Neutral impulsive shunting inhibitory cellular neural networks with time-varying coefficients and leakage delays

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Abstract In this article, we consider a class of neutral impulsive shunting inhibitory cellular neural networks with time varying coefficients and leakage delays. We study the existence and the exponential stability of the piecewise differentiable pseudo almost-periodic solutions and establish sufficient conditions for the existence and exponential stability of such solutions. An example is provided to illustrate the theory developed in this work.

Keywords Piecewise pseudo almost-periodic functions · Impulsive shunting inhibitory cellular neural networks · Exponential stability · Mixed delays · Leakage delays

Mathematics Subject Classification 34C27 · 37B25 · 92C20

Introduction

Neural networks have been extensively applied, recently, in many areas such as optimization problems, pattern recognition, signal processing, image processing, associative memories and many other fields (Abbas et al. 2014;

Ammar et al. 2012; Aouiti et al. 2016; Bai 2008, 2009; Benucci et al. 2007; Cao et al. 2016; Chua and Yang 1988; Huang et al. 2002; Mattia and Sanchez-Vives 2012; Roska and Chua 1992; Tagluk and Tekin 2014; Wei et al. 2014; Zhu et al. 2015).

As is well known, both in biological and man-made neural networks, delays are inevitable, due to various reasons. For instance, time delays can be caused by the finite switching speed of amplifier circuits in neural networks. Time delays in the neural networks make the dynamic behaviors become more complex, and may destabilize the stable equilibria and admit oscillations, bifurcation and chaos. Thus, it is very important to study the dynamics of neural networks delay (Ammar et al. 2012; Aouiti et al. 2016; Bai 2008, 2009; Chua and Yang 1988; Huang et al. 2002; Jiang and Cao 2006; Jiang and Teng 2004; Roska and Chua 1992; Xia 2014; Xiang and Cao 2009; Yang and Cao 2012; Zhao 2008). In particular, the time delay in the negative feedback terms which is known as leakage has a tendency to destabilize the system (Gopalsamy 2007; Hu et al. 2014; Song and Cao 2012) and have great impact on the dynamical behavior of neural networks (Li and Cao 2010). This is to say, it is necessary to consider the effect of leakage delays when studying the stability of state estimation of neural networks. Up to now, recurrent neural networks with leakage delays have been widely investigated in the literature (Gao et al. 2014).

In real world, many evolutionary processes are characterized by abrupt changes at certain time. These short-term perturbations are often treated as having acted instantaneously or in the form of impulses. These phenomena are included in many fields such as physics, chemistry, population dynamics, optimal control (Chérif 2014; Kumar et al. 2016; Pan and Cao 2011;

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Rakkiyappan et al. 2010; Xia 2016; Xia and Wang 2015, 2016). It is well known that the study on dynamical systems not only involve a discussion of stability, but also involve many dynamical behaviors such as pseudo almost periodic phenomenon, bifurcation and chaos (Chen and Cao 2002; Chen and Zhao 2008; Chérif 2012; Fan and Shao 2010; Gao et al. 2014). A central subject in this paper is the concept of pseudo-almost periodicity which it was introduced by Zhang (1994, 1995). Note that the concept of pseudo almost periodic function generalizes the class of almost periodic functions in the sense of Bohr. Shunting inhibitory cellular neural networks (SICNNs), which introduced by Bouzerdoum and Pinter (1993), have been studied and developed extensively in the past few decades by many authors due to their extensive applications in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, image processing and many other fields. We refer the reader to Bouzerdoum and Pinter (1993), Chen and Cao (2002), Chen and Zhao (2008), Chérif (2012), Ding and Ye (2009), Fan and Shao (2010), Liu et al. (2007, 2006), Ou (2009), Shao et al. (2009) and Wu (2011). Consider a two-dimensional grid of processing cells, let C_{ij} denote the cell at the (i, j) position of the lattice, the r -neighborhood $N_r(i, j)$ of C_{ij} is given as

$$N_r(i, j) = \{C_{kl} : \max(|k - i|, |l - j|) \leq r, 1 \leq k \leq m, 1 \leq l \leq n\}.$$

In SICNNs, neighboring cells exert mutual inhibitory interactions of the shunting type. The dynamics of a cell C_{ij} are described by the following nonlinear ordinary differential equation:

$$x'_{ij}(t) = -a_{ij}x_{ij}(t) - \sum_{C_{kl} \in N_q(i, j)} B_{ij}^{kl} g(x_{kl}(t)) x_{ij}(t) + L_{ij}(t)$$

In a recent paper, Li et al. (2012) considered the following SICNN with continuously distributed delays of neutral type:

$$\begin{aligned} x'_{ij}(t) = & -a_{ij}(t)x_{ij}(t) \\ & - \sum_{C_{kl} \in N_r(i, j)} C_{ij}^{kl}(t) \int_0^{+\infty} K_{ij}(u) f(x'_{kl}(t-u)) du x_{ij}(t) \\ & - \sum_{C_{kl} \in N_s(i, j)} D_{ij}^{kl}(t) \int_0^{+\infty} J_{ij}(u) g(x'_{kl}(t-u)) du x_{ij}(t) \\ & + L_{ij}(t) \end{aligned}$$

Inspired by these works, in this paper we consider the following impulsive shunting inhibitory cellular neural networks with time-varying coefficients and leakage delays of neutral type (NISICNN):

$$\begin{aligned} x'_{ij}(t) = & -a_{ij}(t)x_{ij}(t - \rho(t)) \\ & - \sum_{C_{kl} \in N_q(i, j)} B_{ij}^{kl}(t) g(x_{kl}(t)) x_{ij}(t) \\ & - \sum_{C_{kl} \in N_p(i, j)} D_{ij}^{kl}(t) h(x'_{kl}(t - \tau(t))) x_{ij}(t) \\ & - \sum_{C_{kl} \in N_r(i, j)} C_{ij}^{kl}(t) \int_0^{+\infty} K_{ij}(u) f(x'_{kl}(t-u)) du x_{ij}(t) \\ & + L_{ij}(t), t \in \mathbb{R}, t \neq t_w, w \in \mathbb{Z} \\ & \Delta x_{ij}(t_w) = x_{ij}(t_w^+) - x_{ij}(t_w^-) = I_w(x_{ij}(t_w)), \end{aligned} \quad (1)$$

where $N_q(i, j)$ and $N_p(i, j)$ are similarly specified, $x_{ij}(\cdot)$ is the activity of the cell C_{ij} , $L_{ij}(\cdot)$ is the external input to C_{ij} , $a_{ij}(\cdot) > 0$ represents the passive decay rate of the cell activity, $B_{ij}^{kl}(\cdot)$, $D_{ij}^{kl}(\cdot)$ and $C_{ij}^{kl}(\cdot)$ are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij} , and the activity functions $g(\cdot)$, $h(\cdot)$ and $f(\cdot)$ are continuous functions representing the output or firing rate of the cell C_{kl} , $\rho(\cdot)$ and $\tau(\cdot)$ are bounded, continuous function, corresponding to the transmission delays with $0 \leq \rho(t) \leq \rho^+$, $0 \leq \tau(t) \leq \tau^+$ for $t \in \mathbb{R}$ where ρ^+ and τ^+ are constants, the sequence $\{t_w\}$ has no finite accumulation point and $I_w : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

The initial conditions associated with (1) are of the form:

$$\begin{aligned} x_{ij}(s) = & \varphi_{ij}(t), \quad t \in (-\infty, 0], \\ & i = 1, 2, \dots, n, j = 1, 2, \dots, m, \end{aligned}$$

where $\varphi(\cdot)$ are real-valued piecewise continuous functions defined on $(-\infty, 0]$.

In the present paper, the concept of piecewise pseudo almost periodic functions is introduced and some necessary results are obtained, after that, by using fixed point theory in Banach space and Gronwall-Bellmans inequality technique, the existence and exponential stability of pseudo almost periodic solutions for model (1) are established.

Throughout this paper, We introduce the following assumptions:

(H1) There exist positive constant numbers $L^g, L^h, L^f, M^g, M^h, M^f > 0$ such that for all $x, y \in \mathbb{R}$

$$\begin{aligned} |g(x) - g(y)| & < L^g |x - y|, \\ |h(x) - h(y)| & < L^h |x - y|, \\ |f(x) - f(y)| & < L^f |x - y|, \end{aligned}$$

and $|g(u)| \leq M^g, |h(u)| \leq M^h, |f(u)| \leq M^f$, for all $u \in \mathbb{R}$. Furthermore, we suppose that $g(0) = h(0) = f(0) = 0$.

(H2) For all $1 \leq i \leq m, 1 \leq j \leq n$ the kernel $K_{ij} : [0, +\infty[\rightarrow \mathbb{R}$ satisfies, for some positive constant K_{ij}^+, η

$$|K_{ij}(s)| \leq K_{ij}^+ e^{-\eta s},$$

(H3) For all $1 \leq i \leq m$, $1 \leq j \leq n$, the functions $t \mapsto a_{ij}(t)$ and the functions $t \mapsto \rho(t)$ and $t \mapsto \tau(t)$ are almost periodic functions with $\inf_{t \in \mathbb{R}} (a_{ij}(t)) = a_{ij*} > 0$, and for all $t \in \mathbb{R}$ we have $1 - \rho'(t) > 0$ and $1 - \tau'(t) > 0$.

(H4) For all $1 \leq i \leq m$, $1 \leq j \leq n$ the functions $t \mapsto B_{ij}^{kl}(t)$, $t \mapsto D_{ij}^{kl}(t)$, $t \mapsto C_{ij}^{kl}(t)$, $t \mapsto L_{ij}(t)$ are pseudo almost periodic functions.

(H5) $I_w \in PAP(\mathbb{Z}, \mathbb{R}^n)$ and there exists a constant L_1 such that $0 \leq L_1 < 1$ and

$$\|I_w(u) - I_w(v)\| \leq L_1 \|u - v\|, \quad u, v \in \mathbb{R}^n, w \in \mathbb{Z}$$

The rest of this paper is organized as follow. In “Preliminaries” section, we will recall the basic properties of the piecewise pseudo almost periodic functions. In third section, we prove the “Existence of piecewise differentiable pseudo almost periodic solutions” of (1) in the suitable convex set. The 4 “Exponential stability of piecewise differentiable pseudo almost periodic solutions” for (1) is discussed in fourth section. A numerical example is given in “Illustrative example” section to illustrate the effectiveness of our results. Finally, we draw “Conclusion” in last section.

Preliminaries

Throughout this paper, we will first recall some basic definitions and lemmas which are used in what follows.

- $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ stand for the set of natural numbers, integer numbers and real numbers, respectively.
- $C(\mathbb{R}, \mathbb{R}^n)$: the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$.
- $BC(\mathbb{R}, \mathbb{R}^n)$: the set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. Note that $(BC(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space where $\|\cdot\|_\infty$ denotes the sup norm

$$\|f\|_\infty := \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} |f_i(t)|.$$

- Let T be the set consisting of all real sequences $\{t_i\}_{i \in \mathbb{Z}}$ such that $\alpha = \inf_{i \in \mathbb{Z}} (t_{i+1} - t_i) > 0$. It is immediate that this condition implies that $\lim_{i \rightarrow +\infty} t_i = +\infty$ and $\lim_{i \rightarrow -\infty} t_i = -\infty$.
- $PC(\mathbb{R}, \mathbb{R}^n)$: the space of all piecewise continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}^n$ such that
 - $f(\cdot)$ is continuous at t for any $t \notin \{t_i, i \in \mathbb{Z}\}$,
 - $f(t_i^+)$, $f(t_i^-)$ exists and $f(t_i^-) = f(t_i)$ for all $i \in \mathbb{Z}$.
- $PC([- \tau, 0], \mathbb{R}^n) = \{f: [- \tau, 0] \rightarrow \mathbb{R}^n | f(t^-) = f(t), \text{ for } t \in [- \tau, 0], f(t^+) \text{ exists on } \mathbb{R} \text{ and } f(t^+) = f(t) \text{ for all but at most a finite number of points on } [- \tau, 0]\}$,

- $PC^1([- \tau, 0], \mathbb{R}^n) = \{f: [- \tau, 0] \rightarrow \mathbb{R}^n | f'(t^+) \text{ and } f'(t^-) \text{ exist, } f'(t) = f'(t^-) \text{ for } t \in [- \tau, 0], f'(t^+) = f'(t) \text{ for all but at most a finite number of points on } [- \tau, 0]\}$,
- $l^\infty(\mathbb{Z}, \mathbb{R}^n) = \{x: \mathbb{Z} \rightarrow \mathbb{R}^n : \|x\| = \sup_{n \in \mathbb{Z}} \|x(n)\| < \infty\}$.

Definition 1 (Fink 1974) A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ such that every interval of length $L(\varepsilon) > 0$ contains a number τ with the property that $\|f(t + \tau) - f(t)\|_\infty < \varepsilon$, for each $t \in \mathbb{R}$.

The number τ above is called an ε -translation number of f , and the collection of all such functions will be denoted as $AP(\mathbb{R}, \mathbb{R}^n)$.

Definition 2 (Chérif 2012) A sequence $\{x_n\}$ is called almost periodic if for any $\epsilon > 0$, there exists a relatively dense set of its ϵ -periods, i.e., there exists a natural number $l = l(\epsilon)$, such that for $k \in \mathbb{Z}$, there is at least one number p in $[k, k + l]$, for which inequality $\|x_{n+p} - x_n\| < \epsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, \mathbb{R}^n)$, the set of such sequences.

Define

$$PAP_0(\mathbb{Z}, \mathbb{R}^n) := \left\{ x \in l^\infty(\mathbb{Z}, \mathbb{R}^n) : \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{k=-n}^n \|x_k\| = 0 \right\}.$$

Remark 1 Notice that

1. A sequence vanishing at infinity is a $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence.
2. The sequence $(x(n))_{n \in \mathbb{Z}}$ defined by

$$x(n) = \begin{cases} 1, & n = 2^k, \\ 0, & n \neq 2^k, \end{cases}$$

is an example of a $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence which not vanishing at infinity.

3. For $k \in \mathbb{N}$, the sequence $(x(n))_{n \in \mathbb{Z}}$ defined by

$$x(n) = \begin{cases} k, & n = 2^{k^2}, \\ 0, & n \neq 2^{k^2}, \end{cases}$$

is an example of an unbounded $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence.

Definition 3 (Chérif 2012) A sequence $\{x_n\}_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z}, \mathbb{R}^n)$ is called pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, \mathbb{R}^n)$, $x_n^2 \in PAP_0(\mathbb{Z}, \mathbb{R}^n)$. Denote by $PAP(\mathbb{Z}, \mathbb{R}^n)$ the set of such sequences.

For $\{t_i\}_{i \in \mathbb{Z}} \in T$, $\{t_i^j\}$ defined by

$$\{t_i^j = t_{i+j} - t_i\}, \quad i, j \in \mathbb{Z}.$$

It is easy to verify that the numbers t_i^j satisfy

$$t_{i+k}^j - t_i^j = t_{i+j}^k - t_i^k, t_i^j - t_i^k = t_{i+k}^{j-k}, \quad \text{for } i, j, k \in \mathbb{Z}.$$

Definition 4 (Samoilenko and Perestyuk 1995) A function $f \in PC(\mathbb{R}, \mathbb{R}^n)$ is said to be piecewise almost periodic if the following conditions are fulfilled:

1. $\{t_i^j = t_{i+j} - t_i\}$, $i, j \in \mathbb{Z}$ are equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set in $\mathbb{R}$ of ε -almost periods common for all of the sequences $\{t_i^j\}$.
2. For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity of f and $|t' - t''| < \delta$, then $\|f(t') - f(t'')\| < \varepsilon$.
3. For any $\varepsilon > 0$, there exists a relatively dense set Ω_ε in $\mathbb{R}$ such that if $\tau \in \Omega_\varepsilon$, then

$$\|f(t + \tau) - f(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$ which satisfy the condition $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, \mathbb{R}^n)$ the space of all piecewise almost periodic functions. Obviously, $AP_T(\mathbb{R}, \mathbb{R}^n)$ endowed with the supremum norm is a Banach space. Throughout the rest of this paper, we always assume that $\{t_i^j\}$ are equipotentially almost periodic. Let $UPC(\mathbb{R}, \mathbb{R}^n)$ be the space of all functions $f \in PC(\mathbb{R}, \mathbb{R}^n)$ such that f satisfies the condition (2) in Definition 4.

Define

$$PAP_T^0(\mathbb{R}, \mathbb{R}^n) = \left\{ f \in PC(\mathbb{R}, \mathbb{R}^n) : \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|f(t)\| dt = 0 \right\}.$$

Definition 5 (Chérif 2012) A function $f \in PC(\mathbb{R}, \mathbb{R}^n)$ is said to be piecewise pseudo almost periodic if it can be decomposed $f = g + h$, where $g \in AP_T(\mathbb{R}, \mathbb{R}^n)$ and $h \in PAP_T^0(\mathbb{R}, \mathbb{R}^n)$. Denote by $PAP_T(\mathbb{R}, \mathbb{R}^n)$ the set of all such functions. $PAP_T(\mathbb{R}, \mathbb{R}^n)$ is a Banach space when endowed with the supremum norm.

Remark 2 The functions g and h in Definition 5 are respectively called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f . The decomposition given in Definition 5 is unique. For instance, the function

$$f(t) = \begin{cases} \sin^2(\sqrt{3}t) + \cos^2(\pi t) + \frac{1}{1+t^2}, & t \neq t_k, k \in \mathbb{Z}, \\ \frac{1}{4} \sin(\sqrt{3}t) + \frac{1}{1+t^2}, & t = t_k, k \in \mathbb{Z}, \end{cases}$$

is a piecewise pseudo almost periodic function, where

$$t_k = k + \frac{1}{6} |\sin k - \sin \sqrt{2}k|.$$

Hence, it is easy to see that $f(\cdot)$ is more general than our traditional piecewise almost periodic functions since the ergodic perturbations are introduced.

Lemma 1 (Zhang 2003) Suppose that both functions f and its derivative f' are in $PAP(\mathbb{R}, \mathbb{R})$. That is, $f = g + h$ and $f' = g' + h'$, where $g, g' \in AP(\mathbb{R}, \mathbb{R})$ and $h, h' \in PAP_0(\mathbb{R}, \mathbb{R})$. Then the functions g and h are continuous differentiable.

Remark 3 Let $E = \{f \mid f, f' \in PAP(\mathbb{R}, \mathbb{R}^n)\}$ equipped with the induced norm defined by

$$\|f\|_E = \max\{\|f\|_\infty, \|f'\|_\infty\}.$$

Follows from Zhang (2003) that $(PAP(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_E)$ is a Banach space.

Lemma 2 (Shao et al. 2009) Let $c_i(\cdot)$ be an almost periodic function on $\mathbb{R}$ and

$$M[c_i] = \lim_{T \rightarrow +\infty} \int_t^{t+T} c_i(s) ds > 0, \quad i = 1, \dots, n.$$

Then the linear system

$$x'(t) = \text{diag}(-c_1(t), -c_2(t), \dots, -c_n(t))x(t) \quad (2)$$

admits an exponential dichotomy on $\mathbb{R}$.

Lemma 3 (Lin and Lin 2000) The inhomogeneous linear system

$$x'(t) = -c(t)x(t) + f(t)$$

has a unique bounded solution for a vector $f \in C(\mathbb{R}, \mathbb{R}^n)$ if and only if the inhomogeneous linear system (2) has exponential dichotomy.

For all $x(t) = (x_{11}(t), \dots, x_{ij}(t), \dots, x_{mn}(t)) \in \mathbb{R}^{m \times n}$, we define the norm

$$\|x(t)\| = \max_{(i,j)} |x_{ij}(t)|$$

Here, we introduce the phase space $PC((-\infty, 0], \mathbb{R}^{m \times n})$ as a Banach space of piecewise continuously mappings from $(-\infty, 0]$ to $\mathbb{R}^{m \times n}$. For all

$$\varphi = (\varphi_{11}(t), \dots, \varphi_{ij}(t), \dots, \varphi_{mn}(t))^T \in PC((-\infty, 0], \mathbb{R}^{m \times n})$$

we define the norm

$$\|\varphi\|_\infty = \max_{(i,j)} \sup_{-\infty \leq t \leq 0} |\varphi_{ij}(t)|$$

The initial conditions associated with (1) are of the form

$$x_{ij}(s) = \varphi_{ij}(s),$$

where $s \in (-\infty, 0]$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Theorem 1 (Aliprantis and Boreder 1999) (Fubini's Theorem) Assume that $\mu : S_1 \rightarrow [0, \infty]$ and $\nu : S_2 \rightarrow [0, \infty]$ are measures on two semirings of subsets of the sets X and Y , respectively. If $f : X \times Y \rightarrow \mathbb{R}$ is a $\mu \times \nu$ -integrable function, then both iterated integrals $\iint f d\nu d\mu$ and $\iint f d\mu d\nu$ exist and

$$\iint f d(\nu \times \mu) = \iint f d\nu d\mu = \iint f d\mu d\nu.$$

Existence of piecewise differentiable pseudo almost periodic solution

In this section, we establish some results for the existence of the piecewise differentiable pseudo almost periodic solution of (1). For convenience, we introduce the following notations:

$$\begin{aligned} a_{ij}^+ &= \sup_{t \in \mathbb{R}} |a_{ij}(t)|, \quad \bar{C}_{ij}^{kl} = \sup_{t \in \mathbb{R}} |C_{ij}^{kl}(t)|, \\ \bar{D}_{ij}^{kl} &= \sup_{t \in \mathbb{R}} |D_{ij}^{kl}(t)|, \quad \bar{B}_{ij}^{kl} = \sup_{t \in \mathbb{R}} |B_{ij}^{kl}(t)|, \\ \bar{L}_{ij} &= \sup_{t \in \mathbb{R}} |L_{ij}(t)|. \end{aligned}$$

Lemma 4 (Xia 2016) If $\varphi(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$, then $\varphi(\cdot - h) \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$.

Lemma 5 (Xia 2016) If $\varphi, \psi \in PAP_T(\mathbb{R}, \mathbb{R})$, then $\varphi\psi \in PAP_T(\mathbb{R}, \mathbb{R})$.

Lemma 6 If $f(\cdot) \in C(\mathbb{R}, \mathbb{R})$ satisfies the Lipschitz condition, $\varphi(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$, $\varphi'(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$ and $\sigma(\cdot) \in AP_T(\mathbb{R}, \mathbb{R})$ such that $1 - \sigma'(t) > 0$ for all $t \in \mathbb{R}$ then $f(\varphi(\cdot - \sigma(\cdot))) \in PAP_T(\mathbb{R}, \mathbb{R})$

Proof See “Appendix 1”. $\square$

Theorem 2 Assume that assumptions (H1)–(H2) hold and for all $1 \leq k \leq m$, $1 \leq l \leq n$, $x_{kl}(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$ and $x'_{kl}(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$, then for all $1 \leq i \leq m$, $1 \leq j \leq n$ the function

$$\begin{aligned} \phi_{ij} : t &\mapsto \int_0^\infty K_{ij}(s)f(x'_{kl}(t-s))ds \\ &= \int_{-\infty}^t K_{ij}(t-s)f(x'_{kl}(s))ds \end{aligned}$$

belongs to $PAP_T(\mathbb{R}, \mathbb{R})$.

Proof See “Appendix 2”. $\square$

Lemma 7 Suppose that assumptions (H1)–(H4) hold. Define the nonlinear operator $X_\varphi(\cdot)$ as follows, for each $\varphi = (\varphi_{11}, \dots, \varphi_{mn}) \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$ and $\varphi' = (\varphi'_{11}, \dots, \varphi'_{mn}) \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$

$$X_\varphi(t) = \begin{pmatrix} \int_{-\infty}^t e^{-\int_s^t a_{11}(u)du} F_{11}(s)ds \\ \vdots \\ \int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du} F_{ij}(s)ds \\ \vdots \\ \int_{-\infty}^t e^{-\int_s^t a_{mn}(u)du} F_{mn}(s)ds \end{pmatrix}$$

and

$$\begin{aligned} F_{ij}(s) &= a_{ij}(s) \int_{s-\rho(s)}^s \varphi'_{ij}(s)ds \\ &\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s)g(\varphi_{kl}(s))\varphi_{ij}(s) \\ &\quad - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(s)h(\varphi'_{kl}(s - \tau(s)))\varphi_{ij}(s) \\ &\quad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \\ &\quad \times \int_0^{+\infty} K_{ij}(u)f(\varphi'_{kl}(s-u))du\varphi_{ij}(s) \\ &\quad + L_{ij}(s), \end{aligned}$$

then X_φ maps $PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$ into itself.

Proof See “Appendix 3”. $\square$

Lemma 8 Suppose that assumption (H5) hold. For each $\varphi = (\varphi_{11}, \dots, \varphi_{mn}) \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$, we have

$$\sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u)du} I_w(\varphi_{ij}(t_w)) \in PAP_T(\mathbb{R}, \mathbb{R}).$$

Proof See “Appendix 4”. $\square$

Lemma 9 Suppose that assumptions (H1)–(H5) hold. Define the nonlinear operator Γ as follows, for each $\varphi = (\varphi_{11}, \dots, \varphi_{mn}) \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$, $\varphi' = (\varphi'_{11}, \dots, \varphi'_{mn}) \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$, $1 \leq i \leq m$, $1 \leq j \leq n$:

$$(\Gamma_\varphi)_{ij}(t) := (x_\varphi)_{ij}(t) + \sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u)du} I_w(\varphi_{ij}(t_w)),$$

then Γ maps $PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$ into itself and

$$\begin{aligned} (\Gamma_\varphi)'_{ij}(t) &:= F_{ij}(t) - a_{ij}(t) \int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du} F_{ij}(s)ds \\ &\quad - a_{ij}(t) \sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u)du} I_w(\varphi_{ij}(t_w)), \end{aligned}$$

then Γ' maps $PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$ into itself.

Theorem 3 Under the conditions (H1)–(H5) and assume that there exist nonnegative constants L, δ and q such that $\delta < 1, q < 1$ then the impulsive delayed SICNNs (1) has a unique piecewise differentiable pseudo almost periodic solution in the region $\mathbb{B} = \{\varphi, \varphi' \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n}), \|\varphi - \varphi_0\|_E \leq \frac{\delta L}{(1-\delta)}\}$, where

$$\varphi_0(t) = \begin{pmatrix} \int_{-\infty}^t e^{-\int_s^t a_{11}(u)du} L_{11}(s)ds \\ \vdots \\ \int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du} L_{ij}(s)ds \\ \vdots \\ \int_{-\infty}^t e^{-\int_s^t a_{mn}(u)du} L_{mn}(s)ds \end{pmatrix}.$$

Proof See “Appendix 5”. $\square$

Exponential stability of piecewise differentiable pseudo almost periodic solution

To study the exponential stability of (1), we need the following lemma and notations. So, for a continuous function $g(\cdot)$, we denote

$$\bar{g}(t) = \sup_{t-\tau^+ \leq s \leq t} |g(s)|,$$

(H6) There exist positive constants p_{ij} and q_{ij} , such that

$$\begin{aligned} 0 &< p_{ij}a_{ij}(t) - q_{ij}a_{ij}(t)\rho^+ - \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(t)| \\ &\times \left(L^g p_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij}M^g \right) - \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(t)| \\ &\times \left(L^h q_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij}M^h \right) - \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(t)| \\ &\times \frac{K_{ij}^+}{\eta} \left(L^f q_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij}M^f \right), \\ 0 &< q_{ij} - p_{ij}a_{ij}(t) - q_{ij}a_{ij}(t)\rho^+ - \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(t)| \\ &\times \left(L^g p_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij}M^g \right) - \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(t)| \\ &\times \left(L^h q_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij}M^h \right) - \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(t)| \\ &\times \frac{K_{ij}^+}{\eta} \left(L^f q_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij}M^f \right), \end{aligned}$$

for $t \in [0, \infty)$, $i = 1, 2, \dots, n$

Lemma 10 Let $\tau \geq 0$ be a given real constant. Assume that $p(t)$ and $q_i(t)$ ($i = 1, 2$) be continuous functions on

$[0, +\infty)$, $k(s)$ be nonnegative function on $[0, +\infty)$ and satisfies that $\int_0^{+\infty} k(s)ds \leq k$ and $\int_0^{+\infty} k(s)e^{\mu s}ds \leq +\infty$ for positive constant μ .

Moreover, assume that there exist positive constants η and M such that

$$\begin{aligned} p(t) - q_1(t) - kq_2(t) &\geq \eta > 0, \\ 0 \leq q_1(t) \leq M, 0 \leq q_2(t) \leq M, &\forall t \geq 0, \end{aligned}$$

then

$$\begin{aligned} \lambda^* = \inf_{t \geq 0} \{ &\lambda > 0, \lambda - p(t) + q_1(t)e^{\lambda \tau} \\ &+ q_2(t) \int_0^{+\infty} k(s)e^{\lambda s}ds = 0 \} > 0. \end{aligned}$$

Proof See “Appendix 6”. $\square$

Then we have

Lemma 11 Assume that (H1)–(H6) hold and there exist nonnegative vector functions $(V_{11}(t), \dots, V_{ij}(t), \dots, V_{mn}(t))^T$ and $(W_{11}(t), \dots, W_{ij}(t), \dots, W_{mn}(t))^T \in PC([- \rho^+, 0], \mathbb{R}^{m \times n})$, where $V_{ij}(t)$ is continuous at $t \neq t_k$ ($k \in \mathbb{N}^*$), such that

$$\begin{aligned} D^-V_{ij}(t^-) &\leq -a_{ij}(t)V_{ij}(t^-) + a_{ij}(t) \int_{t-\rho(t^-)}^{t^-} W_{ij}(s)ds \\ &+ \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(t)| \times \left(L^g V_{kl}(t^-) \left(\frac{L}{1-\delta} \right) + M^g V_{ij}(t^-) \right) \\ &+ \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(t)| \times \left(L^h \bar{W}_{kl}(t^-) \left(\frac{L}{1-\delta} \right) + M^h V_{ij}(t^-) \right) \\ &+ \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(t)| \times \left(L^f \int_0^\infty |K_{ij}(u)| W_{kl}(t^- - u)du \left(\frac{L}{1-\delta} \right) \right. \\ &\left. + M^f \frac{K_{ij}^+}{\eta} V_{ij}(t^-) \right) \end{aligned} \quad (3)$$

$$\begin{aligned} W_{ij}(t^+) &\leq a_{ij}(t)V_{ij}(t^+) + a_{ij}(t) \int_{t^+-\rho(t^+)}^{t^+} W_{ij}(s)ds \\ &+ \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(t)| \times \left(L^g V_{kl}(t^+) \left(\frac{L}{1-\delta} \right) + M^g V_{ij}(t^+) \right) \\ &+ \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(t)| \times \left(L^h \bar{W}_{kl}(t^+) \left(\frac{L}{1-\delta} \right) + M^h V_{ij}(t^+) \right) \\ &+ \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(t)| \times \left(L^f \int_0^\infty |K_{ij}(u)| W_{kl}(t^+ - u)du \left(\frac{L}{1-\delta} \right) \right. \\ &\left. + M^f \frac{K_{ij}^+}{\eta} V_{ij}(t^+) \right), \end{aligned} \quad (4)$$

$$V_{ij}(t_w^+) \leq L_1 V_{ij}(t^+) \quad (5)$$

for $t > 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ and $w \in \mathbb{N}^*$. Then for all $t \geq 0$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, there exists a positive constant $\tilde{L}$ such that

$$V_{ij}(t) \leq \tilde{L} \sum_{l=1}^m \sum_{\ell=1}^n \max\{\bar{V}_{l\ell}(0), \bar{W}_{l\ell}(0)\} e^{-\lambda^* t}, \quad (6)$$

where λ^* is defined, respectively, as

$$\lambda^* = \min\{\lambda_{ij}^*, \hat{\lambda}_{ij}^* \mid i = 1, 2, \dots, m, j = 1, 2, \dots, n\}, \quad (7)$$

$$\begin{aligned} \lambda_{ij}^* = \inf_{t \geq 0} \{ & \lambda(t) > 0, \lambda(t) - a_{ij}(t) + \frac{q_{ij}}{p_{ij}} a_{ij}(t) \rho^+ e^{\lambda(t) \rho^+} \\ & + \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(t)| \times \left(L^g \frac{p_{kl}}{p_{ij}} \left(\frac{L}{1-\delta} \right) + M^g \right) \\ & + \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(t)| \times \left(L^h \frac{q_{kl}}{p_{ij}} \left(\frac{L}{1-\delta} \right) + M^h \right) e^{\lambda(t) \tau^+} \\ & + \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(t)| \times \left(L^f \frac{q_{kl}}{p_{ij}} \int_0^\infty |K_{ij}(u)| \right. \\ & \left. e^{\lambda(t) u} du \left(\frac{L}{1-\delta} \right) + M^f \frac{K_{ij}^+}{\eta} \right) = 0\} > 0, \end{aligned} \quad (8)$$

$$\begin{aligned} \hat{\lambda}_{ij}^* = \inf_{t \geq 0} \{ & \lambda(t) > 0, -q_{ij} + p_{ij} a_{ij}(t) + q_{ij} a_{ij}(t) \rho^+ e^{\lambda(t) \rho^+} \\ & + \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(t)| \times \left(L^g p_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij} M^g \right) \\ & + \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(t)| \times \left(L^h q_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij} M^h \right) e^{\lambda(t) \tau^+} \\ & + \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(t)| \times \left(L^f q_{kl} \int_0^\infty |K_{ij}(u)| e^{\lambda(t) u} du \right. \\ & \left. \left(\frac{L}{1-\delta} \right) + p_{ij} M^f \frac{K_{ij}^+}{\eta} \right) = 0\} > 0, \end{aligned} \quad (9)$$

$$\left(\frac{L}{1-\delta} \right) + p_{ij} M^f \frac{K_{ij}^+}{\eta} = 0\} > 0, \quad (10)$$

Proof See “Appendix 7”. $\square$

Theorem 4 Assume that (H1)–(H6) hold, then the unique piecewise differentiable pseudo almost periodic solution of system (1) is globally exponentially stable.

Proof See “Appendix 8”. $\square$

Illustrative example

In this section, we will give an example to illustrate the feasibility and effectiveness of our main results obtained in previous sections. Considering the following neutral impulsive shunting inhibitory cellular neural networks with time-varying coefficients and leakage delays, for $1 \leq i \leq 2$ and $1 \leq j \leq 2$,

$$\begin{aligned} \{x'_{ij}(t) = & -a_{ij}(t)x_{ij}(t - \rho(t)) \\ & - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t)g(x_{kl}(t))x_{ij}(t) \\ & - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(t)h(x'_{kl}(t - \tau(t)))x_{ij}(t) \\ & - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) \int_0^{+\infty} K_{ij}(u)f(x'_{kl}(t - u))du x_{ij}(t) \\ & + L_{ij}(t), t \in \mathbb{R}, t \neq 2w, w \in \mathbb{Z} \\ & \Delta x_{ij}(t_w) = x_{ij}(t_w^+) - x_{ij}(t_w^-) = I_w(x_i(t_w)), \end{aligned} \quad (11)$$

where $K_{ij}(t) = e^{-t} \Rightarrow K_{ij}^+ = 1, \eta = 1$,

$\rho(t) = \tau(t) = \frac{1}{80} |\sin t|$ and $r = q = p = 1$. For all $x \in \mathbb{R}$,

$$g(x) = h(x) = f(x) = \sin(x)$$

$$\Rightarrow L^g = L^h = L^f$$

$$= M^g = M^h = M^f = 1$$

$$\begin{aligned} & \begin{pmatrix} \Delta x_{11}(2w) \\ \Delta x_{12}(2w) \\ \Delta x_{21}(2w) \\ \Delta x_{22}(2w) \end{pmatrix} \\ & = \begin{pmatrix} -\frac{1}{80}x_{11}(2w) + \frac{1}{80}\sin(x_{11}(2w)) + \frac{1}{20} \\ -\frac{1}{80}x_{12}(2w) + \frac{1}{80}\cos(x_{12}(2w)) + \frac{1}{30} \\ -\frac{1}{80}x_{21}(2w) + \frac{1}{80}\sin(x_{21}(2w)) + \frac{1}{40} \\ -\frac{1}{80}x_{22}(2w) + \frac{1}{80}\cos(x_{22}(2w)) + \frac{1}{30} \end{pmatrix} \Rightarrow L = \frac{1}{80}. \end{aligned}$$

and

$$(a_{ij}(t))_{1 \leq i,j \leq 2} = (8 + \cos t)_{1 \leq i,j \leq 2}$$

$$\Rightarrow (a_{ij*})_{1 \leq i,j \leq 2} = (7)_{1 \leq i,j \leq 2},$$

$$(B_{ij}(t))_{1 \leq i,j \leq 2} = \begin{pmatrix} 0.02 \cos t & \frac{0.01}{1+t^2} \\ 0.01 \sin t & 0.01 \cos t + \frac{0.01}{1+t^2} \end{pmatrix}$$

$$\Rightarrow \left(\sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} \right)_{1 \leq i,j \leq 2} = \begin{pmatrix} 0.06 & 0.06 \\ 0.06 & 0.06 \end{pmatrix},$$

$$(D_{ij}(t))_{1 \leq i,j \leq 2} = \begin{pmatrix} 0.02 \sin(t) & 0.01 \cos t + \frac{0.01}{1+t^2} \\ 0.02 \sin(t) & 0.01 \cos t + \frac{0.01}{1+t^2} \end{pmatrix}$$

$$\Rightarrow \left(\sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} \right)_{1 \leq i,j \leq 2} = \begin{pmatrix} 0.08 & 0.08 \\ 0.08 & 0.08 \end{pmatrix},$$

$$\begin{aligned}
& (C_{ij}(t))_{1 \leq i, j \leq 2} \\
&= \begin{pmatrix} 0.01 \sin t + \frac{0.01}{1+t^2} & 0.01 \cos t + \frac{0.01}{1+t^2} \\ 0.01 \cos t + \frac{0.01}{1+t^2} & 0.01 \sin t \end{pmatrix} \\
&\Rightarrow \left(\sum_{C_{kl} \in N_p(i, j)} \bar{C}_{ij}^{kl} \right)_{1 \leq i, j \leq 2} = \begin{pmatrix} 0.07 & 0.07 \\ 0.07 & 0.07 \end{pmatrix}, \\
& (L_{ij}(t))_{1 \leq i, j \leq 2} \\
&= \begin{pmatrix} 0.05 \sin t + \frac{0.01}{1+t^2} & 0.06 \cos t + \frac{0.01}{1+t^2} \\ 0.07 \cos t + \frac{0.01}{1+t^2} & 0.02 \cos t \end{pmatrix} \\
&\Rightarrow (\bar{L}_{ij})_{1 \leq i, j \leq 2} = \begin{pmatrix} 0.06 & 0.07 \\ 0.08 & 0.02 \end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
\delta &= \max_{(i, j)} \max \left\{ a_{ij*}^{-1} \left[a_{ij}^+ \rho^+ + \sum_{C_{kl} \in N_q(i, j)} \bar{B}_{ij}^{kl} M^g \right. \right. \\
&\quad + \sum_{C_{kl} \in N_p(i, j)} \bar{D}_{ij}^{kl} M^h + \sum_{C_{kl} \in N_r(i, j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta} \left. \right] \\
&\quad + \frac{L_1}{1 - e^{-a_{ij*}}} \left(1 + \frac{a_{ij}^+}{a_{ij*}} \right) \left[a_{ij}^+ \rho^+ \right. \\
&\quad + \sum_{C_{kl} \in N_q(i, j)} \bar{B}_{ij}^{kl} M^g + \sum_{C_{kl} \in N_p(i, j)} \bar{D}_{ij}^{kl} M^h \\
&\quad + \sum_{C_{kl} \in N_r(i, j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta} \left. \right] + \frac{a_{ij}^+ L_1}{1 - e^{-a_{ij*}}} \left. \right\} \\
&= \max\{0.0586, 0.8497\} < 1, \\
q &= \max_{(i, j)} \max \left\{ a_{ij*}^{-1} \left[a_{ij}^+ \rho^+ \right. \right. \\
&\quad + \sum_{C_{kl} \in N_q(i, j)} \bar{B}_{ij}^{kl} \left(L^g \left(\frac{L}{1-\delta} \right) + M^g \right) \\
&\quad + \sum_{C_{kl} \in N_p(i, j)} \bar{D}_{ij}^{kl} \left(L^h \left(\frac{L}{1-\delta} \right) + M^h \right) \\
&\quad + \sum_{C_{kl} \in N_r(i, j)} \bar{C}_{ij}^{kl} \frac{K_{ij}^+}{\eta} \left(L^f \left(\frac{L}{1-\delta} \right) + M^f \right) \left. \right] \\
&\quad + \frac{L_1}{1 - e^{-a_{ij*}}} \left(1 + \frac{a_{ij}^+}{a_{ij*}} \right) \left[a_{ij}^+ \rho^+ \right. \\
&\quad + \sum_{C_{kl} \in N_q(i, j)} \bar{B}_{ij}^{kl} \left(L^g \left(\frac{L}{1-\delta} \right) + M^g \right) \\
&\quad + \sum_{C_{kl} \in N_p(i, j)} \bar{D}_{ij}^{kl} \left(L^h \left(\frac{L}{1-\delta} \right) + M^h \right) \\
&\quad + \sum_{C_{kl} \in N_r(i, j)} \bar{C}_{ij}^{kl} \frac{K_{ij}^+}{\eta} \left(L^f \left(\frac{L}{1-\delta} \right) + M^f \right) \left. \right] \\
&\quad + \frac{a_{ij}^+ L_1}{1 - e^{-a_{ij*}}} \left. \right\} \\
&= \max\{0.0646, 0.9456\} < 1.
\end{aligned}$$

Fig. 1 The orbits for the system

Fig. 2 The phase system for the system

Fig. 3 Transient response of state variables X_{11} , X_{12} , X_{21} and X_{22} for the system

Let $p_{11} = p_{12} = p_{21} = p_{22} = 1$ and $q_{11} = q_{12} = q_{21} = q_{22} = 17$, from the above assumption, the (H6) is satisfied. Therefore, all conditions from Theorems 3, 4 are satisfied, then the neutral impulsive shunting inhibitory cellular neural networks with time-varying coefficients and leakage delays (27) has a unique piecewise differentiable pseudo almost-periodic solution (Figs. 1, 2 and 3) in the region

$$\mathbb{B} = \{\varphi, \varphi' \in PAP_T(\mathbb{R}, \mathbb{R}^4, \rho), \|\varphi - \varphi_0\|_E \leq 1.0340\}.$$

Conclusion

In recent years impulsive neural networks have been intensively researched and some qualitative properties (oscillation, asymptotic behavior and stability) are investigated. So, in this paper sufficient conditions for existence, uniqueness and exponential stability of piecewise differentiable pseudo almost periodic solutions for neutral impulsive shunting inhibitory cellular neural networks with time-varying coefficients and leakage delays are obtained. The results are obtained by fixed point theorem, Lyapunov functional method and differential inequality techniques. We also provide numerical simulations to support the theoretical result.

Appendix 1: Proof of the Lemma 6

Proof We have $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in AP_T(\mathbb{R}, \mathbb{R})$ and $\varphi_2 \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Let

$$\begin{aligned} E(t) &= f(\varphi(t - \sigma(t))) \\ &= f(\varphi_1(t - \sigma(t)) + [f(\varphi_1(t - \sigma(t)) \\ &\quad + \varphi_2(t - \sigma(t))) - f(\varphi_1(t - \sigma(t))]) \\ &= E_1(t) + E_2(t). \end{aligned}$$

Firstly, it follows from Theorem 2.11 in Li and Wang (2011) that $E_1(\cdot) \in AP_T(\mathbb{R}, \mathbb{R})$. Then, we show that $E_2(t) \in PAP_T^0(\mathbb{R}, \mathbb{R})$ because

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |E_2(t)| dt &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\varphi_1(t - \sigma(t)) \\ &\quad + \varphi_2(t - \sigma(t))) - f(\varphi_1(t - \sigma(t)))| dt \\ &\leq \lim_{T \rightarrow \infty} \frac{L_f}{2T} \int_{-T}^T |\varphi_2(t - \sigma(t))| dt = 0. \end{aligned}$$

Set $\psi(t, z) = \varphi_2(t - z)$. By Theorem 5.3 in Zhang (2003, p. 58), Definition 5.7 in Zhang (2003, p. 59) and Theorem 3.1 in Liu and Zhang (2013), we can obtain that $\psi \in PAP_T^0(\mathbb{R} \times \Omega)$ and ψ is continuous in $z \in K$ and uniformly in $t \in \mathbb{R}$ for all compact subset K of Ω . This together with $\sigma \in AP_T(\mathbb{R}, \mathbb{R})$, Theorem 5.11 in Zhang (2003, p. 60) and Theorem 3.1 in Liu and Zhang (2013), implies that $\varphi_2(\cdot - \sigma(\cdot)) \in PAP_T^0(\mathbb{R}, \mathbb{R})$

So,

$$\lim_{T \rightarrow \infty} \frac{L_j^g}{2T} \int_{-T}^T |\varphi_2(t - \sigma(t))| dt = 0.$$

Thus $E_2(\cdot) \in PAP_T^0(\mathbb{R}, \mathbb{R})$. So, $E(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$. The proof is complete. $\square$

Appendix 2: Proof of the Theorem 2

Proof For $x_{kl}(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$ and $x'_{kl}(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$, it is not difficult to see that $f(x'_{kl}(\cdot)) \in PAP_T(\mathbb{R}, \mathbb{R})$ by Lemma 6. Let $f(x'_{kl}(\cdot)) = u_{kl}(\cdot) + v_{kl}(\cdot)$, where $u_{kl} \in AP_T(\mathbb{R}, \mathbb{R})$ and $v_{kl} \in PAP_T^0(\mathbb{R}, \mathbb{R})$, then

$$\begin{aligned} \phi_i(t) &= \int_{-\infty}^t K_{ij}(t-s)f(x'_{kl}(s))ds \\ &= \int_{-\infty}^t K_{ij}(t-s)u_{kl}(s)ds + \int_{-\infty}^t K_{ij}(t-s)v_{kl}(s)ds \\ &= \phi_{ij}^1(t) + \phi_{ij}^2(t). \end{aligned}$$

First, $\phi_{ij}^1(\cdot) \in AP_T(\mathbb{R}, \mathbb{R})$. It is not difficult to see that $\phi_{ij}^1(\cdot) \in UPC(\mathbb{R}, \mathbb{R})$.

Let $t_w < t \leq t_{w+1}$. For $\varepsilon > 0$, let Ω_ε be a relatively dense set of $\mathbb{R}$ formed by ε -periods of u_{kl} . For $\tau \in \Omega_\varepsilon$ and $0 < h < \min\{\varepsilon, \frac{\varepsilon}{2}\}$,

$$\begin{aligned} &|\phi_{ij}^1(t+\tau) - \phi_{ij}^1(t)| \\ &\leq \int_{-\infty}^t |K_{ij}(t-s)| |u_{kl}(s+\tau) - u_{kl}(s)| ds \\ &\leq \sum_{\ell=-\infty}^{w-1} \int_{t_\ell+h}^{t_{\ell+1}-h} |K_{ij}(t-s)| \times |u_{kl}(s+\tau) - u_{kl}(s)| ds \\ &\quad + \sum_{\ell=-\infty}^{w-1} \int_{t_\ell}^{t_{\ell+1}} |K_{ij}(t-s)| \times |u_{kl}(s+\tau) - u_{kl}(s)| ds \\ &\quad + \sum_{\ell=-\infty}^{w-1} \int_{t_{\ell+1}-h}^{t_{\ell+1}} |K_{ij}(t-s)| \times |u_{kl}(s+\tau) - u_{kl}(s)| ds \\ &\quad + \int_{t_w}^t |K_{ij}(t-s)| |u_{kl}(s+\tau) - u_{kl}(s)| ds. \end{aligned}$$

Since $u_{kl} \in AP_T(\mathbb{R}, \mathbb{R})$, one has $|u_{kl}(t+\tau) - u_{kl}(t)| \leq \varepsilon$, for all $t \in [t_\ell + h, t_{\ell+1} - h]$ and $\ell \in \mathbb{Z}$, $\ell \leq w$, then

$$\begin{aligned} &\sum_{\ell=-\infty}^{w-1} \int_{t_\ell+h}^{t_{\ell+1}-h} |K_{ij}(t-s)| |u_{kl}(s+\tau) - u_{kl}(s)| ds \\ &\leq \varepsilon \sum_{\ell=-\infty}^{w-1} \int_{t_\ell+h}^{t_{\ell+1}-h} |K_{ij}(t-s)| ds \\ &\leq \varepsilon K_{ij}^+ \sum_{\ell=-\infty}^{k-1} \int_{t_\ell+h}^{t_{\ell+1}-h} e^{-\eta(t-s)} ds \\ &\leq \frac{\varepsilon K_{ij}^+}{\eta} \sum_{\ell=-\infty}^{w-1} e^{-\eta(t-t_{\ell+1}+h)} \\ &\leq \frac{\varepsilon K_{ij}^+}{\eta} \sum_{\ell=-\infty}^{w-1} e^{-\eta\alpha(W-\ell-1)} \\ &\leq \frac{\varepsilon K_{ij}^+}{\eta(1-e^{-\eta\alpha})}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \sum_{\ell=-\infty}^{w-1} \int_{t_\ell}^{t_{\ell+h}} |K_{ij}(t-s)| |u_{kl}(s+\tau) - u_{kl}(s)| ds \\
 & \leq 2K_{ij}^+ |u_{kl}|_\infty \sum_{\ell=-\infty}^{w-1} \int_{t_\ell}^{t_{\ell+h}} e^{-\eta(t-s)} ds \\
 & \leq 2K_{ij}^+ |u_{kl}|_\infty \varepsilon e^{\eta h} \sum_{\ell=-\infty}^{w-1} e^{-\eta(t-t_\ell)} ds \\
 & \leq 2K_{ij}^+ |u_{kl}|_\infty \varepsilon e^{\eta h} e^{-\eta(t-t_w)} \sum_{\ell=-\infty}^{w-1} e^{-\eta\alpha(w-\ell)} ds \\
 & \leq \frac{2K_{ij}^+ |u_{kl}|_\infty \varepsilon e^{\frac{\eta\alpha}{2}}}{1 - e^{-\eta\alpha}}
 \end{aligned}$$

Similarly, one has

$$\begin{aligned}
 & \sum_{\ell=-\infty}^{w-1} \int_{t_{\ell+1}-h}^{t_{\ell+1}} |K_{ij}(t-s)| |u_{kl}(s+\tau) - u_{kl}(s)| ds \\
 & \leq M_1 \varepsilon, \\
 & \int_{t_w}^t |K_{ij}(t-s)| |u_{kl}(s+\tau) - u_{kl}(s)| ds \\
 & \leq M_2 \varepsilon,
 \end{aligned}$$

where M_1, M_2 are some positive constants. Hence, $\phi_{ij}^1(\cdot) \in AP_T(\mathbb{R}, \mathbb{R})$.

Second $\phi_{ij}^2(\cdot) \in PAP_T^0(\mathbb{R}, \mathbb{R})$. In fact, for $r > 0$, one has

$$\begin{aligned}
 & \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r |\phi_{ij}^2(t)| dt \\
 & = \lim_{r \rightarrow \infty} \frac{1}{2r} \times \int_{-r}^r | \int_{-\infty}^t K_{ij}(t-s) v_{kl}(s) ds | dt \\
 & = \lim_{r \rightarrow \infty} \frac{1}{2r} \times \int_{-r}^r | \int_0^\infty K_{ij}(s) v_{kl}(t-s) ds | dt \\
 & \leq \lim_{r \rightarrow \infty} \frac{1}{2r} \times \int_{-r}^r \left(\int_0^\infty K_{ij}^+ e^{-\eta s} |v_{kl}(t-s)| ds \right) dt \\
 & \leq \int_0^\infty K_{ij}^+ e^{-\eta s} \times \left(\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r |v_{kl}(t-s)| dt \right) ds.
 \end{aligned}$$

Since $v_{kl}(t) \in PAP_T^0(\mathbb{R}, \mathbb{R})$, it follows that $v_{kl}(\cdot - s) \in PAP_T^0(\mathbb{R}, \mathbb{R})$ for each $s \in \mathbb{R}$ by Lemma 4. Using the Lebesgue's dominated convergence theorem, we have $\phi_{ij}^2(t) \in PAP_T^0(\mathbb{R}, \mathbb{R})$. This completes the proof. $\square$

Appendix 3: Proof of the Lemma 7

Proof First, note that, for all $1 \leq i \leq m$, $1 \leq j \leq n$ the function

$$\begin{aligned}
 F_{ij} : s \mapsto & a_{ij}(s) \int_{s-\rho(s)}^s \phi'_{ij}(s) ds \\
 & - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) g(\phi_{kl}(s)) \phi_{ij}(s) \\
 & - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(s) h(\phi'_{kl}(s - \tau(s))) \phi_{ij}(s) \\
 & - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \int_0^{+\infty} K_{ij}(u) f(\phi'_{kl}(s-u)) du \phi_{ij}(s) \\
 & + L_{ij}(s)
 \end{aligned}$$

is in $PAP_T(\mathbb{R}, \mathbb{R})$, by using Lemmas 4, 5, 6 and Theorem 2.

Consequently, for all $1 \leq i \leq m$, $1 \leq j \leq n$, F_{ij} can be expressed as

$$F_{ij} = F_{ij}^1 + F_{ij}^2,$$

where $F_{ij}^1 \in AP_T(\mathbb{R}, \mathbb{R})$ and $F_{ij}^2 \in PAP_T^0(\mathbb{R}, \mathbb{R})$. So

$$\begin{aligned}
 (X_\phi)_{ij}(t) &= \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} F_{ij}^1(s) ds \\
 &+ \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} F_{ij}^2(s) ds \\
 &= G_{ij}^1(t) + G_{ij}^2(t).
 \end{aligned}$$

(i) Let $t', t'' \in (t_w, t_{w+1})$, $k \in \mathbb{Z}$, $t'' < t'$, then

$$\begin{aligned}
 & |G_{ij}^1(t') - G_{ij}^1(t'')| \\
 &= \left| \int_{-\infty}^{t'} e^{-\int_s^{t'} a_{ij}(u) du} F_{ij}^1(s) ds - \int_{-\infty}^{t''} e^{-\int_s^{t''} a_{ij}(u) du} F_{ij}^1(s) ds \right| \\
 &\leq \left| \int_{-\infty}^{t''} [e^{-\int_s^{t'} a_{ij}(u) du} - e^{-\int_s^{t''} a_{ij}(u) du}] F_{ij}^1(s) ds \right| \\
 &+ \left| \int_{t''}^{t'} e^{-\int_s^{t'} a_{ij}(u) du} F_{ij}^1(s) ds \right| \\
 &\leq |e^{-\int_{t''}^{t'} a_{ij}(u) du} - 1| \times \int_{-\infty}^{t''} e^{-\int_s^{t''} a_{ij}(u) du} |F_{ij}^1(s)| ds \\
 &+ \int_{t''}^{t'} e^{-\int_s^{t'} a_{ij}(u) du} |F_{ij}^1(s)| ds \\
 &\leq ((t' - t'') a_{ij}^+) \int_{-\infty}^{t''} e^{-(t''-s)a_{ij}^+} |F_{ij}^1(s)| ds \\
 &+ \int_{t''}^{t'} e^{-(t'-s)a_{ij}^+} |F_{ij}^1(s)| ds,
 \end{aligned}$$

it is easy to see that for any $\varepsilon > 0$, there exists

$$0 < \delta < \min \left\{ \frac{a_{ij}^+ \varepsilon}{2a_{ij}^+ |F_{ij}^1|_\infty}, \frac{\varepsilon}{2|F_{ij}^1|_\infty} \right\}$$

and for a suitable t', t'' satisfying $0 < t' - t'' < \delta$ one has $|G_{ij}^1(t') - G_{ij}^1(t'')|$

$$\begin{aligned} &\leq [(t' - t'')a_{ij}^+ \int_{-\infty}^{t''} e^{-(t''-s)a_{ij}^*} ds \\ &\quad + \int_{t''}^{t'} e^{-(t'-s)a_{ij}^*} ds] |F_{ij}^1|_{\infty} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which implies that $G_{ij}^1(\cdot) \in UPC(\mathbb{R}, \mathbb{R})$.

(ii) Since $F_{ij}^1 \in AP_T(\mathbb{R}, \mathbb{R})$, for $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that for $\tau \in \Omega_\varepsilon$, $t \in \mathbb{R}$, $|t - t_w| > \varepsilon$, $w \in \mathbb{Z}$, then

$$\begin{aligned} G_{ij}^1(t + \tau) - G_{ij}^1(t) &= \int_{-\infty}^{t+\tau} e^{-\int_s^{t+\tau} a_{ij}(u) du} F_{ij}^1(s) ds - \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} F_{ij}^1(s) ds \\ &= \int_{-\infty}^{t+\tau} e^{-\int_{s-\tau}^s a_{ij}(m+\tau) dm} F_{ij}^1(s) ds - \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} F_{ij}^1(s) ds \\ &= \int_{-\infty}^t e^{-\int_s^t a_{ij}(m+\tau) dm} F_{ij}^1(s + \tau) ds - \int_{-\infty}^t e^{-\int_s^t a_{ij}(m+\tau) dm} F_{ij}^1(s) ds \\ &\quad + \int_{-\infty}^t e^{-\int_s^t a_{ij}(m+\tau) dm} F_{ij}^1(s) ds - \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} F_{ij}^1(s) ds \\ &= \int_{-\infty}^t e^{-\int_s^t a_{ij}(u+\tau) du} (F_{ij}^1(s + \tau) - F_{ij}^1(s)) ds \\ &\quad + \int_{-\infty}^t (e^{-\int_s^t a_{ij}(u+\tau) du} - e^{-\int_s^t a_{ij}(u) du}) F_{ij}^1(s) ds \end{aligned}$$

So there exists $\theta \in]0, 1[$ such that

$$\begin{aligned} |G_{ij}^1(t + \tau) - G_{ij}^1(t)| &\leq |F_{ij}^1|_{\infty} \int_{-\infty}^t \left(e^{-\int_s^t a_{ij}(u+\tau) du} - e^{-\int_s^t a_{ij}(u) du} \right) ds \\ &\quad + \int_{-\infty}^t e^{-\int_s^t a_{ij}(u+\tau) du} |F_{ij}^1(s + \tau) - F_{ij}^1(s)| ds \\ &\leq \int_{-\infty}^t \left(e^{-\left[\int_s^t a_{ij}(u+\tau) du + \theta \left(\int_s^t a_{ij}(u) du - \int_s^t a_{ij}(u+\tau) du \right) \right]} \right. \\ &\quad \times \int_s^t |a_{ij}(u) - a_{ij}(u + \tau)| du \Big) ds |F_{ij}^1|_{\infty} \\ &\quad + \int_{-\infty}^t e^{-\int_s^t a_{ij}(u+\tau) du} |F_{ij}^1(s + \tau) - F_{ij}^1(s)| ds \\ &\leq \int_{-\infty}^t \left(e^{-\left[\int_s^t a_{ij}(u+\tau) du + \theta \left(\int_s^t a_{ij}(u) du - \int_s^t a_{ij}(u+\tau) du \right) \right]} \right. \end{aligned}$$

$$\begin{aligned} &\times \int_s^t |a_{ij}(u) - a_{ij}(u + \tau)| du \Big) ds |F_{ij}^1|_{\infty} \\ &\quad + \int_{-\infty}^t e^{-a_{ij}^*(t-s)} |F_{ij}^1(s + \tau) - F_{ij}^1(s)| ds \\ &\leq \int_{-\infty}^t \left\{ e^{-a_{ij}^*(t-s)} e^{-\theta \left(\int_s^t |a_{ij}(u) - a_{ij}(u + \tau)| du \right)} \right. \\ &\quad \times \int_s^t |a_{ij}(u) - a_{ij}(u + \tau)| du \Big\} ds |F_{ij}^1|_{\infty} \\ &\quad + \int_{-\infty}^t e^{-a_{ij}^*(t-s)} |F_{ij}^1(s + \tau) - F_{ij}^1(s)| ds \\ &\leq |F_{ij}^1|_{\infty} \int_{-\infty}^t \left\{ e^{-a_{ij}^*(t-s)} \times \int_s^t |a_{ij}(u) - a_{ij}(u + \tau)| du \right\} ds \\ &\quad + \int_{-\infty}^t e^{-a_{ij}^*(t-s)} |F_{ij}^1(s + \tau) - F_{ij}^1(s)| ds \\ &= \int_{-\infty}^t \Phi_{ij}(t, s) ds + \int_{-\infty}^t \Psi_{ij}(t, s) ds \end{aligned}$$

where

$$\Phi_i(t, s) = e^{-a_{ij}^*(t-s)} |F_{ij}^1|_{\infty} \int_s^t |a_{ij}(u) - a_{ij}(u + \tau)| du$$

and

$$\Psi_i(t, s) = e^{-a_{ij}^*(t-s)} |F_{ij}^1(s + \tau) - F_{ij}^1(s)|,$$

we obtain immediately that, $G_{ij}^1 \in AP_T(\mathbb{R}, \mathbb{R})$.

Now, we turn our attention to G_{ij}^2 . So

$$\begin{aligned} &\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r |G_{ij}^2(t)| dt \\ &\leq \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \int_{-\infty}^t e^{-(t-s)a_{ij}^*} |F_{ij}^2(s)| ds dt \\ &\leq I_1 + I_2, \end{aligned}$$

where

$$I_1 = \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left(\int_{-r}^t e^{-(t-s)a_{ij}^*} |F_{ij}^2(s)| ds \right) dt$$

and

$$I_2 = \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left(\int_{-\infty}^{-r} e^{-(t-s)a_{ij}^*} |F_{ij}^2(s)| ds \right) dt.$$

Pose $m = t - s$, then by Fubini's Theorem one has

$$\begin{aligned}
I_1 &= \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left(\int_0^{t+r} e^{-ma_{ij^*}} |F_{ij}^2(t-m)| dm \right) dt \\
&\leq \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left(\int_0^{+\infty} e^{-ma_{ij^*}} |F_{ij}^2(t-m)| dm \right) dt \\
&\leq \int_0^{+\infty} e^{-ma_{ij^*}} \left(\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r |F_{ij}^2(t-m)| dt \right) dm \\
&= \int_0^{+\infty} e^{-ua_{ij^*}} \left(\frac{1}{2r} \int_{-r-u}^{r-u} |F_{ij}^2(v)| dv \right) du \\
&\leq \int_0^{+\infty} e^{-ua_{ij^*}} \left(\frac{r+u}{r} \frac{1}{2(r+u)} \int_{-r-u}^{r+u} |F_{ij}^2(v)| dv \right) du
\end{aligned}$$

Since the function $F_{ij}^2(\cdot) \in PAP_T^0(\mathbb{R}, \mathbb{R})$, the function Ω_r defined by

$$\Omega_r(u) = \frac{1}{2(r+u)} \int_{-r-u}^{r+u} |F_{ij}^2(v)| dv$$

is bounded and satisfies $\lim_{r \rightarrow \infty} \Omega_r(u) = 0$. So, by the Lebesgue dominated convergence theorem, we obtain

$$I_1 = \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left(\int_{-r}^t e^{-(t-s)a_{ij^*}} |F_{ij}^2(s)| ds \right) dt = 0.$$

On the other hand, notice that $|F_{ij}^2|_\infty = \sup_{t \in \mathbb{R}} |F_{ij}^2(t)| < \infty$ then

$$\begin{aligned}
I_2 &= \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left(\int_{-\infty}^{-r} e^{-(t-s)a_{ij^*}} |F_{ij}^2(s)| ds \right) dt \\
&\leq \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-\infty}^{-r} e^{sa_{ij^*}} |F_{ij}^2(s)| ds \int_{-r}^r e^{-ta_{ij^*}} dt \\
&= \lim_{r \rightarrow \infty} \frac{|F_{ij}^2|_\infty}{2ra_{ij^*}} \int_{-r}^r e^{-(r+t)a_{ij^*}} dt \\
&= 0,
\end{aligned}$$

then

$$\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left| \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} F_{ij}^2(s) ds \right| dt = 0.$$

Consequently, the function $G_{ij}^2(\cdot)$ belongs to $PAP_T^0(\mathbb{R}, \mathbb{R})$. So, (X_φ) belongs to $PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$. This completes the proof. $\square$

Appendix 4: Proof of the Lemma 8

Proof We will show that:

$$\sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_w(\varphi_{ij}(t_w)) \in PAP_T(\mathbb{R}, \mathbb{R}).$$

It is not difficult to see that $\sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_w(\varphi_{ij}(t_w)) \in UPC(\mathbb{R}, \mathbb{R})$.

By Corollary 2.1 (see Xia 2016), $I_w(\varphi_{ij}(t_w)) \in PAP(\mathbb{Z}, \mathbb{R})$, then let $I_k(\varphi_{ij}(t_w)) = I_w^1 + I_w^2$ where $I_w^1 \in AP(\mathbb{Z}, \mathbb{R})$ and $I_w^2 \in PAP_0(\mathbb{Z}, \mathbb{R})$, so

$$\begin{aligned}
&\sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_k(\varphi_{ij}(t_w)) \\
&= \sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_w^1 + \sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_w^2 \\
&= \Phi_1(t) + \Phi_2(t).
\end{aligned}$$

Since $\{t_j^w\}, w, j \in \mathbb{Z}$ are equipotentially almost periodic, then (see Lemma 3.2, Xia 2016), for any $\varepsilon > 0$, there exists relative dense sets of real numbers Ω_ε and integers Q_ε , such that for $t_w < t \leq t_{w+1}$, $\tau \in \Omega_\varepsilon$, $q \in Q_\varepsilon$, $|t - t_w| > \varepsilon$, $|t - t_{w+1}| > \varepsilon$, $w \in \mathbb{Z}$, one has

$$\begin{aligned}
t + \tau &> t_w + \varepsilon + \tau > t_{w+q}, \\
t_{w+q+1} &> t_{w+1} - \varepsilon + \tau > t + \tau,
\end{aligned}$$

that is $t_{w+q} > t + \tau > t_{w+q+1}$; then

$$\begin{aligned}
&\|\Phi_1(t + \tau) - \Phi_1(t)\| \\
&= \left\| \sum_{t_w < t + \tau} e^{-\int_{t_w}^{t+\tau} a_{ij}(u) du} I_w^1 - \sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_w^1 \right\| \\
&\leq \sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} \|I_{w+q}^1 - I_w^1\| \\
&\leq \varepsilon \sum_{t_w < t} e^{-(t-t_w)a_{ij^*}} \\
&\leq \varepsilon \frac{1}{1 - e^{-a_{ij^*}}},
\end{aligned}$$

So, $\Phi_1(\cdot) \in AP_T(\mathbb{R}, \mathbb{R})$.

Next, we show that $\Phi_2(\cdot) \in PAP_T^0(\mathbb{R}, \mathbb{R})$. For a given $w \in \mathbb{Z}$, define the function $G(t)$ by

$$G(t) = e^{-\int_{t_w}^t a_{ij}(u) du} I_w^2, \quad t_w < t \leq t_{w+1},$$

then

$$\begin{aligned}
\lim_{t \rightarrow \infty} \|G(t)\| &= \lim_{t \rightarrow \infty} \|e^{-\int_{t_w}^t a_{ij}(u) du} I_w^2\| \\
&\leq \lim_{t \rightarrow \infty} e^{-(t-t_w)a_{ij^*}} \sup_{w \in \mathbb{Z}} \|I_w^2\| = 0,
\end{aligned}$$

then $G \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Define $G_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$G_n(t) = e^{-\int_{t_{w-n}}^t a_{ij}(u) du} I_{w-n}^2, \quad t_w < t \leq t_{w+1}, \quad n \in \mathbb{N}.$$

So $G_n \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Moreover,

$$\begin{aligned}
\|G_n(t)\| &= \|e^{-\int_{t_{w-n}}^t a_{ij}(u) du} I_{w-n}^2\| \\
&\leq e^{-(t-t_{w-n})a_{ij^*}} \sup_{w \in \mathbb{Z}} \|I_w^2\| \\
&\leq e^{-(t-t_w)a_{ij^*}} e^{-a_{ij^*}zw} \sup_{w \in \mathbb{Z}} \|I_w^2\|.
\end{aligned}$$

therefore, the series $\sum_{n=1}^{\infty} G_n$ is uniformly convergent on $\mathbb{R}$.

By Lemma 2.2 (see Xia 2016), one has

$$\Phi_2(t) = \sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_w^2 = \sum_{n=0}^{\infty} G_n \in PAP_T^0(\mathbb{R}, \mathbb{R}).$$

So, $\sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_w(\varphi_{ij}(t_w)) \in PAP_T(\mathbb{R}, \mathbb{R})$. This This completes the proof. $\square$

Appendix 5: Proof of the Theorem 3

Proof One has

$$\begin{aligned} \|\varphi_0\|_E &= \max_{(i,j)} \max_{t \in \mathbb{R}} \sup \left\{ \left| \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} L_{ij}(s) ds \right|, \right. \\ &\quad \left. \left| L_{ij}(t) - a_{ij}(t) \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} L_{ij}(s) ds \right| \right\} \\ &\leq \max_{(i,j)} \max \left\{ \frac{\bar{L}_{ij}}{a_{ij}^+}, \left(1 + \frac{a_{ij}^+}{a_{ij}^*} \right) \bar{L}_{ij} \right\} = L. \end{aligned}$$

After

$$\begin{aligned} \|\varphi\|_E &\leq \|\varphi - \varphi_0\|_E + \|\varphi_0\|_E \\ &\leq \|\varphi - \varphi_0\|_E + L. \end{aligned}$$

Set

$$\begin{aligned} \mathbb{B} &= \mathbb{B}(\varphi_0, \delta) = \{\varphi, \varphi' \in PAP_T(\mathbb{R}, \mathbb{R}^{m \times n}), \|\varphi - \varphi_0\|_E \\ &\quad \leq \frac{\delta L}{(1 - \delta)}\} \end{aligned}$$

. Clearly, $\mathbb{B}$ is a closed convex subset of $PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$ and, therefore, for any $\varphi \in \mathbb{B}$ by using the estimate just obtained Lemmas 2, 3, 7 and 8 we see that

$$\begin{aligned} \|\Gamma\varphi - \varphi_0\|_{\infty} &= \sup_{t \in \mathbb{R}} \max_{(i,j)} \left\{ \left| \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} F_{ij}(s) ds - \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} \right. \right. \\ &\quad \left. \left. L_{ij}(s) ds + \sum_{t_w < t} e^{-\int_{t_w}^t a_{ij}(u) du} I_w(\varphi_{ij}(t_w)) \right| \right\} \\ &\leq \max_{(i,j)} \sup_{t \in \mathbb{R}} \left\{ \int_{-\infty}^t e^{-(t-s)a_{ij}^*} \times |a_{ij}(s)| \int_{s-\rho(s)}^s \varphi'_{ij}(s) ds \right. \\ &\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) g(\varphi_{kl}(s)) \varphi_{ij}(s) - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(s) \\ &\quad \left. h(\varphi'_{kl}(s - \tau(s))) \varphi_{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \right. \\ &\quad \left. \times \int_0^{\infty} K_{ij}(u) f(\varphi'_{kl}(s - u)) du \varphi_{ij}(s) \right| ds \\ &\quad \left. + \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} |I_w(\varphi_{ij}(t_w))| \right\} \\ &\leq \max_{(i,j)} \sup_{t \in \mathbb{R}} \left\{ \int_{-\infty}^t e^{-(t-s)a_{ij}^*} [a_{ij}^+ \rho^+ + \varphi'_{ij}(s)] \right. \end{aligned}$$

$$\begin{aligned} &+ \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} M^g |\varphi_{ij}(s)| + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} M^h |\varphi_{ij}(s)| \\ &+ \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta} |\varphi_{ij}(s)| \Big] ds \\ &+ \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} L_1 |\varphi_{ij}(t_w)| \Big\} \\ &\leq \max_{(i,j)} \left\{ a_{ij}^{-1} \left[a_{ij}^+ \rho^+ + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} M^g \right. \right. \\ &\quad \left. + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} M^h + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta} \right] \\ &\quad \left. + \frac{L_1}{1 - e^{-a_{ij}^*}} \right\} \|\varphi\|_E \end{aligned}$$

On the other hand

$$\begin{aligned} &\|(\Gamma\varphi - \varphi_0)'\|_{\infty} \\ &\leq \max_{(i,j)} \sup_{t \in \mathbb{R}} \left\{ |a_{ij}(t)| \int_{t-\rho(t)}^t \varphi'_{ij}(u) du \right. \\ &\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) g(\varphi_{kl}(t)) \varphi_{ij}(t) \\ &\quad - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(t) h(\varphi'_{kl}(t - \tau(t))) \varphi_{ij}(t) \\ &\quad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) \times \int_0^{\infty} K_{ij}(u) f(\varphi'_{kl}(t - u)) du \varphi_{ij}(t) \Big| \\ &\quad + a_{ij}^+ \int_{-\infty}^t e^{-(t-s)a_{ij}^*} \times |a_{ij}(s)| \int_{s-\rho(s)}^s \varphi'_{ij}(u) du \\ &\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) g(\varphi_{kl}(s)) \varphi_{ij}(s) \\ &\quad - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(s) h(\varphi'_{kl}(s - \tau(s))) \varphi_{ij}(s) \\ &\quad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \times \int_0^{\infty} K_{ij}(u) f(\varphi'_{kl}(s - u)) du \varphi_{ij}(s) \Big| ds \\ &\quad \left. + a_{ij}^+ \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} |I_w(\varphi_{ij}(t_w))| \right\} \\ &\leq \max_{(i,j)} \sup_{t \in \mathbb{R}} \left\{ \left[a_{ij}^+ \rho^+ + \varphi'_{ij}(s) \right] + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} M^g |\varphi_{ij}(s)| \right. \\ &\quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} M^h |\varphi_{ij}(s)| + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta} |\varphi_{ij}(s)| \Big] \\ &\quad \left. + a_{ij}^+ \int_{-\infty}^t e^{-(t-s)a_{ij}^*} [a_{ij}^+ \rho^+ + \varphi'_{ij}(s)] + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} M^g |\varphi_{ij}(s)| \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} M^h | \varphi_{ij}(s) | + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta} | \varphi_{ij}(s) |] ds \\
& + a_{ij}^+ \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} L_1 | \varphi_{ij}(t_w) | \Big\} \\
& \leq \max_{(i,j)} \left\{ \left(1 + \frac{a_{ij}^+}{a_{ij}^*} \right) [a_{ij}^+ \rho^+ + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} M^g \right. \right. \\
& \quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} M^h + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta}] \\
& \quad \left. + \frac{a_{ij}^+ L_1}{1 - e^{-a_{ij}^*}} \right\} \| \varphi \|_E
\end{aligned}$$

where

$$\| \Gamma \varphi - \varphi_0 \|_E \leq \delta \| \varphi \|_E$$

and

$$\begin{aligned}
\delta = \max_{(i,j)} \max \left\{ a_{ij}^{-1} [a_{ij}^+ \rho^+ + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} M^g \right. \\
+ \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} M^h + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta}] \\
+ \frac{L_1}{1 - e^{-a_{ij}^*}} \left(1 + \frac{a_{ij}^+}{a_{ij}^*} \right) [a_{ij}^+ \rho^+ \\
+ \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} M^g + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} M^h \\
+ \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} M^f \frac{K_{ij}^+}{\eta}] + \frac{a_{ij}^+ L_1}{1 - e^{-a_{ij}^*}} \Big\} < 1,
\end{aligned}$$

it implies that $\Gamma \in \mathbb{B}$. So, the mapping Γ is a self-mapping from $\mathbb{B}$ to $\mathbb{B}$. Next, we prove that the mapping Γ is a contraction mapping of the $\mathbb{B}$. In fact, in view of (H1), for $\forall \phi, \psi \in \mathbb{B}$, we have

$$\begin{aligned}
& | (\Gamma \phi(t) - \Gamma \psi(t))_{ij} | \\
& \leq \int_{-\infty}^t e^{-(t-s)a_{ij}^*} \times | a_{ij}(s) \int_{s-\rho(s)}^s (\phi'_{ij}(m) - \psi'_{ij}(m)) dm \\
& \quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) (g(\phi_{kl}(s)) \phi_{ij}(s) - g(\psi_{kl}(s)) \psi_{ij}(s)) \\
& \quad - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(s) (h(\phi'_{kl}(s - \tau(s))) \phi_{ij}(s) - \\
& \quad h(\psi'_{kl}(s - \tau(s))) \psi_{ij}(s)) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \\
& \quad \times \left(\int_0^\infty K_{ij}(u) f(\phi'_{kl}(s - u)) du \phi_{ij}(s) \right.
\end{aligned}$$

$$\begin{aligned}
& \left. - \int_0^\infty K_{ij}(u) f(\psi'_{kl}(s - u)) \psi_{ij}(s) \right) | ds \\
& + \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} | I_w(\phi_{ij}(t_w)) - I_w(\psi_{ij}(t_w)) | \\
& \leq \int_{-\infty}^t e^{-(t-s)a_{ij}^*} \times \left[a_{ij}^+ \int_{s-\rho(s)}^s | \phi'_{ij}(m) - \psi'_{ij}(m) | dm \right. \\
& \quad + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} (| g(\phi_{kl}(s)) - g(\psi_{kl}(s)) | | \phi_{ij}(s) | \\
& \quad + | g(\psi_{kl}(s)) | | \psi_{ij}(s) - \phi_{ij}(s) |) \\
& \quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} (| h(\phi'_{kl}(s - \tau(s))) - h(\psi'_{kl}(s - \tau(s))) | \\
& \quad \times | \phi_{ij}(s) | + | h(\psi'_{kl}(s - \tau(s))) | | \psi_{ij}(s) - \phi_{ij}(s) |) \\
& \quad + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} \left(\left| \int_0^\infty K_{ij}(u) (f(\phi_{kl}(s - u)) \right. \right. \\
& \quad \left. \left. - f(\psi_{kl}(s - u))) du \right| | \phi_{ij}(s) | \right. \\
& \quad \left. + \left| \int_0^\infty K_{ij}(u) f(\psi'_{kl}(s - u)) du \right| \times | \psi_{ij}(s) - \phi_{ij}(s) | \right) \Big] ds \\
& + \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} | I_w(\phi_{ij}(t_w)) - I_w(\psi_{ij}(t_w)) | \\
& \leq \int_{-\infty}^t e^{-(t-s)a_{ij}^*} \left[a_{ij}^+ \rho^+ \| \phi'_{ij} - \psi'_{ij} \|_\infty \right. \\
& \quad + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} \left(L^g \| \phi - \psi \|_\infty \left(\frac{L}{1-\delta} \right) + M^g \| \phi - \psi \|_\infty \right) \\
& \quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} \left(L^h \| \phi' - \psi' \|_\infty \left(\frac{L}{1-\delta} \right) + M^h \| \phi - \psi \|_\infty \right) \\
& \quad + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} \left(L^f K_{ij}^+ \| \phi' - \psi' \|_\infty \left(\frac{L}{1-\delta} \right) \right. \\
& \quad \left. + M^f K_{ij}^+ \| \phi - \psi \|_\infty \right) \Big] ds + \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} L_1 \| \phi - \psi \|_\infty
\end{aligned}$$

$$\begin{aligned}
& \leq \left\{ a_{ij}^{-1} \left[a_{ij}^+ \rho^+ + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} \left(L^g \left(\frac{L}{1-\delta} \right) + M^g \right) \right. \right. \\
& \quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} \left(L^h \left(\frac{L}{1-\delta} \right) + M^h \right) + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} K_{ij}^+ \\
& \quad \times \left(L^f \left(\frac{L}{1-\delta} \right) + M^f \right) \Big] + \frac{L}{1 - e^{-a_{ij}^*}} \Big\} \| \phi - \psi \|_E
\end{aligned}$$

where $i = 1, 2, \dots, m, j = 1, 2, \dots, n$ it follows that

On the other hand

$$| (\Gamma \phi(t) - \Gamma \psi(t))_{ij} |$$

$$\begin{aligned}
&\leq |a_{ij}(t) \int_{t-\rho(t)}^t (\phi'_{ij}(m) - \psi'_{ij}(m)) dm \\
&\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) (g(\phi_{kl}(t)) \phi_{ij}(t) \\
&\quad - g(\psi_{kl}(t)) \psi_{ij}(t)) - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(t) (h(\phi'_{kl}(t - \tau(t))) \\
&\quad \phi_{ij}(t) - h(\psi'_{kl}(t - \tau(t))) \psi_{ij}(t)) \\
&\quad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) \times \left(\int_0^\infty K_{ij}(u) f(\phi'_{kl}(t - u)) du \phi_{ij}(s) \right. \\
&\quad \left. - \int_0^\infty K_{ij}(u) f(\psi'_{kl}(t - u)) \psi_{ij}(s) \right) | \\
&\quad + a_{ij}^+ \int_{-\infty}^t e^{-(t-s)a_{ij}^*} \times |a_{ij}(s) \int_{s-\rho(s)}^s (\phi'_{ij}(m) - \psi'_{ij}(m)) dm \\
&\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) (g(\phi_{kl}(s)) \phi_{ij}(s) - g(\psi_{kl}(s)) \psi_{ij}(s)) \\
&\quad - \sum_{C_{kl} \in N_p(i,j)} D_{ij}^{kl}(s) (h(\phi'_{kl}(s - \tau(s))) \phi_{ij}(s) \\
&\quad - h(\psi'_{kl}(s - \tau(s))) \psi_{ij}(s)) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) \\
&\quad \times \left(\int_0^\infty K_{ij}(u) f(\phi'_{kl}(s - u)) du \phi_{ij}(s) \right. \\
&\quad \left. - \int_0^\infty K_{ij}(u) f(\psi'_{kl}(s - u)) \psi_{ij}(s) \right) | ds \\
&\quad + a_{ij}^+ \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} \times |I_w(\phi_{ij}(t_w)) - I_w(\psi_{ij}(t_w))| \\
&\leq \left[a_{ij}^+ \rho^+ \|\phi'_{ij} - \psi'_{ij}\|_\infty \right. \\
&\quad + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} (L^g \|\phi - \psi\|_\infty \left(\frac{L}{1-\delta} \right) \\
&\quad + M^g \|\phi - \psi\|_\infty) \\
&\quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} \left(L^h \|\phi' - \psi'\|_\infty \left(\frac{L}{1-\delta} \right) \right. \\
&\quad \left. + M^h \|\phi - \psi\|_\infty \right) \\
&\quad + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} \left(L^f \frac{K_{ij}^+}{\eta} \|\phi' - \psi'\|_\infty \left(\frac{L}{1-\delta} \right) \right. \\
&\quad \left. + M^f \frac{K_{ij}^+}{\eta} \|\phi - \psi\|_\infty \right) \Big] \\
&\quad + a_{ij}^+ \int_{-\infty}^t e^{-(t-s)a_{ij}^*} \left[a_{ij}^+ \rho^+ \|\phi'_{ij} - \psi'_{ij}\|_\infty \right. \\
&\quad + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} (L^g \|\phi - \psi\|_\infty \left(\frac{L}{1-\delta} \right) \\
&\quad + M^g \|\phi - \psi\|_\infty) \\
&\quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} (L^h \|\phi' - \psi'\|_\infty \left(\frac{L}{1-\delta} \right) \\
&\quad + M^h \|\phi - \psi\|_\infty) \\
&\quad + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} \frac{K_{ij}^+}{\eta} \left(L^f \|\phi' - \psi'\|_\infty \left(\frac{L}{1-\delta} \right) \right. \\
&\quad \left. + M^f \|\phi - \psi\|_\infty \right) \Big] ds \\
&\quad + a_{ij}^+ \sum_{t_w < t} e^{-(t-t_w)a_{ij}^*} L_1 \|\phi - \psi\|_\infty
\end{aligned}$$

$$\begin{aligned}
&\leq \left\{ \left(1 + \frac{a_{ij}^+}{a_{ij}^*} \right) \left[a_{ij}^+ \rho^+ \right. \right. \\
&\quad + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} \left(L^g \left(\frac{L}{1-\delta} \right) + M^g \right) \\
&\quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} \left(L^h \left(\frac{L}{1-\delta} \right) + M^h \right) \\
&\quad \left. + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} \frac{K_{ij}^+}{\eta} \left(L^f \left(\frac{L}{1-\delta} \right) + M^f \right) \right] \\
&\quad \left. + \frac{a_{ij}^+ L_1}{1 - e^{-a_{ij}^*}} \right\} \|\phi - \psi\|_E
\end{aligned}$$

so

$$\|\Gamma\phi - \Gamma\psi\|_E \leq q \|\phi - \psi\|_E,$$

with

$$\begin{aligned}
q = \max_{(i,j)} \max \left\{ a_{ij}^{-1} \left[a_{ij}^+ \rho^+ \right. \right. \\
&\quad + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} \left(L^g \left(\frac{L}{1-\delta} \right) + M^g \right) \\
&\quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} \left(L^h \left(\frac{L}{1-\delta} \right) + M^h \right) \\
&\quad + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} \frac{K_{ij}^+}{\eta} \left(L^f \left(\frac{L}{1-\delta} \right) + M^f \right) \Big] \\
&\quad + \frac{L}{1 - e^{-a_{ij}^*}}, \left(1 + \frac{a_{ij}^+}{a_{ij}^*} \right) \left[a_{ij}^+ \rho^+ \right. \\
&\quad + \sum_{C_{kl} \in N_q(i,j)} \bar{B}_{ij}^{kl} \left(L^g \left(\frac{L}{1-\delta} \right) + M^g \right) \\
&\quad + \sum_{C_{kl} \in N_p(i,j)} \bar{D}_{ij}^{kl} \left(L^h \left(\frac{L}{1-\delta} \right) + M^h \right) \\
&\quad + \sum_{C_{kl} \in N_r(i,j)} \bar{C}_{ij}^{kl} \frac{K_{ij}^+}{\eta} \left(L^f \left(\frac{L}{1-\delta} \right) + M^f \right) \Big] \\
&\quad \left. + \frac{a_{ij}^+ L_1}{1 - e^{-a_{ij}^*}} \right\} < 1,
\end{aligned}$$

which prove that Γ is a contraction mapping. Then, by virtue of the Banach fixed point theorem, Γ has a unique fixed point which corresponds to the solution of (1) in $\mathbb{B} \subset PAP_T(\mathbb{R}, \mathbb{R}^{m \times n})$. This completes the proof. $\square$

Appendix 6: Proof of the Lemma 10

Proof Consider the following equation:

$$G(\lambda) = \lambda - p(t) + q_1(t)e^{\lambda\tau} + q_2(t) \int_0^{+\infty} k(s)e^{\lambda s} ds. \quad (12)$$

Because

$$G(0) \leq -p(t) + q_1(t) + kq_2(t) < 0,$$

$$\frac{dG}{d\lambda} = 1 + q_1(t)\tau e^{\lambda\tau} + q_2(t) \int_0^{+\infty} k(s)se^{\lambda s} ds > 0$$

and $G(+\infty) > 0$, we follow that $G(\lambda)$ is a strictly monotone increasing function.

Therefore, for any $t \geq 0$, there is a unique positive $\lambda(t)$ such that

$$\lambda(t) - p(t) + q_1(t)e^{\lambda\tau} + q_2(t) \int_0^{+\infty} k(s)e^{\lambda s} ds = 0.$$

Moreover, λ^* exists and $\lambda^* \geq 0$.

Now, we will prove $\lambda^* > 0$. Suppose this is not true. Pick $\varepsilon \in (0, \mu)$ such that $\varepsilon < \{\frac{\eta}{3}, \frac{1}{\tau} \ln(1 + \frac{\eta}{3M})\}$ and $\int_0^{+\infty} k(s)e^{\varepsilon s} ds \leq k + \frac{\eta}{3M}$. Then there exist $t^* > 0$ such that $\lambda^*(t^*) < \varepsilon$ and

$$\lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*) \int_0^{+\infty} k(s)e^{\lambda^*(t^*)s} ds = 0.$$

Now we have

$$\begin{aligned} 0 &= \lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*) \int_0^{+\infty} k(s)e^{\lambda^*(t^*)s} ds \\ &< \lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*) \int_0^{+\infty} k(s)e^{\varepsilon s} ds \\ &< \varepsilon - p(t^*) + q_1(t^*) \left(1 + \frac{\eta}{3M}\right) + q_2(t^*) \left(k + \frac{\eta}{3M}\right) \\ &< \frac{\eta}{3} - (p(t^*) - q_1(t^*) - kq_2(t^*)) + (q_1(t^*) + q_2(t^*)) \frac{\eta}{3M} \\ &< \frac{\eta}{3} - \eta + \frac{2\eta}{3} = 0, \end{aligned}$$

which is a contradiction. Hence, $\lambda^* > 0$. The proof of this lemma is completed.

Appendix 7: Proof of the Lemma 11

Proof By the similar analysis in Lemma 10, we can deduce that $\lambda_{ij}^* > 0$ and $\hat{\lambda}_{ij}^* > 0$ exist uniquely. Choose a positive constant θ such that

$$\min\{p_{ij}, q_{ij} | i = 1, 2, \dots, m, j = 1, 2, \dots, n\} \theta > 1.$$

Let for $t \in \mathbb{R}^+$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

$$\Phi_{ij}(t) = \max\left\{\frac{1}{p_{ij}} V_{ij}(t), \frac{1}{q_{ij}} W_{ij}(t)\right\}.$$

$$\Psi(t) = \theta \sum_{l=1}^m \sum_{\ell=1}^n \max\{\bar{V}_{l\ell}(0), \bar{W}_{l\ell}(0)\} e^{-\lambda^* t}. \quad (13)$$

Then for all $t \in (-\infty, 0]$ and $\gamma > 1$, we have

$$\begin{aligned} \gamma \Psi(t) &= \gamma \theta \sum_{l=1}^m \sum_{\ell=1}^n \max\{\bar{V}_{l\ell}(0), \bar{W}_{l\ell}(0)\} e^{-\lambda^* t} \\ &> \Phi_{ij}(t). \end{aligned} \quad (14)$$

Then for $t \in [0, \infty)$, $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$, we have:

$$\Phi_{ij}(t) < \gamma \Psi(t), \quad (15)$$

For the sake of contradiction, assume that there exist $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$ and $\bar{t} > 0$ such that

$$\Phi_{ij}(\bar{t}^+) \geq \gamma \Psi(\bar{t}), \Phi_{l\ell}(t) < \gamma \Psi(t), \quad (16)$$

for $t \in [0, \bar{t})$, $l \in \{1, 2, \dots, m\}$ and $\ell \in \{1, 2, \dots, n\}$.

Then we have the following cases. (I) $\frac{1}{p_{ij}} V_{ij}(\bar{t}^+) \geq \gamma \Psi(\bar{t})$ then we have the following subcases. (i) $\bar{t} \neq t_w, w \in \mathbb{N}^*$. So $V_{ij}(t)$ is continuous at $\bar{t}$. By 16, we have

$$\frac{1}{p_{ij}} V_{ij}(\bar{t}) = \gamma \Psi(\bar{t}), \frac{1}{p_{ij}} D^- V_{ij}(\bar{t}) > \gamma \Psi'(\bar{t}). \quad (17)$$

From (H6), (16) and the definition of λ^* , we have

$$\begin{aligned} &\frac{1}{p_{ij}} D^- V_{ij}(\bar{t}) - \gamma \Psi'(\bar{t}) \\ &\leq -a_{ij}(\bar{t}) \gamma \Psi(\bar{t}) + \frac{q_{ij}}{p_{ij}} a_{ij}(\bar{t}) \int_{\bar{t}-\rho(\bar{t})}^{\bar{t}} \gamma \Psi(s) ds \\ &\quad + \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(\bar{t})| \\ &\quad \times \left(L^g \frac{p_{kl}}{p_{ij}} \left(\frac{L}{1-\delta} \right) + M^g \right) \gamma \Psi(\bar{t}) \\ &\quad + \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(\bar{t})| \\ &\quad \times \left(L^h \frac{q_{kl}}{p_{ij}} \left(\frac{L}{1-\delta} \right) + M^h \right) \gamma \Psi(\bar{t} - \tau^+) \\ &\quad + \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(\bar{t})| \\ &\quad \times \left(L^f \frac{q_{kl}}{p_{ij}} \int_0^\infty |K_{ij}(u)| \gamma \Psi(\bar{t} - u) du \left(\frac{L}{1-\delta} \right) \right. \\ &\quad \left. + M^f \frac{K_{ij}^+}{\eta} \gamma \Psi(\bar{t}) \right) + \lambda^* \gamma \Psi(\bar{t}) \\ &\leq \gamma \Psi(\bar{t}) (\lambda^* - a_{ij}(\bar{t}) + \frac{q_{ij}}{p_{ij}} a_{ij}(\bar{t}) \rho^+ e^{\lambda^* \rho^+} \\ &\quad + \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(\bar{t})| \times \left(L^g \frac{p_{kl}}{p_{ij}} \left(\frac{L}{1-\delta} \right) + M^g \right) \\ &\quad + \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(\bar{t})| \times \left(L^h \frac{q_{kl}}{p_{ij}} \left(\frac{L}{1-\delta} \right) + M^h \right) e^{\lambda^* \tau^+} \\ &\quad + \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(\bar{t})| \end{aligned}$$

$$\begin{aligned} & \times (L^f \frac{q_{kl}}{p_{ij}} \int_0^\infty |K_{ij}(u)| e^{\lambda^* u} du \left(\frac{L}{1-\delta} \right) \\ & + M^f \frac{K_{ij}^+}{\eta}) < 0, \end{aligned} \quad (18)$$

which is a contradiction with (17).

(ii) There exists $w_0 \in \mathbb{N}^*$ such that $\bar{t} = t_{w_0}$. By (16), we have

$$\frac{1}{p_{ij}} V_{ij}(\bar{t}) \leq \gamma \Psi(\bar{t}) \leq \frac{1}{p_{ij}} V_{ij}(\bar{t}^+). \quad (19)$$

Noting $\frac{1}{p_{ij}} V_{ij}(\bar{t}^-) \neq \frac{1}{p_{ij}} V_{ij}(\bar{t}^+)$, we have $\frac{1}{p_{ij}} V_{ij}(\bar{t}^-) < \gamma \Psi(\bar{t})$ or $\gamma \Psi(\bar{t}) < \frac{1}{p_{ij}} V_{ij}(\bar{t}^+)$. Without loss of generality, we assume that $\gamma \Psi(\bar{t}) < \frac{1}{p_{ij}} V_{ij}(\bar{t}^+)$. From (5) and (19) we get that

$$\gamma \Psi(\bar{t}) < \frac{1}{p_{ij}} V_{ij}(\bar{t}^+) \leq \gamma L_1 \Psi(\bar{t}). \quad (20)$$

Simplifying (20), we obtain $L_1 > 1$, which contradict that $L_1 < 1$.

If (I) does not hold, then

(II) For $t \in [0, \bar{t}]$, $l = 1, \dots, m$, $\ell = 1, \dots, n$,

$$\begin{aligned} \frac{1}{q_{ij}} W_{ij}(\bar{t}^+) & \geq \gamma \Psi(\bar{t}), \quad \frac{1}{q_{l\ell}} W_{l\ell}(t) < \gamma \Psi(t), \\ \frac{1}{p_{l\ell}} W_{l\ell}(t) & \geq \gamma \Psi(t). \end{aligned} \quad (21)$$

Then from (4) and (H6) we have

$$\begin{aligned} 0 & \leq -W_i(\bar{t}^+) + a_{ij}(\bar{t}) V_{ij}(\bar{t}^+) \\ & + a_{ij}(\bar{t}) \int_{\bar{t}^+ - \rho(\bar{t}^+)}^{\bar{t}^+} W_{ij}(s) ds \\ & + \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(\bar{t})| \\ & \times \left(L^g V_{kl}(\bar{t}^+) \left(\frac{L}{1-\delta} \right) + M^g V_{ij}(\bar{t}^+) \right) \\ & + \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(\bar{t})| \\ & \times \left(L^h \bar{W}_{kl}(\bar{t}^+) \left(\frac{L}{1-\delta} \right) + M^h V_{ij}(\bar{t}^+) \right) \\ & + \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(\bar{t})| \\ & \times \left(L^f \int_0^\infty |K_{ij}(u)| W_{kl}(\bar{t}^+ - u) du \left(\frac{L}{1-\delta} \right) \right. \\ & \left. + M^f \frac{K_{ij}^+}{\eta} V_{ij}(\bar{t}^+) \right) \\ & \leq \gamma \Psi(\bar{t}) (-q_{ij} + p_{ij} a_{ij}(\bar{t}) \\ & + q_{ij} a_{ij}(\bar{t}) \rho^+ e^{\lambda^* \rho^+} \\ & + \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(\bar{t})| \end{aligned} \quad (22)$$

$$\begin{aligned} & \times (L^g p_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij} M^g) \\ & + \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(\bar{t})| \\ & \times (L^h q_{kl} \left(\frac{L}{1-\delta} \right) + p_{ij} M^h) e^{\lambda^* \tau^+} \\ & + \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(\bar{t})| \\ & \times \left(L^f q_{kl} \int_0^\infty |K_{ij}(u)| e^{\lambda^* u} du \left(\frac{L}{1-\delta} \right) \right. \\ & \left. + p_{ij} M^f \frac{K_{ij}^+}{\eta} \right) < 0 \end{aligned} \quad (22)$$

which is a contradiction. From (I) and (II), (15) holds. Letting $\gamma \rightarrow 1^+$ in (15), we have

$$\Phi_{ij}(t) \leq \gamma \Psi(t), \quad (23)$$

for $t \in [0, \infty)$, $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

So $\frac{1}{p_{ij}} V_{ij}(t) \leq \Psi(t)$ for all $t \in [0, \infty)$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. Let $\tilde{L} = \max_{i,j} \{p_{ij} \theta\}$ the for $t \geq 0$ and $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, we have

$$V_{ij}(t) \leq \tilde{L} \sum_{l=1}^m \sum_{\ell=1}^n \max\{\bar{V}_{l\ell}(0), \bar{W}_{l\ell}(0)\} e^{-\lambda^* t}, \quad (24)$$

The proof is complete. $\square$

Appendix 8: Proof of the Theorem 4

Proof It follows from Theorem 3 that system (1) has a unique piecewise differentiable pseudo almost periodic solution

$$x^*(t) = (x_{11}^*(t), \dots, x_{ij}^*(t), \dots, x_{mn}^*(t))^T \in \mathbb{B}$$

with initial value $\phi^*(t)$.

Let $x(t) = (x_{11}(t), \dots, x_{ij}(t), \dots, x_{mn}(t))^T$ be an arbitrary solution of system (1) with initial value $\phi(t)$.

Let $V_{ij}(t) = |x_{ij}^*(t) - x_{ij}(t)|$, $W_{ij}(t) = |x_{ij}'^*(t) - x_{ij}'(t)|$ for $i = 1 \dots m$ and $j = 1 \dots n$, then,

$$\begin{aligned} & D^- V_{ij}(t^-) \\ & \leq -a_{ij}(t) V_{ij}(t^-) + a_{ij}(t) \int_{t^- - \rho(t^-)}^{t^-} W_{ij}(s) ds \\ & + \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(t)| \\ & \times \left(L^g V_{kl}(t^-) \left(\frac{L}{1-\delta} \right) + M^g V_{ij}(t^-) \right) \\ & + \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(t)| \end{aligned}$$

$$\begin{aligned}
& \times \left(L^h \overline{W}_{kl}(t^-) \left(\frac{L}{1-\delta} \right) + M^h V_{ij}(t^-) \right) \\
& + \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(t)| \\
& \times \left(L^f \int_0^\infty |K_{ij}(u)| W_{kl}(t^- - u) du \left(\frac{L}{1-\delta} \right) \right. \\
& \left. + M^f \frac{K_{ij}^+}{\eta} V_{ij}(t^-) \right)
\end{aligned} \quad (25)$$

$$\begin{aligned}
& W_{ij}(t^+) \\
& \leq a_{ij}(t) V_{ij}(t^+) + a_{ij}(t) \int_{t^+ - \rho(t^+)}^{t^+} W_{ij}(s) ds \\
& + \sum_{C_{kl} \in N_q(i,j)} |B_{ij}^{kl}(t)| \\
& \times \left(L^g V_{kl}(t^+) \left(\frac{L}{1-\delta} \right) + M^g V_{ij}(t^+) \right) \\
& + \sum_{C_{kl} \in N_p(i,j)} |D_{ij}^{kl}(t)| \\
& \times \left(L^h \overline{W}_{kl}(t^+) \left(\frac{L}{1-\delta} \right) + M^h V_{ij}(t^+) \right) \\
& + \sum_{C_{kl} \in N_r(i,j)} |C_{ij}^{kl}(t)| \\
& \times \left(L^f \int_0^\infty |K_{ij}(u)| W_{kl}(t^+ - u) du \left(\frac{L}{1-\delta} \right) \right. \\
& \left. + M^f \frac{K_{ij}^+}{\eta} V_{ij}(t^+) \right),
\end{aligned} \quad (26)$$

By (5) and (H5) we have

$$V_{ij}(t_w^+) \leq L_1 V_{ij}(t_w), \text{ with } L_1 < 1. \quad (27)$$

By (25)–(27), (H1)–(H6) and Lemma 11, there exists a positive constant M such that

$$V_{ij}(t) \leq M \sum_{l=1}^m \sum_{\ell=1}^n \max\{\overline{V}_{l\ell}(0), \overline{W}_{l\ell}(0)\} e^{-\lambda^* t}, \quad (28)$$

where λ^* is defined in (7).

This finishes the proof of Theorem 4. $\square$

References

- Abbas S, Mahto L, Hafayed M, Alimi AM (2014) Asymptotic almost automorphic solutions of impulsive neural network with almost automorphic coefficients. *Neurocomputing* 142:326–334
- Aliprantis CD, Boreder KC (1999) Infinite dimensional analysis, 2nd edn. Springer, Berlin
- Ammar B, Chérif F, Alimi AM (2012) Existence and uniqueness of pseudo almost-periodic solutions of recurrent neural networks with time-varying coefficients and mixed delays. *IEEE Trans Neural Net Learn Syst* 23(1):109–118.
- Aouiti C, M'hamdi MS, Touati A (2016) Pseudo almost automorphic solutions of recurrent neural networks with time-varying

- coefficients and mixed delays. *Neural Process Lett.* doi:10.1007/s11063-016-9515-0
- Bai C (2008) Global stability of almost periodic solutions of Hopfield neural networks with neutral time-varying delays. *Appl Math Comput* 203(1):72–79
- Bai C (2009) Existence and stability of almost periodic solutions of Hopfield neural networks with continuously distributed delays. *Nonlinear Anal: Theory Methods Appl* 71(11):5850–5859
- Benucci A, Verschure PF, König P (2007) Dynamical features of higher-order correlation events: impact on cortical cells. *Cogn Neurodyn* 1(1):53–69
- Bouzerdoum A, Pinter RB (1993) Shunting inhibitory cellular neural networks: derivation and stability analysis. *IEEE Trans Circuits Syst I: Fundam Theory Appl* 40(3):215–221
- Cao J, Rakkiyappan R, Maheswari K, Chandrasekar A (2016) Exponential H_∞ filtering analysis for discrete-time switched neural networks with random delays using sojourn probabilities. *Sci China Technol Sci* 59(3):387–402
- Chen A, Cao J (2002) Almost periodic solution of shunting inhibitory CNNs with delays. *Phys Lett A* 298(2):161–170
- Chen L, Zhao H (2008) Global stability of almost periodic solution of shunting inhibitory cellular neural networks with variable coefficients. *Chaos, Sol Fractals* 35(2):351–357
- Chérif F (2012) Existence and global exponential stability of pseudo almost periodic solution for SICNNs with mixed delays. *J Appl Math Comput* 39(1–2):235–251
- Chérif F (2014) Pseudo almost periodic solutions of impulsive differential equations with delay. *Differ Equ Dyn Syst* 22(1):73–91
- Chua LO, Yang L (1988) Cellular neural networks: applications. *IEEE Trans Circuits Syst* 35(10):1273–1290
- Ding HS, Ye GR (2009) Exponential convergence of solutions of SICNNs with mixed delays. *Electron J Differential Equ* 2009(41):1–7
- Fan Q, Shao J (2010) Positive almost periodic solutions for shunting inhibitory cellular neural networks with time-varying and continuously distributed delays. *Commun Nonlinear Sci Numer Simul* 15(6):1655–1663
- Fink AM (1974) Almost periodic differential equations. Springer, New York
- Gao J, Wang QR, Zhang LW (2014) Existence and stability of almost-periodic solutions for cellular neural networks with time-varying delays in leakage terms on time scales. *Appl Math Comput* 237:639–649
- Gopalsamy K (2007) Leakage delays in BAM. *J Math Anal Appl* 325(2):1117–1132
- Hu M, Cao J, Hu A (2014) Exponential stability of discrete-time recurrent neural networks with time-varying delays in the leakage terms and linear fractional uncertainties. *IMA J Math Contr Inform* 31(3):345–362
- Huang H, Cao J, Wang J (2002) Global exponential stability and periodic solutions of recurrent neural networks with delays. *Phys Lett A* 298(5):393–404
- Jiang H, Cao J (2006) Global exponential stability of periodic neural networks with time-varying delays. *Neurocomputing* 70(1):343–350
- Jiang H, Teng Z (2004) Global exponential stability of cellular neural networks with time-varying coefficients and delays. *Neural Net* 17(10):1415–1425
- Kumar RS, Sugumaran G, Raja R, Zhu Q, Raja UK (2016) New stability criterion of neural networks with leakage delays and impulses: a piecewise delay method. *Cogn Neurodyn* 10(1):85–98
- Li Y, Wang C (2011) Almost periodic functions on time scales and applications. *Discrete Dyn Nat Soc* 2011:727068
- Li L, Fang Z, Yang Y (2012) A shunting inhibitory cellular neural network with continuously distributed delays of neutral type. *Nonlinear Anal: Real World Appl* 13(3):1186–1196

- Li X, Cao J (2010) Delay-dependent stability of neural networks of neutral type with time delay in the leakage term. *Nonlinearity* 23(7):1709–1726
- Lin Z, Lin YX (2000) Linear systems exponential dichotomy and structure of sets of hyperbolic points. World Scientific
- Liu Y, You Z, Cao L (2006) On the almost periodic solution of generalized shunting inhibitory cellular neural networks with continuously distributed delays. *Phys Lett A* 360(1):122–130
- Liu Y, You Z, Cao L (2007) Almost periodic solution of shunting inhibitory cellular neural networks with time varying and continuously distributed delays. *Phys Lett A* 364(1):17–28
- Liu J, Zhang C (2013) Composition of piecewise pseudo almost periodic functions and applications to abstract impulsive differential equations. *Adv Differ Equ* 2013(1):1–21
- Mattia M, Sanchez-Vives MV (2012) Exploring the spectrum of dynamical regimes and timescales in spontaneous cortical activity. *Cogn Neurodyn* 6(3):239–250
- Ou C (2009) Almost periodic solutions for shunting inhibitory cellular neural networks. *Nonlinear Anal: Real World Appl* 10(5):2652–2658
- Pan L, Cao J (2011) Anti-periodic solution for delayed cellular neural networks with impulsive effects. *Nonlinear Anal: Real World Appl* 12(6):3014–3027
- Rakkiyappan R, Balasubramaniam P, Cao J (2010) Global exponential stability results for neutral-type impulsive neural networks. *Nonlinear Anal: Real World Appl* 11(1):122–130
- Roska T, Chua LO (1992) Cellular neural networks with non-linear and delay-type template elements and non-uniform grids. *Int J Circuit Theory Appl* 20(5):469–481
- Samoilenko AM, Perestyuk NA (1995) Impulsive differential equations, 14th edn. World Scientific, Singapore
- Shao J, Wang L, Ou C (2009) Almost periodic solutions for shunting inhibitory cellular neural networks without global Lipschitz activity functions. *Appl Math Model* 33(6):2575–2581
- Song Q, Cao J (2012) Passivity of uncertain neural networks with both leakage delay and time-varying delay. *Nonlinear Dynam* 67(2):1695–1707
- Tagluk ME, Tekin R (2014) The influence of ion concentrations on the dynamic behavior of the Hodgkin–Huxley model-based cortical network. *Cogn Neurodyn* 8(4):287–298
- Wei X, Chen Y, Lu M, Deng B, Yu H, Wang J, Che Y, Han C (2014) An ephaptic transmission model of CA3 pyramidal cells: an investigation into electric field effects. *Cogn Neurodyn* 8(3):177–197
- Wu H (2011) Pseudo almost-periodic solution of shunting inhibitory cellular neural networks with delay. *J Appl Math* 2011:510789
- Xia Z (2014) Weighted pseudo almost automorphic solutions of hyperbolic semilinear integro-differential equations. *Nonlinear Anal: Theory, Methods Appl* 95:50–65
- Xia Z (2016) Pseudo almost periodic mild solution of nonautonomous impulsive integro-differential equations. *Mediterr J Math* 13(3):1065–1086
- Xiang H, Cao J (2009) Almost periodic solutions of recurrent neural networks with continuously distributed delays. *Nonlinear Anal: Theory, Methods Appl* 71(12):6097–6108
- Xia Z, Wang D (2015) Piecewise weighted pseudo almost periodic solutions of impulsive integro-differential equations via fractional operators. *Electronic J Differ Equ* 2015(185):1–18
- Xia Z, Wang D (2016) Pseudo almost periodic solution for impulsive Hematopoiesis model with infinite delays and multiple linear harvesting terms. *Int J Biomath*. doi:[10.1142/S1793524516500789](https://doi.org/10.1142/S1793524516500789)
- Xinsong Y, Jinde C (2012) Adaptive pinning synchronization of coupled neural networks with mixed delays and vector-form stochastic perturbations. *Acta Math Sci* 32(3):955–977
- Zhang CY (1994) Pseudo almost periodic solutions of some differential equations. *J Math Anal Appl* 181(1), 62–76
- Zhang CY (1995) Pseudo almost periodic solutions of some differential equations, II. *J Math Anal Appl* 192(2):543–561
- Zhang C (2003) Almost periodic type functions and ergodicity. Science Press, Beijing
- Zhao W (2008) Dynamics of Cohen–Grossberg neural network with variable coefficients and time-varying delays. *Nonlinear Anal: Real World Appl* 9(3):1024–1037
- Zhu Q, Cao J, Hayat T, Alsaadi F (2015) Robust stability of Markovian jump stochastic neural networks with time delays in the leakage terms. *Neural Process Lett* 41(1):1–27