The membrane potential process of a single neuron seen as a cumulative damage process

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Abstract A simple integrate-and-fire mechanism of a single neuron can be compared with a cumulative damage process, where the spiking process is analogous to rupture sequences of a material under cycles of stress. Although in some cases lognormal-like patterns can be recognized in the inter-spike times under a simple integrate-and-fire mechanism, fatigue life models as the inverse Gaussian distribution and the Birnbaum–Saunders distribution (which was recently introduced in the neural activity framework) provide theoretical arguments that make them more suitable for the modeling of the resulting inter-spike times.

Keywords Lognormal · Inverse Gaussian and Birnbaum–Saunders distributions · Integrate-and-fire model · Inter-spike distribution

In a recent communication, Kish et al. (2015) highlight lognormal-like features in the inter-spike distribution of a single neuron under a simple integrate-and-fire scheme with an additive noise with no long-tail but exponential cutoff. To be more precise, let $V(k)$ be the membrane potential of a typical neuron at discrete time $k \in \mathbb{N}$. Its dynamics is given by:

$$V(k) = V(k-1) + \delta + \sqrt{D}\xi(k), \quad (1)$$

with initial condition $V(0) = V_0$. The injected current is described through the process $\{\xi(k)\}_{k \in \mathbb{N}}$, which represents a white noise process (typically normally or uniformly distributed), and the parameters δ and D , which represent the drift velocity and the diffusion coefficient, respectively. Here, the condition $V_0 \leq V(k)$ is enforced during the whole motion (i.e., V_0 is a reflecting boundary). The process, then, stops at time $\kappa = \inf\{k : V(k) > V_{th}\}$, where V_{th} represents a constant (membrane potential) threshold at which the neuron spikes. Once it occurs, the process resets at V_0 . Although this model is too simple to capture a great part of reality, it is still a reasonable approximation when the input rate is high in comparison with the membrane capacitance. A lognormal-like feature in the inter-spike distribution appears when

$$\delta \gg \delta_0 = \frac{D}{V_{th} - V_0} \quad (2)$$

Actually, this is not a theoretical result but a graphical similarity. Indeed, if we assume that $\{\xi(k)\}_{k \in \mathbb{N}}$ in (1) is a standard normal process, when no condition on the motion is imposed, it is well-known that the resulting inter-spike times are inverse Gaussian distributed (which is already mentioned in Kish et al. 2015). The fact that the inverse Gaussian and the lognormal distributions are graphically similar under some specific values of its parameters is just an akin of the pictures of its probability density functions. In Takagi et al. (1997) the identical conclusion is obtained: consider $V_0 = 0$ and $V_{th} > 0$, and let $\tau_{V_{th}} := \inf\{t : V_t > V_{th}\}$. Then, $\tau_{V_{th}}$ follows an inverse Gaussian distribution with expectation $E(\tau_{V_{th}}) = V_{th}/\delta$, variance $V(\tau_{V_{th}}) = V_{th}D/\delta^3$ and coefficient of variation $CV(\tau_{V_{th}}) = \sqrt{V(\tau_{V_{th}})/E(\tau_{V_{th}})^2} = \sqrt{D/(V_{th}\delta)}$. The

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probability density of $\tau_{V_{th}}$ agrees well with a lognormal distribution when $CV(\tau_{V_{th}}) \gg 1$, which means that $\delta \gg D/V_{th}$.

Recently in Leiva et al. (2015), it is argued that a spike generation is analogous to a material rupture under a cumulative damage process, where such a cumulative damage process will come to represent the injected current in the neuron charging, under a simple integrate-and-fire mechanism. A new distribution was also introduced in this context, the Birnbaum–Saunders distribution (Birnbaum and Saunders 1969), or *the fatigue life distribution*, which produced a well goodness-of-fit of a real inter-spike time data set. Moreover, as in the inverse Gaussian model, it has a theoretical argument in which it naturally arises for modeling the inter-spike time data under a dynamical mechanism like (1). Specifically, for δ big and D small enough such that $V_0 \leq V(k)$ occurs with high probability (i.e., V_0 is near a reflecting boundary), the Birnbaum–Saunders distribution approximately arises independent of which white noise distribution is chosen in (1). This has the advantage that both the inverse Gaussian and the Birnbaum–Saunders distributions appear almost naturally under simple integrate-and-fire neurons. Additionally, they have a close relationship (Leiva et al. 2015; Desmond 1986; Balakrishnan et al. 2009) where its distributions become closer as $P(V_0 \leq V(k))$ becomes higher, for any $k > 0$, and thus, inverse Gaussian and Birnbaum–Saunders families can be considered as belonging to the family of fatigue life distributions. When such probability decreases the goodness-of-fit of the Birnbaum–Saunders distribution remains high but one of its parameters, the median, losses its interpretation (Leiva et al. 2015). In this situation, the Birnbaum–Saunders distribution can still be used as a predictive tool as in the case of a regression model with high coefficient of determination R^2 where some of its parameters are non-significative.

Finally, notice that condition (2) may imply that V_0 acts as a reflecting boundary with high probability, so in such a case it is expected that inverse Gaussian, Birnbaum–Saunders and lognormal distributions are close each other (see Fig. 1), although inverse Gaussian and Birnbaum–Saunders distributions can be recognized from a theoretical point of view, making them explicative models. Also, those models may be taken into consideration where similar schemes to the simple integrate-and-fire ones appeared (see, e.g., Söderlund et al. 1998; Kiss et al. 1999), or for extending them to a neural network (see, e.g., Heinze et al. 2007) in a simply way, where excitatory Poisson inputs may lead a Birnbaum–Saunders distribution for the inter-spike distribution (Fierro et al. 2013). The metaphor regarding the material rupture under a cumulative damage process comes from that the Birnbaum–Saunders

Fig. 1 This histogram was generated by using (1) with $\{\xi(k)\}_{k \in \mathbb{N}}$ a sequence of independent and identically standard normal distributed random variables, $D = 0, 25$, $\delta = 1$, $V_0 = 0$ and $V_{th} = 20$, so that (2) holds and $V(k) > 0$ occurs with high probability, for all $k \in \mathbb{N}$ ($P(\delta + \sqrt{D}\xi(1) < 0) \approx 0, 02$). 2000 spikes was simulated, and the log-likelihoods of the corresponding inter-spike times for each distribution were: $-4484,396$ for the inverse Gaussian distribution (the exact distribution), $-4484,424$ for the Birnbaum–Saunders distribution and $-4484,738$ for the lognormal distribution. As expected, they are close, although Birnbaum–Saunders distribution is closer to the exact distribution than the lognormal distribution, as was anticipated and theoretically explained in Leiva et al. (2015)

distribution was originally conceived inspired in fatigue life distributions (Birnbaum and Saunders 1969). Nevertheless, we are conscious that from a meticulous physical point of view there are other models which give a better description of what happen when materials are undergoing some cumulative damage (see, e.g., Pennetta et al. 1997), but this discussion is not the issue of this manuscript.

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