Published in final edited form as:

Phys Med Biol. 2007 October 21; 52(20): 6165–6179. doi:10.1088/0031-9155/52/20/006.

A Fourier reconstruction algorithm with constant attenuation compensation using 180° acquisition data for SPECT

Qiulin Tang^{1,4}, Gengsheng L Zeng², and Grant T Gullberg³

¹Department of Physics, University of Utah, Salt Lake City, UT 84112, USA

²Department of Radiology, University of Utah, Salt Lake City, UT 84108, USA

³E O Lawrence Berkeley National Laboratory, One Cyclotron Road, Mail Stop 55R0121, Berkeley, CA 94720, USA

Abstract

In this paper, we develop an approximate analytical reconstruction algorithm that compensates for uniform attenuation in 2D parallel-beam SPECT with a 180° acquisition. This new algorithm is in the form of a direct Fourier reconstruction. The complex variable central slice theorem is used to derive this algorithm. The image is reconstructed with the following steps: first, the attenuated projection data acquired over 180° are extended to 360° and the value for the uniform attenuator is changed to a negative value. The Fourier transform (FT) of the image in polar coordinates is obtained from the Fourier transform of an analytic function interpolated from an extension of the projection data according to the complex central slice theorem. Finally, the image is obtained by performing a 2D inverse Fourier transform. Computer simulations and comparison studies with a 360° full-scan algorithm are provided.

1. Introduction

There exist many analytical parallel-beam reconstruction algorithms that correct for the uniform attenuation with projection data which are acquired over a range of 360° (Bellini *et al* 1979, Tretiak and Metz 1980, Gullberg and Budinger 1981, Clough and Barrett 1983, Markoe *et al* 1984, Hawkins *et al* 1988, Inouye *et al* 1989). Although iterative reconstruction methods show that an activity distribution can be recovered from 180° data, the analytical parallel-beam reconstruction algorithm had remained unknown until Noo and Wagner (2001) showed that an angular range of 180° is sufficient to reconstruct uniform attenuated parallel-beam projections and provided a reconstruction formula which uses only 180° data. Noo and Wagner provided a reconstruction formula which uses only 180° data (Noo and Wagner 2001). Pan *et al* (2002) extended Noo and Wagner's results into a family of π -scheme reconstruction formulae for the exponential Radon transform of an object function (the full angular range of 2π is divided into a number of non-overlapping angular intervals without conjugate views whose summation is equal to π). Later Rullgård (2004) developed an explicit expression using a holomorphic function. Our paper derives an approximate

tangql@physics.utah.edu, larry@uclaf.med.utah.edu and gtgullberg@lbl.gov

⁴Present address: Radiation Medicine Program, Princess Margaret Hospital, 610 University Avenue, Toronto, ON M5G 2M9, Canada.

analytical attenuation-corrected reconstruction algorithm for uniform attenuated data acquired over 180° by using a complex variable Fourier central slice theorem.

As pointed out in Noo and Wagner (2001) and Pan *et al* (2002) a reconstruction problem can be expressed by a Fredholm integral equation of the second type, and it has been shown that the integral equation has a unique and stable solution. The solution can be written in the form of a Neumann series, and can be implemented as a recursion procedure (Noo and Wagner 2001, Pan *et al* 2002).

Rullgård also developed an explicit inversion formula for the exponential Radon transform using data from 180° (Rullgård 2004). Rullgård's algorithm involves an inverse kernel which is a holomorphic function for which an explicit expression has not been found. In implementation this holomorphic function can be approximated by a polynomial expression of a rather low degree (Rullgård 2004).

To avoid the recursion method as used in Noo and Wagner (2001) and Pan *et al* (2002) and to avoid the uncertain holomorphic function used in Rullgård (2004), this paper presents an approximate reconstruction algorithm with uniform attenuation compensation for data acquired over 180° . For projections over 360° , Metz and Pan developed a quasi-optimal parallel-beam method which has demonstrated superior noise properties (Pan and Metz 1995); this paper extends their 360° method to a parallel-beam algorithm over 180° . In our algorithm, the projection data acquired over 180° are first extended to 360° with the method proposed in Noo and Wagner (2001). The complex variable central slice theorem similar to that presented in Bellini *et al* (1979), Inouye *et al* (1989) and Noo and Wagner (2001) is used to build a relationship between the image and the extended projections in the complex frequency domain. Then the Fourier series expansion is performed on the Fourier transform of the extended data. Finally, a 2D Fourier transform is performed to obtain the image.

The proposed algorithm is in the form of a direct Fourier reconstruction. The implementation does not require any exponential weighting factor in the backprojection. This is the main difference between our algorithm and other existing 180° -FBP algorithms that compensate for constant attenuation.

2. Algorithm

2.1. Preparation of projection data

Let the 2D radioactivity distribution be $f(\underline{x})$, which is a smooth, bounded and compactly supported square-integrable function defined on $\mathbb{R}^2$. The attenuator has a convex boundary and the attenuation coefficient is zero outside the boundary. The attenuated sinogram $q(s, \phi)$ obtained with parallel projection is given by

$$q(s, \phi) = \int_{-\infty}^{\infty} f(s\underline{\phi} + t\underline{\phi}^\perp) e^{-\mu_0[t+D(s,\phi)]} dt, \quad (1)$$

where $D(s, \phi)$ is the distance from the point $(s \cos \phi, s \sin \phi)$ to the boundary of the attenuator in the direction of the projection as shown in figure 1 and μ_0 is the attenuation coefficient inside the convex boundary.

Let the modified attenuated sinogram (i.e. the exponential Radon transform) be $p_0(s, \phi)$. The subscript 0 of p_0 and μ_0 indicates that the data have not yet been extended to $[0, 2\pi]$. The modified projection data $p_0(s, \phi)$ are only available over 180° , that is, $p_0(s, \phi)$ is defined for the range $\phi \in [0, \pi)$ and $s \in \mathbb{R}$ by

$$p_0(s, \phi) = \int_{-\infty}^{\infty} f(s\underline{\phi} + t\underline{\phi}^\perp) e^{-\mu_0 t} dt, \quad (2)$$

where

$$\underline{\phi} = (\cos \phi, \sin \phi), \quad (3a)$$

$$\underline{\phi}^\perp = (-\sin \phi, \cos \phi). \quad (3b)$$

We now extend the modified projection data $p_0(s, \phi)$ to $p_e(s, \phi)$ over $[\pi, 2\pi)$ as in Noo and Wagner (2001):

$$p_e(s, \phi) = \int_{-\infty}^{\infty} f(s\underline{\phi} + t\underline{\phi}^\perp) e^{-\mu(\phi)t} dt, \quad (4)$$

with $\phi \in [0, 2\pi)$ and

$$\mu(\phi) = \begin{cases} \mu_0 & \text{if } \phi \in [0, \pi), \\ -\mu_0 & \text{if } \phi \in [\pi, 2\pi). \end{cases} \quad (5)$$

With this definition of $\mu(\phi)$,

$$p_e(s, \phi) = \begin{cases} p_0(s, \phi) & \text{if } \phi \in [0, \pi), \\ p_0(-s, \phi - \pi) & \text{if } \phi \in [\pi, 2\pi), \end{cases} \quad s \in \mathbb{R}. \quad (6)$$

It is well known that projections without attenuation have the symmetry property $p(s, \phi) = p(-s, \phi - \pi)$, while this symmetry is not satisfied normally in SPECT because of the attenuation. Thus, extended projections $p_e(s, \phi)$ obtained by (6) are normally discontinuous and not differentiable when the view angle ϕ is equal to 0° and 180° because the property $p_e(s, 0) = p_e(-s, \pi)$ is not normally satisfied.

2.2. Complex variable central slice theorem

Inouye *et al* (1989) developed a ‘backward’ complex central slice theorem in which the Fourier transform of the image F has complex variables, while the variables of the Fourier transform P of attenuated projections acquired over 360° are real:

$$P(\omega_s, \phi) = F(\omega, \phi + v), \quad (7)$$

where ω_s and ω are frequencies and ϕ is the view angle. The variables ω_s , ω , and ϕ are all real and v is imaginary. The variables satisfy the following relationship:

$$\omega = \sqrt{\omega_s^2 - \mu_0^2}, \quad (8)$$

$$v = -\frac{i}{2} \ln \left(\frac{\omega_s + \mu_0}{\omega_s - \mu_0} \right), \quad (9)$$

where μ_0 is the attenuation coefficient and $i = \sqrt{-1}$.

Bellini *et al* (1979) developed a ‘forward’ complex variable central slice theorem which established a relationship between the Fourier transform F of the image and the Fourier transform P of attenuated projections also acquired over 360° . The formula is as follows:

$$F(\omega, \phi) = P \left(\sqrt{\omega^2 + \mu_0^2}, \phi + i \sinh^{-1} \left(\frac{\mu_0}{\omega} \right) \right), \quad (10)$$

where μ_0 is the attenuation coefficient, ω is the frequency and ϕ is the view angle. Bellini’s ‘forward’ complex variable central slice theorem is for a 360° acquisition. A similar ‘forward’ complex variable central slice theorem for the 180° data acquisition is derived as follows.

The one-dimensional Fourier transform of the extended modified projections $p_e(s, \phi)$ with respect to the variable s is defined as follows:

$$P_e(\omega, \phi) = \int_{-\infty}^{\infty} p_e(s, \phi) e^{-is\omega} ds, \quad (11)$$

where $\omega \in \mathbb{R}$ and $\phi \in [0, 2\pi)$. Substituting (4) into (11) yields

$$P_e(\omega, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s\phi + t\phi^\perp) e^{(-is\omega - \mu(\phi)t)} ds dt. \quad (12)$$

Introducing polar coordinates (ρ, θ) to express the image function $f(x, y)$, we have

$$x=s \cos \phi-t \sin \phi=\rho \cos \theta, \quad (13a)$$

$$y=s \sin \phi+t \cos \phi=\rho \sin \theta, \quad (13b)$$

as illustrated in figure 1. From (13) the relationships between (s, t) and (ρ, θ, ϕ) are obtained as follows:

$$s=\rho \cos (\theta-\phi), \quad (14a)$$

$$t=\rho \sin (\theta-\phi). \quad (14b)$$

Substituting (14) into (12) yields

$$P_e(\omega, \phi)=\int_0^{2\pi} \int_0^\infty f(\rho, \theta) e^{-\mu(\phi) \rho \sin (\theta-\phi)} e^{-i \omega \rho \cos (\theta-\phi)} \rho d \rho d \theta. \quad (15)$$

As we stated, the extended projections $p_e(s, \phi)$ are discontinuous and not differentiable when angle ϕ is equal to 0° and 180° ; thus, the Fourier transform of the extended projections $P_e(\omega, \phi)$, where ϕ is a real variable, is normally discontinuous and not differentiable with respect to the angle ϕ at these two points (0° and 180°). In practice, the projections obtained in simulations or from clinical acquisitions are discrete. The projections are sampled at view angles ϕ_k , where k indicates discrete view angle samples. The Fourier transform of the extended projections are identified as $\hat{P}_e(\omega, \phi_k)$. There exists a continuous and differentiable function $P_{cd}(\omega, \phi)$, where the variables ω and ϕ are defined in the real space and the subscript *cd* indicates that the function is continuous and differentiable. The analytic function satisfies the relationship $P_{cd}(\omega, \phi_k)=\hat{P}_e(\omega, \phi_k)$. It is obvious that this interpolated analytic function $P_{cd}(\omega, \phi)$ is not unique. It can be obtained easily by many methods such as using Sinc interpolation or even using a Chebyshev polynomial expansion (Press *et al* 1988). Our algorithm is an approximation because we make the assumption that the interpolated analytic function $P_{cd}(\omega, \phi)$ still satisfies (15). Now, it is possible to find an analytic continuation of $P_{cd}(\omega, \phi)$ in a complex space of ϕ . We perform an analytic continuation of the function $P_{cd}(\omega, \phi)$ to obtain $\tilde{P}_e(\omega, z)$ with respect to the variable ϕ , where $z \in \mathbb{C}$.

Our continuation procedure is described as follows. In order to change the right-hand side of (15) into a Fourier transform of $f(\rho, \theta)$, a continuation is performed on the function $P_e(\omega, \phi)$. The variables (ω, ϕ) are replaced by (ω_s, z) with

$$z=\phi+v(\phi), \quad (16a)$$

$$\omega_s = \omega \cos v(\phi), \quad (16b)$$

$$i\mu(\phi) = \omega \sin v(\phi), \quad (16c)$$

where ω_s and v are generally complex valued. Then, an analytic continuation $\tilde{P}_e(\omega, \phi + v(\phi))$ can be expressed as

$$\begin{aligned} \tilde{P}_e(\omega_s, \phi + v(\phi)) &= \int_0^{2\pi} \int_0^\infty f(\rho, \theta) e^{-\mu(\phi)\rho \sin(\theta - \phi - v(\phi))} e^{-i\omega_s \rho \cos(\theta - \phi - v(\phi))} \rho \, d\rho \, d\theta \\ &= \int_0^{2\pi} \int_0^\infty f(\rho, \theta) e^{-i\rho W} \rho \, d\rho \, d\theta, \end{aligned} \quad (17)$$

where

$$W = \frac{-\mu(\phi)}{-i} \sin(\theta - \phi - v(\phi)) + \omega_s \cos(\theta - \phi - v(\phi)). \quad (18)$$

Solving (16) one obtains

$$\omega = [\omega_s^2 - \mu(\phi)^2]^{\frac{1}{2}}, \quad (19a)$$

$$v(\phi) = \frac{i}{2} \ln \left[\frac{\omega_s + \mu(\phi)}{\omega_s - \mu(\phi)} \right]. \quad (19b)$$

Substituting (5) into (19b) we have

$$v(\phi) = \begin{cases} \frac{i}{2} \ln \left(\frac{\omega_s + \mu_0}{\omega_s - \mu_0} \right) & \text{if } \phi \in [0, \pi), \\ \frac{i}{2} \ln \left(\frac{\omega_s - \mu_0}{\omega_s + \mu_0} \right) & \text{if } \phi \in [\pi, 2\pi). \end{cases} \quad (20)$$

Applying trigonometry identities, (18) becomes

$$\begin{aligned} W = \frac{-\mu(\phi)}{-i} & [\sin(\theta - \phi) \cos v(\phi) - \cos(\theta - \phi) \sin v(\phi)] \\ & + \omega_s [\cos(\theta - \phi) \cos v(\phi) + \sin(\theta - \phi) \sin v(\phi)]. \end{aligned} \quad (21)$$

Substituting (16) and (19) into (21) yields

$$\begin{aligned}
 W &= \frac{\mu(\phi)}{i} \left[\sin(\theta - \phi) \frac{\omega_s}{\omega} - \cos(\theta - \phi) \frac{i\mu(\phi)}{\omega} \right] + \omega_s \left[\cos(\theta - \phi) \frac{\omega_s}{\omega} + \sin(\theta - \phi) \frac{i\mu(\phi)}{\omega} \right] \\
 &= -\frac{(\mu(\phi))^2}{\omega} \cos(\theta - \phi) + \frac{\omega_s^2}{\omega} \cos(\theta - \phi) \\
 &= \omega \cos(\theta - \phi).
 \end{aligned} \tag{22}$$

Therefore, (17) becomes

$$\tilde{P}_e(\omega_s, \phi + v(\phi)) = \int_0^{2\pi} \int_0^\infty f(\rho, \theta) e^{-i\rho\omega \cos(\theta - \phi)} \rho \, d\rho \, d\theta. \tag{23}$$

The right-hand side of (23) is the Fourier transform of f . Thus,

$$\tilde{P}_e(\omega_s, \phi + v(\phi)) = F(\omega, \phi). \tag{24}$$

Substituting (20) into (24) yields

$$F(\omega, \phi) = \begin{cases} \tilde{P}_e(\omega_s, \phi - v) & \text{if } \phi \in [0, \pi), \\ \tilde{P}_e(\omega_s, \phi + v) & \text{if } \phi \in [\pi, 2\pi). \end{cases} \tag{25}$$

Equation (25) is the complex central slice theorem for uniform attenuation imaging. The imaginary part of the second variable of P is constant as a function of the view angle in Bellini's complex variable central slice theorem (Bellini *et al* 1979), while $\tilde{P}_e$ varies with the view angle in our more general central slice theorem, see (20), which is the 'forward' version of Noo's formula (Noo and Wagner 2001).

2.3. The Fourier transform of the image

Metz and Pan (1995) derived their quasi-optimal 360° parallel-beam method by performing a Fourier series expansion of the projections with respect to the view angle. They established a frequency domain relationship between the image and projections acquired over 360°. Our paper adopts the same procedure to derive a reconstruction algorithm for 180° data.

With the same procedure as in Metz and Pan (1995), expanding the left-hand side of (25) as a Fourier series with respect to the polar angle ϕ , we obtain

$$F(\omega, \phi) = \sum_{k=-\infty}^{\infty} F_k(\omega) e^{ik\phi}, \tag{26}$$

where

$$F_k(\omega) = \frac{1}{2\pi} \int_0^{2\pi} F(\omega, \phi) e^{-ik\phi} \, d\phi. \tag{27}$$

Substituting (25) into (27) yields

$$\begin{aligned} F_k(\omega) &= \frac{1}{2\pi} \left[\int_0^\pi F(\omega, \phi) e^{-ik\phi} d\phi + \int_\pi^{2\pi} F(\omega, \phi) e^{-ik\phi} d\phi \right] \\ &= \frac{1}{2\pi} \left[\int_0^\pi \tilde{P}_e(\omega_s, \phi - v) e^{-ik\phi} d\phi + \int_\pi^{2\pi} \tilde{P}_e(\omega_s, \phi + v) e^{-ik\phi} d\phi \right]. \end{aligned} \quad (28)$$

Here, $\tilde{P}_e(\omega_s, \phi + v(\phi))$ is a holomorphic function of period 2π with respect to the second variable and $\text{Im}(\phi + v(\phi))$ is finite. Thus, $\tilde{P}_e(\omega_s, \phi + v(\phi))$ has a Fourier series expansion in the form (Bellini *et al* 1979, Sansone and Gerretsen 1960)

$$\tilde{P}_e(\omega_s, \phi + v) = \sum_{k=-\infty}^{\infty} \tilde{P}_k(\omega_s) e^{ik(\phi+v)}, \quad (29)$$

where

$$\tilde{P}_k(\omega_s) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{P}_e(\omega_s, \phi) e^{-ik\phi} d\phi. \quad (30)$$

Substituting (29) into (28), we obtain

$$F_k(\omega) = \frac{1}{2\pi} \left[\int_0^\pi \sum_{m=-\infty}^{\infty} \tilde{P}_m(\omega_s) e^{im(\phi-v)} e^{-ik\phi} d\phi + \int_\pi^{2\pi} \sum_{n=-\infty}^{\infty} \tilde{P}_n(\omega_s) e^{in(\phi+v)} e^{-ik\phi} d\phi \right]. \quad (31)$$

Changing the order of integration and summation and evaluating e^{-imv} and e^{inv} using (20) yields

$$\begin{aligned} F_k(\omega) &= \frac{1}{2\pi} \left[\sum_{m=-\infty}^{\infty} \tilde{P}_m(\omega_s) \int_0^\pi e^{-i(k-m)\phi} \left(\frac{\omega_s + \mu_0}{\omega_s - \mu_0} \right)^{-\frac{m}{2}} d\phi \right. \\ &\quad \left. + \sum_{n=-\infty}^{\infty} \tilde{P}_n(\omega_s) \int_\pi^{2\pi} e^{-i(k-n)\phi} \left(\frac{\omega_s - \mu_0}{\omega_s + \mu_0} \right)^{-\frac{n}{2}} d\phi \right]. \end{aligned} \quad (32)$$

Let

$$\gamma_+ = \left(\frac{\omega_s + \mu_0}{\omega_s - \mu_0} \right)^{-\frac{1}{2}}, \quad (33a)$$

$$\gamma_- = \left(\frac{\omega_s - \mu_0}{\omega_s + \mu_0} \right)^{-\frac{1}{2}}. \quad (33b)$$

Then, (32) can be expressed as

$$F_k(\omega) = \frac{1}{2\pi} \left[\sum_{m=-\infty}^{\infty} \tilde{P}_m(\omega_s) \gamma_+^m \int_0^\pi e^{-i(k-m)\phi} d\phi + \sum_{n=-\infty}^{\infty} \tilde{P}_n(\omega_s) \gamma_-^n \int_\pi^{2\pi} e^{-i(k-n)\phi} d\phi \right]. \quad (34)$$

Replacing m by n in the first term of (34), we have

$$F_k(\omega) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \tilde{P}_n(\omega_s) \left[\gamma_+^n \int_0^\pi e^{-i(k-n)\phi} d\phi + \gamma_-^n \int_\pi^{2\pi} e^{-i(k-n)\phi} d\phi \right]. \quad (35)$$

Evaluating (35) yields

$$F_k(\omega) = \frac{1}{2\pi} \sum_n \gamma_{k,n} \tilde{P}_n(\omega_s), \quad (36)$$

where

$$\gamma_{k,n} = \begin{cases} \pi(\gamma_+^n + \gamma_-^n) & \text{if } k - n = 0, \\ \frac{-2i}{k-n}(\gamma_+^n - \gamma_-^n) & \text{if } k - n = \text{odd}, \\ 0 & \text{if } k - n = \text{even}. \end{cases} \quad (37)$$

Therefore, the image frequency components $F_k(\omega)$ can be calculated using (36) from the frequency components $\tilde{P}_n(\omega_s)$ of the data. The image can be reconstructed by synthesizing these components using (26).

In fact, from our algorithm we can readily deduce Metz and Pan's quasi-optimal 360° method. The attenuated coefficient $\mu(\phi)$ does not vary with the view angle ϕ when the 360° projections are used instead of the extended projections, which means that γ_- is equal to γ_+ . Therefore,

$$\gamma_{k,n} = \begin{cases} \gamma_+^n & \text{if } k - n = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (38)$$

and (36) becomes formula (27) in (Metz and Pan 1995)

$$F_k(\omega) = \gamma_+^k P_k(\omega_s). \quad (39)$$

Through linear combination of positive and negative frequency components (i.e. $P_k(\omega_s)$ and $P_k(-\omega_s)$), a family of algorithms can be obtained (Metz and Pan 1995). Different linear

combinations of positive and negative frequency components can also be performed in (36) to obtain a family of 180° algorithms, but this is not included in this work.

Implementing (37) and (38) requires $O(N^3)$ and $O(N^2)$ operations, respectively. Metz and Pan's quasi-optimal 360° algorithm and the proposed algorithm both require $O(N^3)$ operations. Implementing (37) instead of (38) does not significantly increase the computational cost.

3. Simulation results

In our computer-simulation studies, the projection data were generated from a modified 2D Shepp–Logan phantom with a uniform attenuator. The parameters of the mathematical phantom are shown in table 1. The units used were in terms of the projection bin size. The uniform attenuator in the simulations had the same circular shape as the exterior of the modified Shepp–Logan phantom and an attenuation coefficient of 0.05 unit⁻¹. This value is approximately the attenuation coefficient of water at 140 keV when one unit is 3.6 mm. The projection data were noise free and were generated with 128 views over 180°. At each view, there were 129 projection samples in the detector.

3.1. Numerical implementation

3.1.1. Modifying projections—First, the line-integral projection data were multiplied by an exponential factor $e^{\mu D(s,\phi)}$, obtaining modified attenuated data. The modified attenuated data over the first 128 views were then extended to 256 views using (6).

3.1.2. Evaluating $F_k(\omega)$ —For a given sampling frequency ω , the shifted frequency

$\omega_s = \sqrt{\omega^2 + \mu_0^2}$ was calculated; then $\tilde{P}_k(\omega_s)$ was obtained by

$$\tilde{P}_k(\omega_s) = \int_0^{2\pi} \int_{-\infty}^{\infty} p_e(s, \phi) e^{-i\omega_s s} e^{-ik\phi} ds d\phi, \quad (40)$$

where k represents the angular frequency. When ω is small and the absolute value of k is large, one of the functions (γ_+^k or γ_-^k) in (36) will become large and the algorithm becomes unstable. In order to regularize the function $\gamma_{k,n}$ we applied a Hann window:

$$W(\omega, k) = \begin{cases} 1, & \text{for } \gamma_{\pm}^k < 10, \\ 0.5 + 0.5 \cos(\pi(\gamma_{\pm}^k - 10)/20), & \text{for } 10 < \gamma_{\pm}^k < 20 \\ 0, & \text{otherwise.} \end{cases} \quad (41)$$

This window function is commonly used, for example, with a ramp filter in the FBP algorithm. Thus, (36) becomes

$$F_k(\omega) = \frac{1}{2\pi} \sum_n \gamma_{k,n} \tilde{P}_n(\omega_s) W(\omega, k). \quad (42)$$

3.1.3. Obtaining the reconstruction $f(\underline{x})$ —An inverse Fourier transform of $F_k(\omega)$ with respect to k was performed to obtain $F(\omega, \theta)$ by

$$F(\omega, \phi) = \sum_{k=-\infty}^{\infty} F_k(\omega) e^{ik\phi}. \quad (43)$$

Then $F(\omega, \theta)$ was expressed and evaluated in Cartesian coordinates. Finally $f(\underline{x})$ was obtained by performing a 2D inverse Fourier transform.

For comparison purposes, images were also reconstructed by Metz and Pan's quasi-optimal method (Metz and Pan 1995) from attenuated data over 360° using the same Hann window.

3.2. Noise-free results

Figure 2 shows the reconstruction image from noiseless data: (a) the true phantom, (b) the image reconstructed by Metz and Pan's quasi-optimal 360° reconstruction and (c) the image reconstructed by the proposed algorithm from projections over 180° . Comparing (a) and (c), we can see qualitatively that the proposed algorithm provides a good reconstruction.

Figure 3 shows the profiles of the reconstructed image and the true phantom shown in figure 2. The solid line indicates the true phantom, the dashed line indicates the image reconstructed by Metz and Pan's quasi-optimal 360° reconstruction, and the dotted line indicates the image reconstructed by the proposed algorithm from projections over 180° . We can see that the profiles of the image reconstructed by the proposed 180° algorithm closely match the profiles of the true phantom, and the proposed approximate method performs as well as Metz and Pan's quasi-optimal 360° method.

Figure 4 shows the close-up portion of the profiles within the box in figure 3(b). It is evident that the full-width at half maximum (FWHM) of the proposed algorithm is equal to or may be slightly larger than the FWHM of Metz and Pan's quasi-optimal 360° method. These two implementations thus result in the same spatial resolution.

3.3. Noise results

To illustrate the noise properties of the proposed 180° algorithm, Poisson noise was added to the projections. Images were reconstructed by the proposed 180° algorithm and Metz and Pan's quasi-optimal 360° method.

3.3.1. Half total counts—The noisy projections were generated over 360° (3×10^6 total counts). The whole data set was used by Metz and Pan's quasi-optimal 360° method to reconstruct an image. Two data sets from 0° to 180° and from 180° to 360° were used by the proposed 180° algorithm, separately, to reconstruct two different images. Thus, the projections used by Metz and Pan's quasi-optimal 360° method to reconstruct the image had twice the counts as the projections used by the proposed 180° method.

Figure 5 shows images reconstructed from attenuated projections containing Poisson noise (10^6 total counts): image (a) is reconstructed by Metz and Pan's quasi-optimal 360° method,

image (b) is reconstructed by the proposed 180° method with projections in the range of $[0^\circ, 180^\circ]$ and image (c) is reconstructed by the proposed method with projections in the range of $[180^\circ, 360^\circ]$. We should point out that the 360° reconstruction used the data set containing about twice the photon counts as in each of the 180° reconstructions. The white circles show the sampled region for the calculations of the average normalized standard deviations in table 2.

Standard deviation and mean images were calculated for the noisy reconstructions using 100 independent noise realizations in the projections. The normalized noise image was the ratio image of the standard deviation image to the mean image.

In figure 6, column 1 was the normalized noise image by Metz and Pan's quasi-optimal 360° method, column 2 was the normalized noise image by the proposed 180° method and column 3 shows the difference of the two noise images. Column 4 shows the sign image of column 3, which illustrates that Metz and Pan's quasi-optimal 360° method is noisier in the black region and the proposed 180° method is noisier in the white region. The upper row compares Metz and Pan's quasi-optimal 360° method with the proposed method using projections in the range of $[180^\circ, 360^\circ]$, and the lower row compares Metz and Pan's quasi-optimal 360° method with the proposed method using projections in the range of $[0^\circ, 180^\circ]$. The upper part of (b1) and lower part of (b2) have approximately the same noise level as (a), but the lower part of (b1) and upper part of (b2) are noisier than (a). This implies that the proposed algorithm has equivalent noise properties as Metz and Pan's quasi-optimal 360° parallel method in the region nearest the detector and has worse noise properties in the region furthest from the detector.

3.3.2. Same total counts—In this numerical simulation, noise-free projections over 360° were generated first. The noisy projections used by Metz and Pan's quasi-optimal 360° method and the proposed 180° algorithm were generated from the noise-free projections by the following procedures, respectively. Noisy projections used by Metz and Pan's quasi-optimal 360° method were obtained by adding Poisson noise to the whole noise-free projection set. To generate the noisy projections used by the proposed 180° method, the whole noise-free projections were divided into two parts: from 0° to 180° and from 180° to 360° . Each of these two parts of the projections was scaled up to the same counts as the whole 360° set; then Poisson noise was added. These two noisy projection sets were used for the proposed 180° algorithm.

Figure 7 shows images reconstructed from attenuated projections containing Poisson noise (10^6 total counts): image (a) is reconstructed by Metz and Pan's quasi-optimal 360° method (same as figures 6(a)), image (b) is reconstructed by the proposed method with projections in the range of $[0^\circ, 180^\circ]$ and image (c) is reconstructed by the proposed method with projections in the range of $[180^\circ, 360^\circ]$. The average normalized standard deviations in the same regions as figure 5 are shown in table 2.

Normalized noise images were generated as for figure 6, with 100 noise realizations. In figure 8, column 1 shows the normalized noise images for Metz and Pan's quasi-optimal 360° method, column 2 shows the normalized noise images of the proposed 180° method

and column 3 shows the difference of the two images. Column 4 shows the sign image of column 3. Column 4 shows that the quasi-optimal 360° method is noisier in the black region, and the proposed 180° method is noisier in the white region. The upper row compares the quasi-optimal 360° method with the proposed method using projections in the range of [180°, 360°], and the lower row compares the quasi-optimal 360° method with the proposed method using projections in the range of [0°, 180°]. We can see that the upper part of (b1) and lower part of (b2) are less noisy than (a), but the lower part of (b1) and upper part of (b2) have almost the same noise level as (a). The proposed algorithm thus has better noise properties than Metz and Pan's quasi-optimal 360° parallel-beam method in the region where projection data are measured with twice the scanning time and about the same noise level in the region where the data are not measured.

4. Conclusion

In this paper, we presented an approximate 2D parallel-beam image reconstruction algorithm for 180° data acquisition with constant attenuation. The projection data acquired over 180° are first extended into those that would be acquired over 360°. Although the ideal Fourier transform of the extended projections $P_d(\omega, \phi)$ is discontinuous and not differentiable when the view angle ϕ is equal to 0 and π , the real projections $\hat{P}_e(\omega, \phi_k)$ obtained in simulations or from clinical acquisitions are discrete. The projections are sampled at view angles ϕ_k , where k indicates discrete samples. An assumption was made that a continuous and differentiable function $P_{cd}(\omega, \phi)$, satisfying both the relationship $P_{cd}(\omega, \phi_k) = \hat{P}_e(\omega, \phi_k)$ and (15), can be obtained by interpolation methods. Then, analytic continuation is performed on the function $P_{cd}(\omega, \phi)$ to obtain $\tilde{P}_e(\omega, z)$ with respect to the variable ϕ , where $z \in \mathbb{C}$. Thus, a more generalized complex variable central slice theorem (25) is established and used to build the relationship between the image and the analytic continuation $\tilde{P}_e(\omega, z)$. Next Fourier series expansion is performed on the analytic continuation $\tilde{P}_e(\omega, z)$. Finally, a 2D Fourier transform is performed to obtain the image.

Our algorithm is an approximate reconstruction algorithm with data acquired over 180° which differs from the recursion formulation of Noo and Wagner (2001) and Pan *et al* (2002) and the uncertain holomorphic function formulation of Rullgård (2004). Our proposed algorithm is, in fact, a 180° data version of Metz and Pan's 360° data algorithm (Metz and Pan 1995). Simulation results show that the image reconstructed by the proposed algorithm has good error properties. Under the condition of the same total counts, the proposed algorithm has better noise properties than Metz and Pan's quasi-optimal 360° parallel-beam method in the half image plane nearest the detector used to measure the projections, while it has worse noise properties than Metz and Pan's method in the other half image plane.

The proposed algorithm is in the form of a direct Fourier reconstruction. In the implementation, no exponential backprojection weighting factor is required. This is the primary difference between our algorithm and other existing 180°-FBP algorithms that compensate for constant attenuation.

Acknowledgments

We thank Dr Roy Rowley for editing the text. This work was supported by the National Cancer Institute and the National Institute of Biomedical Imaging and Bioengineering of the National Institutes of Health under grants R21CA100181, R01EB00121 and by the Director, Office of Science, Office of Biological and Environmental Research, Medical Sciences Division of the US Department of Energy under contract DE-AC03-76SF00098.

References

- Bellini S, Piacenti M, Caffaro C, Rocca F. Compensation of tissue absorption in emission tomography. *IEEE Trans. Acoust. Speech Signal Process.* 1979; 27:213–8.
- Clough AV, Barrett HH. Attenuated Radon and Abel transforms. *J. Opt. Soc. Am.* 1983; 73:1590–5.
- Gullberg GT, Budinger TF. The use of filtering methods to compensate for constant attenuation in single photon emission computed tomography. *IEEE Trans. Biomed. Eng.* 1981; 28:142–57. [PubMed: 6974681]
- Hawkins WG, Lechner PK, Yang NC. The circular harmonic transform for SPECT reconstruction and boundary conditions on the Fourier transform of the sinogram. *IEEE Trans. Med. Imaging.* 1988; 7:135–48. [PubMed: 18230462]
- Inouye T, Kose K, Hasegawa H. Image reconstruction algorithm for single-photon-emission computed tomography with uniform attenuation. *Phys. Med. Biol.* 1989; 34:299–304. [PubMed: 2784575]
- Markoe A. Fourier inversion of the attenuated X-ray transform. *SIAM J. Math. Anal.* 1984; 15:718–22.
- Metz CE, Pan X. A unified analysis of exact methods of inverting the 2-D exponential Radon transform, with implications for noise control in SPECT. *IEEE Trans. Med. Imaging.* 1995; 17:643–58.
- Noo F, Wagner JM. Image reconstruction in 2D SPECT with 180° acquisition. *Inverse Problems.* 2001; 17:1357–71.
- Pan X, Kao C, Metz CE. A family of π -scheme exponential Radon transforms and the uniqueness of their inverses. *Inverse Problems.* 2002; 18:825–36.
- Pan X, Metz CE. Analysis of noise properties of a class of exact methods of inverting the 2-D exponential Radon transform. *IEEE Trans. Med. Imaging.* 1995; 14:659–68. [PubMed: 18215870]
- Press, WH., Flannery, BP., Teukolsky, SA., Vetterling, WT. *Numerical Recipes in C—the Art of Scientific Computing.* Cambridge University Press; Cambridge: 1988.
- Rullgård H. An explicit inverse formula for the exponential Radon transform using data from 180°. *Ark. Mat.* 2004; 42:353–62.
- Sansone, G., Gerretsen, J. *Lectures on the Theory of Functions of a Complex Variable.* Noordhoff-Groningen; Amsterdam, The Netherlands: 1960.
- Tretiak O, Metz CE. The exponential Radon transform. *SIAM J. Appl. Math.* 1980; 39:341–54.

Figure 1.
Illustration of parameters used in the definition of the exponential Radon transform.

Figure 2. Reconstructions with noiseless data. (a) The true phantom, (b) the image reconstructed by Metz and Pan's quasi-optimal method from projections over 360° and (c) the image reconstructed by the proposed algorithm from projections over 180° .

Figure 3.

Profiles of the reconstructed images and the true phantom in figure 2. The solid line indicates the true phantom, the dotted line indicates the image reconstructed using Metz and Pan's quasi-optimal 360° method and the dashed line indicates the image reconstructed using the proposed 180° algorithm. (a) Profiles drawn from $x = 65$ in figure 2 and (b) profiles drawn from $y = 65$ in figure 2.

Figure 4.

A 'close-up' of the portion of the profiles within the boxes in figure 3(b). The solid line indicates the true phantom, the dotted line indicates the image reconstructed using Metz and Pan's quasi-optimal 360° method and the dashed line indicates the image reconstructed using the proposed 180° algorithm. The dashed and dotted horizontal lines indicate the half maximum of the peaks of the dashed and dotted curves, respectively.

Figure 5.

Images reconstructed from attenuated projections containing Poisson noise (3×10^6 total counts) with the 'half total counts' setup. The first row shows the view ranges of the projections, which are used to reconstruct the corresponding images in the second row. (a) The image reconstructed by the Metz and Pan quasi-optimal 360° method using twice the counts as in (b) and (c), (b) the image reconstructed by the proposed method with projections in the range of $[0^\circ, 180^\circ]$ and (c) the image reconstructed by the proposed method with projections in the range of $[180^\circ, 360^\circ]$. The white circles show the sampled region for the calculations of the average normalized standard deviations.

Figure 6.

Normalized noise image with the ‘half total counts’ setup. Column 1 shows the normalized noise image of the Metz and Pan quasi-optimal 360° method, column 2 shows the normalized noise images of the proposed 180° method and column 3 shows the difference between images (a) and (b). Column 4 shows the sign image of column 3, which illustrates that the quasi-optimal 360° method is noisier in the black region and the proposed 180° method is noisier in the white region. The upper row compares the quasi-optimal 360° method with the proposed method with projections in the range of $[180^\circ, 360^\circ]$, and the lower row compares with the proposed method with projections in the range of $[0^\circ, 180^\circ]$.

Figure 7.

Images reconstructed from attenuated projections containing Poisson noise (10^6 total counts) with the 'same total counts' setup. (a) The image reconstructed by Metz and Pan's quasi-optimal 360° method, (b) the image reconstructed by the proposed method with projections in the range of $[0^\circ, 180^\circ]$ and (c) the image reconstructed by the proposed method with projections in the range of $[180^\circ, 360^\circ]$.

Figure 8.

Normalized noise images with the ‘same total counts’ setup. Column 1 shows the normalized noise image of Metz and Pan’s quasi-optimal 360° method, column 2 shows the normalized noise images of the proposed 180° method and column 3 shows the difference between images (a) and (b). Column 4 shows the sign image of column 3. Column 4 shows that the quasi-optimal 360° method is noisier in the black region, and the proposed 180° method is noisier in the white region. The upper row compares the quasi-optimal 360° method with the proposed method with projections in the range of $[180^\circ, 360^\circ]$, and the lower row compares with the proposed method with projections in the range of $[0^\circ, 180^\circ]$.

Table 1

Parameters of the modified Shepp–Logan phantom used in our numerical simulations. Here, x and y are the coordinates of the center of the ellipse, a and b are the major and minor axes of the ellipse, respectively, and ϕ is the rotation angle related to the x axis.

Phantom	x	y	a	b	ϕ	Activity function
	0.00	0.00	50.00	50.00	90.00	2.00
	0.00	0.00	47.50	47.50	90.00	−1.00
	11.00	0.00	15.50	5.50	72.00	−0.50
	−11.00	0.00	20.50	8.00	108.00	−0.50
	0.00	17.50	12.50	10.15	90.00	−0.50
	0.00	5.00	5.30	2.30	0.00	0.50
	0.00	−5.00	5.30	2.30	0.00	1.00
	0.00	−30.25	2.15	1.15	180.00	0.50

Table 2

Average voxelwise standard deviation (STD) in the selected area shown in figure 5. The sampled regions for average standard deviation calculation are shown by white circles in figure 5.

Methods			Average normalized STD
Metz and Pan's method	In upper region		0.45
	In bottom region		0.41
Half total counts	Projections from $[0^\circ, 180^\circ]$	In upper region	0.53
		In bottom region	1.12
	Projections from $[180^\circ, 360^\circ]$	In upper region	1.33
		In bottom region	0.44
The proposed method	Projections from $[0^\circ, 180^\circ]$	In upper region	0.34
		In bottom region	0.58
	Projections from $[180^\circ, 360^\circ]$	In upper region	0.68
		In bottom region	0.32