruction based scatter compensation in fully 3D SPECT image reconstruction. *Phys Med Biol.* 1998; 43:857–873. [PubMed: 9572510]
6. Glick SJ, Soares EJ. Noise characteristics of SPECT iterative reconstruction with a mismatched projector-backprojector pair. *IEEE Trans Nucl Sci.* 1998; 45:2183–2188.
7. Atkinson, KE. *An Introduction to Numerical Analysis.* 2. New York: Wiley; 1987.
8. Lalush DS, Tsui BMW. Improving the convergence of iterative filtered backprojection algorithm. *Med Phys.* 1994; 21:1283–1286. [PubMed: 7799872]
9. Xu XL, Liow JS, Strother SC. Iterative algebraic reconstruction algorithms for emission computed tomography: A unified framework and its application to positron emission tomography. *Med Phys.* 1993; 20:1675–1684. [PubMed: 8309440]
10. Shepp A, Vardi Y. Maximum likelihood reconstruction for emission tomography. *IEEE Trans Med Imag.* 1982; 1:133–122.
11. Zeng GL, Gullberg GT, Tsui BMW, Terry JA. Three-dimensional iterative reconstruction algorithms with attenuation and geometric point response correction. *IEEE Trans Nucl Sci.* 1991; 38:693–702.

Fig. 1.

Singular value spectra for projector/backprojector pairs. The projector models attenuation and system point response function. Usually, a flatter curve gives a faster convergent rate if the algorithm converges. Refer to Table I for curve labels.

Fig. 2. Normalized mean square error (log scale) versus the number of iterations in *Landweber* reconstructions with *noiseless* projection data. The central slice of the reconstructed image for each projector/backprojector pair is shown at the iteration that gives a minimal normalized mean square error. A profile (solid line) is drawn horizontally across the center of the image and is compared with the ideal profile (dotted line). Refer to Table I for curve labels.

Fig. 3.

Normalized mean square error (log scale) versus the number of iterations in *Landweber* reconstructions with *noisy* projection data. The central slice of the reconstructed image for each projector/backprojector pair is shown at the iteration that gives a minimal normalized mean square error. A profile (solid line) is drawn horizontally across the center of the image and is compared with the ideal profile (dotted line). Refer to Table I for curve labels.

Fig. 4. Normalized mean square error (log scale) versus the number of iterations in *ML-EM* reconstructions with *noiseless* projection data. The central slice of the reconstructed image for each projector/backprojector pair is shown at the iteration that gives a minimal normalized mean square error. A profile (solid line) is drawn horizontally across the center of the image and is compared with the ideal profile (dotted line). Refer to Table I for curve labels.

Fig. 5.

Normalized mean square error (log scale) versus the number of iterations in *ML-EM* reconstructions with *noisy* projection data. The central slice of the reconstructed image for each projector/backprojector pair is shown at the iteration that gives a minimal normalized mean square error. A profile (solid line) is drawn horizontally across the center of the image and is compared with the ideal profile (dotted line). Refer to Table I for curve labels.

TABLE I

Results of Projector/Backprojector Pair Validation

Projector C	Backprojector B^T	Smallest eigenvalue of $B^T C$	Valid?
Line-length weighting with attenuation and system point response modeling [11]	1) C^T	Positive	Yes
	2) $C^T R$ = Ramp filtering followed by C^T	Positive	Yes
	3) B_a^T = attenuation modeling only	Negative	No
	4) $B_a^T R$ = Ramp filtering followed by B_a^T	Negative	No
	5) B_p^T = system point response modeling only	Positive	Yes
	6) $B_p^T R$ = Ramp filtering followed by B_p^T	Positive	Yes
	7) B_{no}^T = no attenuation modeling and no system point response modeling	Negative	No
	8) $B_{no}^T R$ = Ramp filtering followed by B_{no}^T	Negative	No