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Selection of Fiducial Locations and Performance Metrics for Point-Based Rigid-Body Registration

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Abstract

A method is described to select the location and number of fiducials used in point-based, rigid-body registration of two coordinate frames. Two indices are introduced which are used to search for the optimum configuration of fiducials. They can be used to quickly evaluate a large number of configurations because no actual registration is involved in their calculation. Furthermore, configurations yielding small values of the indices correlate well with configurations which result in optimum registrations. Three registration performance metrics are discussed, and it is shown that optimization of different metrics leads to different selection of fiducial configurations. If an optimized configuration is selected as a starting configuration of N fiducials, the addition of extra fiducials does not significantly improve the registration in most cases. This work is based on 3D data acquired with three different instruments, each having different noise and bias characteristics.

Keywords

point-based rigid-body registration; performance metrics; noise and bias in positional data

I. Introduction

The aim of a registration procedure is to obtain a transformation between two coordinate frames. Usually, sensors acquire the location of a point in three dimensional (3D) space in their own local coordinate frame. When positional data are obtained by two different sensors or two datasets are acquired with the same instrument placed in two different poses, some of the points may be measured only in one frame but have to be accessed in the other frame. Then, the transformation to map a set of points measured in one frame to the other is needed. The first coordinate frame (from which the data are transformed) will be called the working frame and the second one (to which the data are transformed) the destination frame. In point-based, rigid-body registration, the parameters of the transformation are determined using measurements of the same physical points acquired in both frames. These common points form a list of N pairs of corresponding points called fiducials. In the ideal case when the measurement of the fiducials is noise and bias free, the rigid-body assumption dictates that the distance between any two fiducials in the working frame is equal to the distance of the corresponding two points in the destination frame. In reality, every consecutive i -th measurement of the same points yields a slightly different pair of datasets: $\{\mathbf{X}\}_{N,i}$ in the

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working and $\{\mathbf{Y}\}_{N,i}$ in the destination frames. Then, the transformation is the rotation $\mathbf{R}_i$ and translation $\boldsymbol{\tau}_i$ which minimize the Fiducial Registration Error, FRE_i

$$FRE_i(\{\mathbf{X}\}_{N,i}, \{\mathbf{Y}\}_{N,i}) = \min \left(\frac{1}{N} \sum_{n=1}^N \|\mathbf{R}_i \mathbf{X}_{n,i} + \boldsymbol{\tau}_i - \mathbf{Y}_{n,i}\|^2 \right). \quad (1)$$

One may be tempted to use FRE_i as a metric for the quality of registration. However, in the general problem of fitting a model to noisy data, a large residual value of the error function may be obtained in two different situations: 1) correct model is fitted to data with large noise; 2) wrong model is fitted to data with small noise. Therefore, a better metric is needed to quantify the performance of registration. Once the registration transformation $(\mathbf{R}_i, \boldsymbol{\tau}_i)$ is determined, it can be applied to a target point which is not a fiducial (i.e., not used to calculate the registration transformation) and the Target Registration Error $TRE_i(\mathbf{T}_x)$ is defined as:

$$TRE_i(\mathbf{T}_x) = \|\mathbf{R}_i \mathbf{T}_x + \boldsymbol{\tau}_i - \mathbf{T}_y\|^2, \quad (2)$$

where $\mathbf{T}_x$ and $\mathbf{T}_y$ are the target positions in the working and destination frames. In general, theoretical formulations of $TRE_i(\mathbf{T}_x)$ assume no noise or bias in the target measurements and thus, the only source of uncertainty comes from noisy registration $(\mathbf{R}_i, \boldsymbol{\tau}_i)$. Non-zero values of FRE_i and TRE_i are a consequence of non-zero Fiducial Localization Error FLE_i defined as

$$FLE_i(\{\mathbf{X}\}_{N,i}, \{\mathbf{Y}\}_{N,i}) = \frac{1}{N} \sum_{n=1}^N \left(\|\mathbf{X}_{n,i} - \mathbf{X}_{n,0}\|^2 + \|\mathbf{Y}_{n,i} - \mathbf{Y}_{n,0}\|^2 \right), \quad (3)$$

where $\{\mathbf{X}_0\}_N$ and $\{\mathbf{Y}_0\}_N$ are the true unknown locations of the fiducials.

Much effort has been made to formulate an analytical relation between FLE , FRE , and TRE , where

$$FLE = \langle FLE_i \rangle, FRE = \langle FRE_i \rangle, TRE = \langle TRE_i \rangle, \quad (4)$$

where $\langle \dots \rangle$ indicates averaging over repeated measurements of fiducials acquired in the same experimental conditions, as defined in [1]. Different analytical formulas have been derived. They depend on different models of Gaussian noise perturbing the true locations of the fiducials in the working and destination frames

$$\mathbf{X}_{n,i} = \mathbf{X}_{0,n} + \mathbf{B}_{x,n} + \boldsymbol{\xi}_{n,i}, \quad \mathbf{Y}_{n,i} = \mathbf{Y}_{0,n} + \mathbf{B}_{y,n} + \boldsymbol{\eta}_{n,i}, \quad (5)$$

where $\mathbf{B}_{x,n}$ and $\mathbf{B}_{y,n}$ are systematic biases at the n -th location while $\xi_{n,i}$ and $\eta_{n,i}$ are random perturbations with zero-mean Gaussian distributions. Historically, the oldest and simplest model used is:

$$\text{cov}(\xi_n) = \text{cov}(\eta_n) = \sigma \mathbf{I}_{3 \times 3}, \mathbf{B}_{x,n} = \mathbf{B}_{y,n} = 0 \quad (6)$$

for all $n = 1, \dots, N$. This model corresponds to homogenous (i.e., independent of location n), isotropic (covariance matrix has only equal diagonal elements), zero-mean Gaussian noise. Based on this model, two noteworthy equations were derived. First in [2] and then in [3], it was shown that

$$FRE = \left(1 - \frac{2}{N}\right) FLE, \quad (7)$$

where $N - 3$ is the number of fiducials used for registration. This equation has two rather surprising and counterintuitive implications. First, FRE increases with increasing N ; second, FRE does not depend on the spatial distribution of the fiducials. Since noise is assumed to be homogeneous, there is no good or bad geometrical distribution of fiducials (excluding extreme configurations of nearly collinear points). The second noteworthy equation derived in [3] relates TRE with FLE

$$TRE(\mathbf{T}_x) = FLE \times \left(\frac{1}{N} + \frac{1}{3} \sum_{k=1}^3 \frac{d_k^2}{M_k^2} \right), \quad (8)$$

where M_k^2 is the moment of inertia of the fiducial configuration about the principal k -th axis (defined by fiducials $\{\mathbf{X}_0\}_N$) and $d_k(\mathbf{T}_x)$ is the distance of the target $\mathbf{T}_x$ to the k -th principal axis. A few important conclusions may be derived from (8): 1) contrary to FRE , TRE depends on the geometrical configuration of the fiducials $\{\mathbf{X}_0\}_N$; 2) a target located close to the centroid of $\{\mathbf{X}_0\}_N$ should have small TRE ; 3) as N increases TRE decreases. Subsequent efforts using more realistic noise models led to further modifications of the closed form equations for FRE and TRE [4–8]. Practical use of these analytical expressions is limited due to the fact that FLE cannot be measured experimentally as it depends on the true locations of the fiducials $\{\mathbf{X}_0\}_N$ and $\{\mathbf{Y}_0\}_N$ which are unknown [9, 10]. Attempts to relate $TRE_f(\mathbf{T}_x)$ to measurable FRE_f were hampered by the discovery that $\langle TRE_f, FRE_f \rangle = 0$, i.e., they are statistically uncorrelated [7]. Furthermore, the method of removing bias from the fiducial measurements requires measurements of targets that are bias free [11] and this restriction cannot be satisfied in many realistic experimental settings. Finally, if $TRE_f(\mathbf{T}_x)$ from (2) is used as a metric for registration performance, then the search for the optimum placement of fiducials may be incorrect because noise and bias in the measurement of target $\mathbf{T}_x$ are ignored in (2). Thus, for realistic noise characteristics (anisotropic, heterogeneous, and with non-zero bias), the performance of registration depends on the selection of fiducial locations, and in spite of intensive theoretical efforts, no analytical method exists to guide practitioners in selecting the optimum placement of fiducials.

The problem of finding the best placement of fiducials was studied extensively in the area of image assisted neurosurgery [12–17]. In this paper, we follow the strategy for finding the best fiducial configuration outlined in [18]. The method is based on an exhaustive linear search of all possible combinations of N fiducials from M potential locations. The number of

such combinations $M_N = \frac{M!}{N!(M-N)!}$ grows rapidly with M and N (in our experiments $M=125$, $N=4$, and $M_N=9,691,375$). In principle, for each combination, the corresponding transformation should be calculated first and then applied to the target(s) to gauge the registration quality. However, for a large number of combinations M_N , this approach is time consuming and not practical. Therefore, we propose two proxy indices which do not require the calculation of the transformation matrix and, hence, they both can be quickly determined for a large number of combinations M_N . We show that these indices correlate well with the characteristics of the actual registration. We also show that if a combination of $N=4$ fiducials is optimally selected according to a certain performance metric, then the addition of more fiducials leads to marginal or small improvement in the registration performance. However, this behavior may not hold for a different performance metric and the addition of more fiducials may worsen the performance. The choice of performance metric is a subtle issue which can have far reaching consequences. The metrics used in this study fall in two categories. One category minimizes the uncertainty of a target point T_x transformed into the destination frame, i.e., it minimizes the spread of a point transformed by noisy transformations (R_i, τ_i) obtained from repeated measurements of the fiducials. The second category minimizes the distance between T_x transformed by (R_i, τ_i) and the corresponding target T_y in the destination frame. The two categories reflect the dichotomy of each measurement, i.e., its precision and accuracy. Both are desired but practitioners should be aware that optimizing fiducial placement based on one criterion does not necessarily satisfy the second.

We verify our approach using 3D data acquired with three different instruments, each having different noise characteristics: a laser tracker (LT), a motion tracking system (System A), and a large-scale metrology system (System B). For the purposes of this research, LT is considered to have no noise and no bias, System A has small noise and large bias, while System B has large noise and small bias. The three sets of acquired 3D data (each containing repeated measurements of the same points) allowed us to evaluate three registrations (A to LT, B to LT, and A to B) covering a wide spectrum of realistic conditions.

The paper is organized as follows: in Section II, a brief description of experimental set-up and data processing is provided. Also, in this section, definitions of the proxy indices and metrics which are used to gauge the quality of registration are provided. In Section III, the results are presented, followed by a discussion in Section IV, and final conclusions are presented in Section V.

II. Experiment and Data processing

Positions of 125 3D points distributed on a semi-regular $5 \times 5 \times 5$ grid were measured using three different instruments: System A (motion capture system), System B (large scale metrology), and laser tracker LT [19]. In addition, 16 other points randomly located in the

work volume ($3\text{m} \times 3\text{m} \times 1.8\text{m}$) were also measured by the three instruments. Registrations between three pairs of instruments (A to B, A to LT, and B to LT) were performed for different configurations of N fiducials selected from the 125 grid points. The 16 points were used as targets to evaluate the performance of the registration. Each grid and target point was measured 200 times by Systems A and B (no repeats were collected by LT as the LT measurements are considered noise and bias free for the purpose of this study). The repeated measurements allowed for the calculation of the mean location and the covariance matrix for each grid and target point for Systems A and B. Extra attention was made to ensure that all three systems acquired the same physical points (details of experiment are described in [19]). In this paper, we use the SVD (singular value decomposition) method to calculate the rotation and translation parameters of the registration transformation ($\mathbf{R}$, $\mathbf{t}$), as described in [20, 21].

The data were post-processed using two different methods. In method 1, averaged (over the 200 repeats) locations of fiducials and targets were used to calculate the registration and the performance metrics. Averaging over many repeated measurements may suppress random noise but it does not remove bias, if it exists. In method 2, registration was determined for each repeated measurement along with the performance metrics for each registration. Then, the mean and standard deviation of the performance metrics were calculated. The results presented in this paper were obtained using mostly method 1, unless otherwise noted.

The diagonal elements of the covariance matrix (variances) were used to calculate the magnitude of the noise σ_0 at a given point in 3D space:

$$\sigma_0(x, y, z) = \sqrt{\text{var}(x) + \text{var}(y) + \text{var}(z)}. \quad (9)$$

In addition, three eigenvalues λ (λ_1 λ_2 λ_3) and the corresponding eigenvectors of each covariance matrix were also calculated. The average locations of points were used to check the rigid-body approximation: the distance $d_{i,j}$ between any two points measured by System A or B were compared with the distance $D_{i,j}$ between the same two points measured by the laser tracker LT. Compliance with the rigid-body assumption is quantified by two parameters L and p defined as

$$L = d_{i,j} - D_{i,j}, p = L / \text{std}(L), \quad (10)$$

where $\text{std}(L)$ is the standard deviation of the repeated (i.e., instantaneous) measurements of $d_{i,j}$.

Residual values of the error function RMS_F , which is the square root of FRE , were calculated for the N fiducials $\{\mathbf{X}\}_N$, $\{\mathbf{Y}\}_N$ used for registration:

$$RMS_F(\{\mathbf{X}\}_N, \{\mathbf{Y}\}_N) = \sqrt{\frac{1}{N} \sum_{n=1}^N \|\mathbf{R}\mathbf{X}_n + \mathbf{t} - \mathbf{Y}_n\|^2}. \quad (11)$$

The performance of the registration based on different sets of fiducials is gauged by three metrics calculated for the same set of $K = 16$ targets: RMS_T , index Q , and index W . RMS_T is similar to RMS_F but it is determined for targets $\{\mathbf{T}_x\}_K$ and $\{\mathbf{T}_y\}_K$ measured in working and destination frames, respectively

$$RMS_T(\{\mathbf{T}_x\}_K, \{\mathbf{T}_y\}_K) = \sqrt{\frac{1}{K} \sum_{k=1}^K \|\mathbf{R}\mathbf{T}_{x,k} + \boldsymbol{\tau} - \mathbf{T}_{y,k}\|^2}. \quad (12)$$

RMS_T is an extension of TRE from (2) which was calculated for one target point only. In some applications, gauging the performance of registration based on only one target may be sufficient (for example, in image assisted neurosurgery where the region of interest is relatively small [22]). In manufacturing applications, where assembly tasks need to be accomplished in a relatively large work volume, the average error of many targets is more representative of the registration performance for the work volume.

The other two metrics: index Q and W , are the median of the corresponding q_k and w_k defined below for $k = 1 \dots K$. For every k -th target in the working frame $\mathbf{T}_{x,k}$, eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ of the respective covariance matrix are calculated and the volume of the uncertainty ellipse $V(\mathbf{T}_{x,k})$ is calculated as

$$V(\mathbf{T}_{x,k}) = \frac{4}{3} \pi \lambda_1 \lambda_2 \lambda_3. \quad (13)$$

Similarly, for a target transformed from the working to the destination frame $\tilde{\mathbf{T}}_{x,k} = \mathbf{R}\mathbf{T}_{x,k} + \boldsymbol{\tau}$, the corresponding eigenvalues $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\}$ and the volume $V(\tilde{\mathbf{T}}_{x,k})$ are determined. Then, q_k is defined as

$$q_k = \sqrt[3]{V(\tilde{\mathbf{T}}_{x,k})/V(\mathbf{T}_{x,k})} \quad (14)$$

and it describes the expansion ratio of the radii of two spheres with volumes equal to $V(\tilde{\mathbf{T}}_{x,k})$ and $V(\mathbf{T}_{x,k})$ caused by imperfect registration. Larger $\{q_k\}$ indicate poorer registration, i.e., the transformed targets $\tilde{\mathbf{T}}_{x,k}$ have larger uncertainties.

The third metric W is the median of all w_k defined as

$$w_k = d_k / \text{std}(d_k), \quad (15)$$

where d_k is the distance between k -th target transformed from the working frame and corresponding target measured in the destination frame

$$d_k = \|\tilde{\mathbf{T}}_{x,k} - \mathbf{T}_{y,k}\|. \quad (16)$$

Values of $w_k < 1$ indicate that the target transformed from the working frame is within one sigma of the target measured in the destination frame. Large values of w_k may indicate the presence of a bias.

Thus, the three performance metrics: RMS_T , index Q , and index W , try to gauge two different aspects of a measurement. Small RMS_T characterizes accurate measurement (transformed targets are close to the ‘true’ target locations in destination frame), small Q corresponds to precise measurement (small dispersion of the transformed targets), and small W accounts for a trade-off between accuracy and precision (w_k is a ratio and a small value may be caused either by small d_k or by a large $std(d_k)$).

As mentioned earlier, targets $\{\mathbf{T}_y\}_K$ are usually not measured in the destination frame and the three metrics described above cannot be calculated. Instead, two proxy indices can be evaluated for two sets of corresponding fiducials $\{\mathbf{X}\}_N, \{\mathbf{Y}\}_N$: index F and $minR_I$. Index F is defined as

$$F(\{\mathbf{X}\}_N, \{\mathbf{Y}\}_N) = \frac{2}{N(N-1)} \sum_{n=2}^N \sum_{m=1}^{n-1} \left(\frac{\sigma_0(\mathbf{X}_n) + \sigma_0(\mathbf{X}_m)}{\|\mathbf{X}_n - \mathbf{X}_m\|} + \frac{\sigma_0(\mathbf{Y}_n) + \sigma_0(\mathbf{Y}_m)}{\|\mathbf{Y}_n - \mathbf{Y}_m\|} \right) \quad (17)$$

and is used to find the optimal fiducials for registration.

The second proxy index is defined as

$$minR_I(\{\mathbf{X}\}_N, \{\mathbf{Y}\}_N) = \sqrt{\frac{1}{N} \sum_{n=1}^N h_{\max,n}^2} \quad (18)$$

where

$$h_{\max,n} = \frac{1}{2} \max\{m=1, \dots, N; m \neq n: h_{m,n}\} \quad (19)$$

and

$$h_{m,n} = \left| \|\mathbf{X}_m - \mathbf{X}_n\| - \|\mathbf{Y}_m - \mathbf{Y}_n\| \right|. \quad (20)$$

Parameter $h_{m,n}$ is closely related to L defined in (10) and both take on non-zero values when the rigid-body assumption is not satisfied. It can be shown that $minR_I$ as defined in (18) is a lower bound for RMS_F (the proof is similar to that given in [19], Appendix B). Neither F nor $minR_I$ require that a registration be performed and they both can be quickly calculated for a large number of different configurations of fiducials $\{\mathbf{X}\}_N, \{\mathbf{Y}\}_N$.

III. Results

The data acquired by three instruments (Systems A, B, and LT) were processed as described in Section II. The results are presented in three groups to address three major objectives: 1) obtain noise characteristics of the instruments used in the experiment; 2) check the validity of the rigid-body assumption; and 3) provide experimental support for the development of a practical procedure to select the location and number of fiducials used in registration.

A. Characteristics of Instrument Noise

The magnitude of the noise σ_0 defined in (1) and determined at each of M grid points (from which N fiducials are later selected) is shown in Figure 1.

In Figure 2(a, b), the directions of the eigenvectors $e_3(m)$ corresponding to the largest eigenvalue λ_3 of the covariance matrix evaluated at the m -th location are shown. In Figure 2(c, d) histograms of noise anisotropy coefficients (evaluated as the ratio λ_3/λ_1) are displayed.

B. Check of Rigid-Body Assumption

The error $L(i, j)$ as defined in (10) versus the corresponding distance $D(i, j)$ measured by the laser tracker between (i, j) pair of grid points is shown in Figure 3. The plot contains all $M(M-1)/2$ unique pairs of points. In Figure 4, a histogram of the dimensionless ratio p defined in (10) is plotted.

C. Results for Registration using $N = 4$ fiducials

Histograms of proxy index F and index Q calculated for each combination of N fiducials selected out of M grid points are shown in Figure 5 for three registrations: System B to LT, System A to LT, and System A to B. Execution times T_F and T_Q of the Matlab¹ script for all M_N combinations for $N = 4$ fiducials were measured, yielding a ratio $T_F/T_Q < 0.01$ for all three pairs of instruments. In Figure 6 the correlation between F and Q is demonstrated by plotting the ten smallest Q values and the ten values of Q corresponding to ten smallest values of F . In a similar way, the correlation between $\min R_I$ defined in (18) and RMS_F defined in (11) is shown in Figure 7 and Figure 8. Execution time T_{RI} of Matlab script was also recorded, yielding $T_{RI}/T_Q \approx 0.016$ for all three pairs of instruments.

In Figure 9, the histograms of RMS_F , RMS_T defined in (11), (12) and their ratio $s = RMS_T/RMS_F$ are plotted for the two processing methods: 1) when registration is based on averaged locations of the fiducials; 2) when 200 registrations are performed using instantaneous measurements of fiducials and for each registration, corresponding RMS_F , RMS_T are determined from which the average RMS_F and RMS_T are calculated. In Figure 10, histograms of index W (median of w_k defined in (15)) are plotted. Figure 11 shows the three performance metrics: index $Q(m)$, $W(m)$, and $RMS_T(m)$ calculated for six different configurations of $N = 4$ fiducials, $m = 1, \dots, 6$. These six configurations yield registrations

¹Certain trade names and company products are mentioned in the text in order to adequately specify the experimental procedure and equipment used. In no case does such an identification imply recommendation or endorsement by the National Institute of Standards and Technology, nor does it imply that the products are necessarily the best available for the purpose.

for which the two proxy indices F and $\min R_I$, three metrics Q , W , and RMS_T , and residual value of the error function RMS_F are minimum. Results shown in Figure 11 are for registering System B to LT, similar graphs are displayed in Figure 12 for registering System A to LT and System A to B in Figure 13.

D. Effect of the increased number of fiducials on registration performance

Different strategies of adding a fifth fiducial to the previously selected $N = 4$ fiducials and their impact on the performance gauged by index Q are displayed in Figure 14. Initially (Case 1) the configuration of $N = 4$ fiducials yielding the smallest proxy index F is selected. Then (Case 2), the fifth fiducial is chosen from the remaining $M - N$ grid points and the combination yielding the smallest index F is selected. Case 3 is an extension of the strategy for Case 2: instead of taking only the best combination of $N = 4$ fiducials, the first 10^6 combinations (rank $k_F = 10^6$) are checked and for each combination, the fifth fiducial from the remaining $M - N$ grid points which yielded the smallest F is selected – then, from the resulting 10^6 combinations of $N = 5$ fiducials, the combinations with the smallest F is chosen. For case 4, the ‘median’ combination of $N = 4$ fiducials corresponding to rank $k_F = M_N / 2$ is chosen (median of the sorted list of F). Case 5 starts with the 4 fiducials from Case 4, and the fifth fiducial is selected from the remaining $M - N$ grid points.

In Figure 15, index Q versus the number N of fiducials is shown. The criterion for adding $N + 1$ fiducials to the previously selected configuration of N fiducials is based on smallness of index Q . As before, two strategies for selecting extra fiducials are used: one based on the best configuration (smallest Q , rank $k_Q = 1$) and the second based on the medium selection (rank k_Q corresponding to the median value of Q). Similar to Figure 15, Figure 16 plots RMS_T versus the number of fiducials N and Figure 17, plots RMS_F versus N . In Figure 18, index W versus the number of fiducials N is shown. In this case, the criterion for adding extra fiducial is based only on the smallest W (i.e., the next fiducial is selected from the remaining $M - N$ grid points for which W is the smallest). Finally, in Figure 19, the mean RMS_T versus N is shown. Figure 19 differs from Figure 16 in that the fiducials in Figure 16 are based on average values (method 1) and the fiducials used in Figure 19 are the 200 instantaneous measurements of fiducials (method 2). For the 200 registrations, the RMS_T values were calculated and then, the mean and standard deviation were calculated for a given combination of N fiducials. Using the previously selected N fiducials, the next $N+1$ fiducial was selected from the remaining $M - N$ grid points, and the RMS_T was calculated for the $M - N$ configurations. The configuration yielding the smallest RMS_T was used as the basis for the next $N+1$ configuration.

IV. Discussion

In ideal conditions where measurements are free of noise and bias, the locations of the fiducials are irrelevant because the rigid-body assumption is perfectly satisfied and any triplet of non-collinear points will yield the same rotation matrix and a unique translation vector. However, in more realistic situations where measurements are affected by noise and/or bias which is dependent on the location of the point, more data need to be acquired to quantify the magnitude of the noise and bias to improve the quality of the registration.

The results presented in Figure 1 clearly indicate that instrument noise for System A and B is non-homogenous, i.e., its magnitude depends on the location of the 3D point. The direction of the eigenvector e_3 of the covariance matrix and the noise anisotropy coefficient evaluated at each nominal location of the grid point also reveal that the noise is dependent on the point location for both Systems A and B, (see Figure 2).

The error L between the distance d_{ij} measured by System A or B and the same distance D_{ij} measured by laser tracker LT reveals substantial differences for System A and small differences for System B, (see Figure 3). The error L for System A shows strong linear correlation with distance D_{ij} and it has mostly positive values, i.e., $L > 0$. This indicates that System A is systematically overestimating the distance d_{ij} between two measured points. This may indicate a scaling problem in the acquired 3D data. The error L for System B takes on both positive and negative values in a relatively narrow range and it is not correlated with distance D_{ij} . It should be noted that the distance d_{ij} is calculated between the mean positions (i, j) averaged over 200 repeated measurements. Thus, the non-zero values of L must be attributed to a systematic bias in the measured 3D data. Indeed, the histograms of the ratio p in Figure 4 confirm that the bias observed in System A is almost two orders of magnitude larger than the noise, while for System B the bias is within one standard deviation of the noise. These results clearly indicate that instrument noise in Systems A and B must be viewed as non-homogenous, anisotropic, and with location dependent bias. Therefore, it is not surprising that the prediction of FRE and TRE [3] which is based on a bias-free, isotropic, and homogenous model of Gaussian noise is not applicable to these data, and the performance of registration depends on the selection of fiducials. In particular, histograms of RMS_F in Figure 9(a, d) are stretched over relatively wide support intervals (approximately 0.2 mm in Figure 9a and 5 mm in Figure 9d) while (7) predicts no spread of RMS_F . The three performance metrics (index Q , RMS_T , and index W) also exhibit wide support intervals, as seen in Figure 5 (d – f), Figure 9(b, e), and Figure 10, and this observation clearly proves a need for finding the optimum configuration of fiducials.

As mentioned earlier, the three performance metrics cannot be evaluated in most practical situations because targets are measured only in the working frame but not in the destination frame. They are also not very useful when a large number of fiducial combinations need to be checked because they require prior calculation of registration transformation for each combination. Therefore, index F and $minR_I$ were introduced as surrogates of the performance metric and were used to search for the best combination of fiducials. The plots of index Q shown in Figure 6 indicate that there is a correlation between small Q and small F and that proxy index F can be used to find the set of fiducials which yield small Q . However, this does not mean that the set of fiducials which yields the smallest F will also yield the smallest Q . For the registration of System B to LT, the configuration which corresponds to the smallest F will yield Q with rank $k_Q = 49$ (see Figure 6a) while for the registration of System A to LT, $k_Q = 8,059$ (see Figure 6b) for the smallest F . In terms of absolute numbers, these ranks are far greater than $k_Q = 1$. However, these ranks should be compared with the maximum (i.e., the worst) rank: $k_Q = M_N$, where M_N is the total number of all combinations (9,691,375). Also, the difference between the two best Q values, i.e., $Q(k_F=1) - Q(k_Q=1)$ (gray vs. black bars in Figure 6) is very small when compared with the

width of the support interval for histograms of Q shown in Figure 5(d – f) which means that $Q(k_F=1)$ is in the left tail of the histogram. Note that the correlation between Q and F holds only for relatively small values of rank k_Q (for $k_Q < 0.01 M_N$) – no correlation is expected for larger ranks.

The proxy index $\min R_I$ is the lower limit of RMS_F and the histogram of their ratio is shown in Figure 7. In Figure 8, the correlation between RMS_F and $\min R_I$ for small ranks k of RMS_F is even better than that for Q and F shown in Figure 6. However, this observation has limited usefulness as RMS_F is not a very good metric for registration.

In Figure 11, all three performance metrics $Q(m)$, $W(m)$, and $RMS_T(m)$, calculated for six different configurations of fiducials ($m = 6$), are plotted for the registration of System B to LT. Similar plots are shown in Figure 12 for the registration of System A to LT and in Figure 13 for the registration of System A to B. The most important conclusion drawn from these three plots is that there is not one combination m^* which simultaneously minimizes all three metrics $Q(m^*)$, $W(m^*)$, and $RMS_T(m^*)$. For example: configuration $m(F)$ in Figure 11 yields small Q but large W and RMS_T . When W and RMS_T are more important, it is better to use the configuration $m(\min R_I)$. On the other hand, configuration $m(RMS_T)$ yields a Q value that is very close to its minimum value and a value of W that is less than the W value for $m(Q)$ configuration. However, as mentioned earlier, targets are not measured in the destination frame and thus, RMS_T cannot be calculated and configuration $m(RMS_T)$ remains unknown in practical applications. The data presented in Figure 11 correspond to registration of System B (small bias/ large noise) to LT. The two other graphs (Figure 12, Figure 13) contain results obtained from registering System A (large bias/small noise) to LT and to System B, respectively, and they support an earlier statement that configuration $m(F)$ yields Q and RMS_T close to their minimum values. However, neither $m(F)$ nor $m(\min R_I)$ configuration yields values of index $W < 1$.

So far, the number of fiducials has been fixed at $N = 4$. A common expectation is that increasing N would lead to better registration, i.e., decreased values of Q , W , and RMS_T . It was found in this study that this expectation is correct only in some cases; when the starting configuration of $N = 4$ fiducials is optimized, it is shown that there is little or no benefit to adding more fiducials. For example, Cases 1 and 2 in Figure 14 show that when the optimal configuration $m(Q)$ is chosen as the starting combination of $N = 4$ fiducials, the addition of a 5th fiducial only marginally improves the registration (small decrease of Q). On the other hand, when the median configuration is selected as the starting configuration (Cases 4 and 5), the addition of a 5th fiducial improves the registration performance substantially (large decrease of Q). Index Q for Cases 2 and 3 is almost the same which indicates that registration performance for the $N+1$ configuration is not very sensitive to the fiducial configuration if the previously selected N configuration was optimal. Therefore, the use of the iterative procedure of adding $N+1$ fiducial to the previously selected best N fiducials out of the remaining $M - N$ points is reasonable. Figure 15 shows that Q is almost independent of N for $N > 20$. Generally, $Q(N)$ is monotonically decreasing and the difference between the starting value of $Q(N = 4)$ and its asymptotic limit for large N is much smaller when the optimal starting configuration of fiducials, i.e., $m(k_Q = 1)$ is used than when the median

configuration, i.e., $m(k_Q = M_N/2)$ is used. In the second case, Figure 15b, $Q(N)$ is not smoothly decreasing (large spike at $N=15$).

The second performance metric (RMS_T) is almost independent of N when the optimum configuration $m(RMS_T)$ (which yields the smallest value of metric) is selected as the starting configuration of $N=4$ fiducials, as shown in Figure 16. This result seems to contradict the theoretical predictions that TRE depends on the target position relative to the centroid of the fiducials and to the three main axes of the moment of inertia (note that the addition of fiducials will affect the location of the centroid and the direction of the axes). There are two reasons why this apparent contradiction between our experimental results and theory is not really a contradiction. First, the closed form equations for TRE either ignore bias in the fiducials location or assume that the bias can be removed. They also assume that there is no bias in the target location. Our experimental results are based on fiducials and targets measurements with bias. Second, the analytical equations for TRE deal with a single target while RMS_T used in our study is the median value of many (16) targets scattered randomly in the work volume. So, as noted earlier, the addition of extra fiducials does not improve the registration when its performance is gauged by RMS_T and when the location of initial configuration is optimized.

A similar conclusion applies to the third metric, index W , shown in Figure 18. For the registration of System B (small bias/ large noise) to LT, the addition of extra fiducials is not beneficial but does not have an adverse effect on W , as can be seen in Figure 18a (values of $W < 1$ indicate good registration as the targets transformed from the working frame are within one sigma of the locations of the corresponding targets in the destination frame). However, for registration of System A (large bias / small noise) to LT or to B, the addition of extra fiducials has a disastrous effect, as could be seen in Figure 18(b, c). In summary: when $m(W)$ (which yields the smallest W) is selected as the starting configuration of fiducials and there is little or no bias in the measurements, the addition of extra fiducials is not necessary. For systems with large bias, addition of extra fiducials should be avoided.

As mentioned earlier, RMS_F is not a good measure of registration performance, but it could be used as an estimate for the upper bound of RMS_T , (see histograms of the ratio s in Figure 9(c, f)). However, it is interesting to see how RMS_F changes with the addition of more fiducials (Figure 17). Two strategies of adding new fiducials were used, similar to the procedure used for investigating $Q(N)$, $W(N)$, and $RMS_T(N)$, i.e., starting from a configuration of $N=4$ fiducials which yields the best solution (smallest RMS_F) and starting from a medium configuration for which the RMS_F rank equals to $M_N/2$. The medium strategy results in an irregular plot with small fluctuations of the RMS_F value, while the best strategy yields a smooth increase of RMS_F . The latter behavior is similar to the analytical prediction provided by (7) which was derived based on a simple homogenous, isotropic, and bias-free model of noise. Recall that this model did not describe the almost constant $RMS_T(N)$ relationship evaluated for the best strategy shown in Figure 16.

The discussion presented above is based on results from data processed using method 1: all registrations and corresponding performance metrics Q , W , and RMS_T were calculated for averaged locations of fiducials and targets. In method 2, for each repeated i -th measurement

of the fiducials and targets ($i = 1, \dots, 200$), registration parameters RMS_{T_i} and RMS_{F_i} were calculated. The resulting means and standard deviations of RMS_T calculated using method 2 is shown in Figure 19. The same starting combination of $N = 4$ fiducials for the best strategy is used in Figure 16 and Figure 19. Two different conclusions can be drawn from results obtained with method 2. For registering System B (small bias/ large noise) to LT, the addition of fiducials is beneficial: mean RMS_T is decreasing, a similar behavior is observed for data processed using method 1, see Figure 19a and Figure 16a. However, for registering System A (large bias/ small noise) to B or to LT, the addition of extra fiducials is clearly disadvantageous, as seen in Figure 19(b, c). We note also that the mean RMS_T is almost three times larger than the RMS_T calculated with method 1 for registration of System B to LT (compare values for $N = 4$ in Figure 19a and Figure 16a) while for registration of System A, both methods yield comparable values of RMS_T for $N = 4$. This is because the large bias overshadows the effect of small noise, and similarly, large noise diminishes the relevance of small bias.

The same argument explains why histograms of RMS_F and RMS_T calculated with method 2 are shifted to the right (towards higher values) when compared with histograms obtained using method 1 for registering System B to LT, (see dotted lines in Figure 9(a, b)). No such shift is observed for registering System A to B or to LT; the dotted lines and solid lines are overlaid in Figure 9(d – f). We note another interesting feature: the width of the support interval for dotted histogram of ratio s for method 2 in Figure 9c is smaller than the width of the support interval for method 1. This means that for data with no bias, an estimate for the upper limit of RMS_T (for targets located within the work volume) can be expressed as $s_{max} \times RMS_F$, where $s_{max} \approx 2$ for method 2 and $s_{max} \approx 3$ for method 1.

V. Summary of proposed procedure

When trying to determine locations of N fiducials for registration, it is advantageous to measure a large number M of potential fiducials as this will result in a higher probability of obtaining an optimum combination of N fiducials for registration. However, if fewer potential fiducials are obtained because of time or other constraints, the chances of getting an optimum combination of fiducials are reduced.

If only few fiducials are obtained, then the procedure proposed in this paper will not be needed as actual registrations may be performed for all possible combinations of fiducials within a reasonable amount of time. The proposed method of using proxy index F and $minR_1$ to select the optimum combination of fiducials for registration is recommended when a large number of fiducials are measured as the time to perform the actual number of registrations of all possible $M_N = M! / [(N!)(M - N)!]$ combinations would be prohibitively large.

The following steps summarize the procedure to select $N = 4$ fiducials for point-based, rigid-body registration.

1. Acquire 3D points at M locations in the working and destination frame (larger M is better). At each location, collect J repeat measurements in the same

environmental conditions. It is good practice to select fiducials distributed as uniformly as possible throughout the work space.

2. At each location, calculate from the J repeats the mean fiducial location, covariance matrix, and magnitude of noise σ_0 using (9).
3. For each unique pair of fiducial (i, j) , where $1 \leq i < j \leq M$, calculate the difference L and dimensionless parameter p using (10) and its mean $\bar{p}$ over all pairs (i, j) .
4. For each unique quadruple $(i1, i2, i3, i4)$ of fiducials, where $1 \leq i1 < i2 < i3 < i4 \leq M$, calculate proxy index F as in (17) and $\min R_I$ as in (18). Find quadruple $m(F)$ which yields the smallest F and quadrupole $m(\min R_I)$ which yields the smallest $\min R_I$.
5. The set of optimum fiducials $(j1, j2, j3, j4)$ is either:
 - a. quadruple $m(\min R_I)$ when $|\bar{p}| < 1$ and accuracy of registration (as gauged by RMS_T) is more important than precision (gauged by Q);
 - b. quadruple $m(F)$ when $|\bar{p}| < 1$ and precision (Q) is more important than accuracy (RMS_T) or $|\bar{p}| > 1$.
6. Calculate the rotation $\mathbf{R}$ and translation $\mathbf{\tau}$ of the rigid-body registration using the mean positions of the quadruple $(j1, j2, j3, j4)$ selected in the Step 5.
7. Calculate for all $i \notin (j1, j2, j3, j4)$ two metrics:
 - a.
$$E_1(i) = \sqrt{\frac{1}{J} \sum_{n=1}^J \| \mathbf{R} \mathbf{X}_{n,i} + \mathbf{\tau} - \mathbf{Y}_i \|^2}$$
 where $\mathbf{X}_{n,i}$ is an n -th instantaneous measurement of i -th fiducial;
 - b.
$$E_2(i) = \| \mathbf{R} \mathbf{X}_i + \mathbf{\tau} - \mathbf{Y}_i \|^2.$$

Once $\{\mathbf{R}, \mathbf{\tau}\}$ is determined, it can be applied to any target point $\mathbf{T}_x$ in the working frame. The $M - 4$ fiducials not used for registration can be used to estimate uncertainty and error of the transformed target.

1. Find the k -th point $\mathbf{X}_k$ closest to $\mathbf{T}_x$ from the set of $M - 4$ points in the working frame, i.e. $\| \mathbf{T}_x - \mathbf{X}_k \|$ is the smallest and $k \notin (j1, j2, j3, j4)$.
2. Use $E_1(k)$ as the estimated uncertainty of transformed target $\mathbf{T}_x$.
3. Use $E_2(k)$ the estimated location error of transformed target $\mathbf{T}_x$.

The use of optimally placed fiducials is most beneficial when the overlap region of the work volumes in the working and destination frames is large. Such a situation occurs in manufacturing where the field of view of a perception system which guides an arm movement of autonomous robot contains the whole work volume accessible to the robot. In other applications, the overlap may be small and the expected benefit may be smaller. Such a situation occurs in image-guided neurosurgery where fiducials can be placed only on a patient's head while the operation is performed on the brain.

VI. Final remarks

The performance of rigid-body registration based on 3D data characterized by non-homogenous, anisotropic noise, and with position dependent bias, depends on the selection of fiducial locations. To aid in determining the fiducial locations, two proxy indices, F and $\min R_1$, are introduced in this paper to rapidly evaluate a large number of fiducial configurations in the work volume. Low values of these indices correspond to configurations which will yield near optimal registrations. Three performance metrics, Q , RMS_T , and W were formulated to gauge two aspects of the registration: its precision (expressed as the spread of the target points transformed from the working to the destination frame) and its accuracy (evaluated as distances between targets transformed from the working frame and the corresponding targets measured in the destination frame). This study shows that the use of different performance metrics will lead to the selection of different fiducial configurations.

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Figure 1. Noise magnitude σ_0 at grid points for: a) System A; b) System B. The size of the marker is proportional to σ_0 . For better visualization and comparison, both datasets were transformed into the coordinate frame of the laser tracker. Histograms of σ_0 for System A and System B are shown in c) and d).

Figure 2.

Top plots: direction of the eigenvector $e_3(m)$ corresponding to the largest eigenvalue λ_3 of the covariance matrix at the m -th grid point for: a) data acquired with System A; b) System B. Grid points are displayed in the local coordinate frame of the respective instrument.

Bottom plots: histogram of noise anisotropy coefficient (ratio of the largest to the smallest eigenvalue) for: c) data acquired with System A; d) System B.

Figure 3.
The error L between the distance between two grid points measured by System A or B and the distance D between the same points as measured by laser tracker LT.

Figure 4.
Histogram of the ratio p for data collected by: a) System A; b) System B.

Figure 5.

Distribution of indices F and Q for all possible combinations of $N=4$ fiducials selected from $M=125$ grid points for registering: System B to LT (a and d); System A to LT (b and e); System A to B (c and f).

Figure 6.

First ten values of index Q sorted in ascending order (dark bars, rank $k_Q = 10$) and ten values of Q corresponding to ten smallest values of F (light gray bars). Numbers on upper horizontal axes of each subplot are ranks k_Q corresponding to ten smallest values of F . Results for registration: a) System B to LT; b) System A to LT; c) System A to B.

Figure 7. Histogram of ratio $RMS_F / minR_I$ for: a) System A to System B; b) System B to LT. The plot of the histogram for the registration of A to LT looks identical to a) when plotted in the same scale.

Figure 8.

First ten values of index RMS_F sorted in ascending order (dark bars) and ten values of RMS_F corresponding to ten smallest values of $minR_I$ (light gray bars). Numbers on upper horizontal axes of each subplot are ranks k of RMS_F corresponding to ten smallest values of $minR_I$. Results for registration: a) System B to LT; b) System A to LT; c) System A to B.

Figure 9.

Histograms of *RMS* based on all possible combinations of $N=4$ fiducials for registrations: (a–c) System B to LT; (d–f) System A to LT. Solid line: *RMS* calculated using method 1, dotted line: using method 2. In (d–f), the solid and dotted lines are overlaid.

Figure 10. Histograms of index W for registering: a) System B to LT; b) System A to LT; c) System A to B.

Figure 11.

The performance metrics: a) index Q ; b) index W ; c) RMS_T for six different configurations of $N=4$ fiducials. Configurations $m(F)$, $m(\min R_1)$, $m(Q)$, $m(W)$, $m(RMS_F)$, and $m(RMS_T)$ yield the best performance as gauged by the smallest value of proxy index F , $\min R_1$, index Q , W , RMS_F and RMS_T , respectively. Results from registration of System B to LT.

Figure 12.

The performance metrics: a) index Q ; b) index W ; c) RMS_T for six different configurations of $N=4$ fiducials. Configurations $m(F)$, $m(\min R_1)$, $m(Q)$, $m(W)$, $m(RMS_F)$, and $m(RMS_T)$ yield the best performance as gauged by the smallest value of proxy index F , $\min R_1$, index Q , W , RMS_F and RMS_T , respectively. Results from registration of System A to LT.

Figure 13.

The performance metrics: a) index Q ; b) index W ; c) RMS_T for six different configurations of $N=4$ fiducials. Configurations $m(F)$, $m(\min R_1)$, $m(Q)$, $m(W)$, $m(RMS_F)$, and $m(RMS_T)$ yield the best performance as gauged by the smallest value of proxy index F , $\min R_1$, index Q , W , RMS_F and RMS_T , respectively. Results from registration of System A to B.

Figure 14. Comparison of registration performance gauged by index Q for $N = 4$ and $N = 5$ fiducials. Results are for registering: a) System B to LT; b) System A to LT; c) System A to B. Selection of fiducials based on proxy index F .

Figure 15.

Index Q vs. the number of fiducials N used in registration. Two criteria for adding $N+1$ fiducials to the previously selected configuration of N fiducials are used: based on the smallest Q (+) and medium Q (*). Individual plots are for registering: a) System B to LT; b) System A to LT; c) System A to B.

Figure 16. RMS_T vs. the number of fiducials N used in registration. Two criteria for adding $N+1$ fiducials to the previously selected configuration of N fiducials are used: based on the smallest $RMS_T(+)$ and medium $RMS_T(*)$. Individual plots are for registering: a) System B to LT; b) System A to LT; c) System A to B.

Figure 17. RMS_F vs. the number of fiducials N used in registration. Two criteria for adding $N+1$ fiducials to the previously selected configuration of N fiducials are used: based on the smallest RMS_F (+) and medium RMS_F (*). Individual plots are for registering: a) System B to LT; b) System A to LT; c) System A to B.

Figure 18.

Index W vs. the number of fiducials N used in registration. The $N+1$ fiducial added to the previously selected configuration of N fiducials yields the smallest W chosen out of the remaining $M - N$ configurations. Individual plots are for registering: a) System B to LT; b) System A to LT; c) System A to B.

Figure 19. Mean RMS_T (calculated using method 2) vs. the number N of fiducials used in registration. Error bars indicate $\pm$ one standard deviation of RMS_T obtained from 200 repeated measurements. Individual plots are for registering: a) System B to LT; b) System A to LT; c) System A to B.