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On the Equivalence of the Edge/Isolate and Edge/Concurrent Tie ERGM Families, and Their Extensions¹

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Abstract

Partnership concurrency is a major driver of permeability of social networks to diffusion, and an important modeling target in the context of sexually transmitted infections. A seemingly unrelated phenomenon of concern in modeling social networks is isolation avoidance—the tendency of individuals to maintain at least one tie. Although concurrency bias and bias in isolate formation would naively seem to be distinct, we here show that their respective ERGM expressions (edge/concurrent tie and edge/isolate families, and their regular extensions) are equivalent, and that both are equivalent to a special case of the geometrically weighted degree families. In addition to being statistically useful, this equivalence provides insight into the essential connection between these apparently different structural phenomena.

Keywords

ERGMs; model parameterization; social isolation; concurrency; Markov graphs; k -stars; geometrically weighted degree

Exponential family random graph models (ERGMs) provide a powerful framework for describing, simulating, and measuring structural biases within social networks associated with social processes (Wasserman and Robins, 2005; Robins and Morris, 2007). Given an order- N random graph, Y , on support $\mathcal{Y}_N$, the probability mass function (pmf) of Y may be written in ERGM form as

$$\Pr(Y=y|\theta, X) = \frac{\exp(\theta^T t(y, X))}{\sum_{y' \in \mathcal{Y}_N} \exp(\theta^T t(y', X))} \mathbb{I}_{\mathcal{Y}_N}(y), \quad (1)$$

where X is a covariate set, $t: \mathcal{Y}_N \rightarrow \mathbb{R}^p$ is a vector of sufficient statistics, $\theta \in \mathbb{R}^p$ is a vector parameters, and $\mathbb{I}_{\mathcal{Y}_N}$ is the counting measure on the support. By making different choices of t , a wide range of properties can be modeled, including both heterogeneity and conditional dependence among edge variables (see, e.g. Frank and Strauss, 1986; Wasserman and

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Pattison, 1996; Robins and Pattison, 2005; Snijders et al., 2006). By turns, studying the properties of a hypothesized model can give one insight into the behavior of the process giving rise to it, including its dependence structure, stability, and extrapolative behavior (Snijders, 2010; Butts, 2011; Schweinberger, 2011).

In this paper, we show that two families of models for seemingly distinct structural biases—concurrency and isolate formation—are in fact formally equivalent. In addition to being statistically useful, this equivalence provides insight into how these two types of structural phenomena are related.

In the development that follows, we will focus on models for simple graphs of fixed order N (i.e., taking $\mathcal{Y}_N$ equal to the set of all undirected loopless graphs on N vertices). This is the case for which concurrency per se is most meaningful, and most typically studied.

Generalization of these results to the directed case is possible, though we do not pursue this here.

1 Two Model Families

We begin by introducing the two model families of interest, showing their equivalence in Section 2.

1.1 Edge/Concurrent Tie Family

Although concurrency can be modeled in various ways, one of the simplest and most natural (implemented e.g. as the concurrentities statistic in statnet (Handcock et al., 2008; Hunter et al., 2008)) is to include a statistic counting the number of *edgewise concurrencies*; i.e., the sum over all vertices of the number of edges incident upon each vertex after the first. When associated with a negative parameter, this statistic expresses the notion that the addition of a tie to an existing graph (respectively, removal) is penalized (respectively favored) for each prospective endpoint that already has at least one other edge. Formally, this statistic is defined as

$$t_c(y) = \sum_{i=1}^N \max \left[0, \sum_{j=1}^N y_{ij} - 1 \right].$$

Let $t_e(y) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N y_{ij}$ be the standard edge count statistic. We may then define the family of (regularly extended) *edge/concurrent tie* models by setting $\theta^C = (\theta_c, \theta_c, \theta_r)$ and $t^C = (t_e, t_c, t_r)$, where t_r is any set of additional statistics, and taking $\Pr(Y = y | \theta^C, X)$ as per Equation 1. θ_c then represents the bias towards or away from the formation of concurrent ties, with $\theta_c < 0$ representing the case of concurrency penalties typically seen in sexual contact networks (Morris and Kretzschmar, 1995).

1.2 Edge/Isolate Family

A bias towards or away from isolate formation is commonly captured by inclusion of a statistic, t_I , counting the number of isolates in the graph. Setting $\theta^I = (\theta_e, \theta_I, \theta_r)$ and $t^I = (t_e, t_I, t_r)$ (with t_e and t_r respectively representing the edge count and any additional statistics) and taking $\Pr(Y = y | \theta^I, X)$ as per Equation 1 then defines the family of regularly extended *edge/isolate* models. Intuitively, $\theta_I < 0$ implies the case of isolation avoidance (studied e.g., by Butts (2015)), while $\theta_I > 0$ implies a higher incidence of isolates than would be expected given the other model terms.

2 Model Equivalence

Our main result is the following:

Theorem 1 (Model Equivalence)

Let $f_C(y | \theta^C)$ be an edge/concurrent tie ERGM pmf and $f_I(y | \theta^I)$ an edge/isolate ERGM pmf on common support $\mathcal{Y}_N$. For all θ^C, θ^I there exist $\theta^{C'}, \theta^{I'}$ such that $f_C(y | \theta^C) = f_I(y | \theta^{C'})$ and $f_I(y | \theta^I) = f_C(y | \theta^{I'})$ for all $y \in \mathcal{Y}_N$.

Proof—Although seemingly distinct, we can prove the model families implied by t^C and t^I to be equivalent by showing that we can write t^C as an affine transformation of t^I and vice versa (implying that, for any θ on t^C , there exists a θ' such that $\theta t^C(y) = \theta' t^I(y) + k$ for all y in the support). Considering that t_e and t_r appear in both vectors, the only non-trivial mapping is from concurrent ties to edges and isolates (and, respectively, from isolates, to edges and concurrent ties). To that end, we observe the following:

$$t_c(y) = \sum_{i=1}^N \left[\sum_{j=1} y_{ij} - 1 \right] + t_I(y)$$

This follows from the fact that the number of concurrent edges on any given vertex of degree greater than 0 is equal to the degree of that vertex minus 1; subtracting 1 from the degree of each vertex yields a “deficit” equal to -1 per isolate, which can be recovered by adding back the isolate count. We then have

$$\begin{aligned} t_c(y) &= \sum_{i=1}^N \left[\sum_{j=1} y_{ij} \right] - N + t_I(y) \\ &= 2t_e(y) - N + t_I(y). \end{aligned}$$

Since N is an additive constant over the support, it has no effect on the associated ERGM pmf and may be dropped, giving us

$$t_c(y) \doteq 2t_e(y) + t_I(y)$$

where the $\doteq$ relation signifies equality up to an additive constant.¹ It follows immediately that

$$t_I(y) \doteq t_c(y) - 2t_e(y)$$

(again, we may ignore the additive constant), and hence $t^C = (t_e, t_c, t_I)$ and $t^I = (t_e, t_I, t_r)$ are related (up to a constant) via the invertible affine mapping

$$t^C \doteq \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 2 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} t^I$$

(with the continuation being the identity matrix), the inverse being

$$t^I \doteq \begin{bmatrix} 1 & 0 & 0 & \cdots \\ -2 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} t^C.$$

It follows that the family of ERGMs on the order- N simple graphs with statistics t^C is equivalent to the ERGM family on the same support with statistics t^I , and hence for any θ^C there must exist a θ^C such that $f_C(y|\theta^C) = f_I(y|\theta^C)$ for all $y \in \mathcal{Y}_N$ (and, likewise, for any θ^I there must exist a θ^I such that $f_I(y|\theta^I) = f_C(y|\theta^I)$).

Via the above, we can easily determine the mapping that takes the parameter vector from the edge/concurrent tie family to the edge/isolate family (and vice versa). Equating the transformed statistics gives us:

$$\begin{aligned} \theta^I t^I(y) &= \theta^C t^C(y) \\ \theta_e^I t_e(y) + \theta_I^I t_I(y) + \theta_r^I t_r(y) &= \theta_e^C t_e(y) + \theta_c^C t_c(y) + \theta_r^C t_r(y) \\ &= \theta_e^C t_e(y) + \theta_c^C (2t_e(y) - N + t_I(y)) + \theta_r^C t_r(y) \\ &\doteq (\theta_e^C + 2\theta_c^C) t_e(y) + \theta_c^C t_I(y) + \theta_r^C t_r(y). \end{aligned}$$

As usual, we drop constant terms in the above, giving us the expression

$\theta^I = (\theta_e^C + 2\theta_c^C, \theta_c^C, \theta_r^C)$. To obtain the edge/concurrent tie parameters from the edge/isolate parameters, we simply solve for θ^C :

¹I.e., $f(x) \doteq g(x) \Leftrightarrow f(x) = g(x) + k$, where k is some quantity that is constant with respect to x .

$$\begin{aligned}
\theta_e^C t_e(y) + \theta_c^C t_c(y) + \theta_r^C t_r(y) &\doteq \theta_e^I t_e(y) + \theta_I^I t_I(y) + \theta_r^I t_r(y) \\
&\doteq \theta_e^I t_e(y) + \theta_I^I (t_c(y) - 2t_e(y)) + \theta_r^I t_r(y) \\
&= (\theta_e^I - 2\theta_I^I) t_e(y) + \theta_I^I t_I(Y) + \theta_r^I t_r(y),
\end{aligned}$$

noting as above the omission of a constant term. The resulting expression for θ^C is

$$\theta^C = (\theta_e^I - 2\theta_I^I, \theta_I^I, \theta_r^I).$$

3 *k*-Star Representation

Since both the edge/isolate and edge/concurrent tie models can be expressed in terms of an isolate count statistic, both belong to the family of Markov graphs (Frank and Strauss, 1986) and can be given a common representation in terms of *k*-star statistics. This representation is especially simple for these model families, and we derive it here. For simplicity, we omit any additional terms and parameters (t_r , θ_r); these can be “added on” to the *k*-star form after the fact without loss of generality.

Let $\zeta = (\zeta_0, \dots, \zeta_{N-1})$ be the vector of degree parameters, and $\psi = (\psi_1, \dots, \psi_{N-1})$ the corresponding vector of *k*-star parameters (such that ζ_i is the parameter for the count of nodes having degree *i*, and ψ_i is the parameter for the count of *i*-stars). For a pure isolate model, with no other degree terms, we have trivially $\zeta_0 = \theta_I$ and $\zeta_i = 0$ for $i > 0$. To find the corresponding ψ vector, we first note that the linear dependence among the degree statistics implies that the *N* degree parameters have only *N* – 1 degrees of freedom; to establish a mapping from ζ to ψ we thus begin by mapping ζ to an “isolate free” parameterization, $\zeta' = (\zeta'_1, \dots, \zeta'_{N-1})$, with no isolate effect (equivalently, $\zeta'_0 = 0$ by construction). This is accomplished by subtracting the isolate parameter from each other degree parameter (here 0), i.e., $\zeta'_1 = 0 - \zeta_0 = -\theta_I$. Given ζ' , we may obtain ψ by the relation

$$\psi_i = \sum_{j=1}^i (-1)^{i-j} \zeta'_j \binom{i}{j}, \quad (2)$$

itself the inverse of the well-known mapping from the *k*-star parameters to the (non-isolate) degree parameters given by Frank and Strauss (1986). For the special case of $\zeta'_i = -\theta_I$, Eq. 2 reduces to the trivial form

$$\psi_i = (-1)^i \theta_I. \quad (3)$$

Observe that we have held out the edge term; since edges are synonymous with 1-stars, we may simply add the edge parameter to ψ_1 to obtain the final parameter vector. Substituting the isolate form of each model into Eq. 3 and adding the edge parameter then gives their *k*-star representations: $\psi^I = (\theta_e^I - 3\theta_I^I, \theta_I^I, -\theta_I^I, \dots)$ and $\psi^C = (\theta_e^C + \theta_c^C, \theta_c^C, -\theta_c^C, \dots)$. In both

families, the dependence parameter affects the successive star parameters as a constant of alternating sign.

4 Discussion

The most immediate implication of Theorem 1 is that the apparently distinct phenomena of isolation avoidance and suppression of concurrent edges have equivalent effects on graph structure; the two effects cannot be separated on the basis of cross-sectional data, and indeed isolate count and concurrent tie statistics cannot be jointly identified in an ERGM with a free edge parameter. Pragmatically, this implies that whichever representation is more mathematically and/or computationally convenient can be employed when using these models. Less sanguinely, this also underscores a basic limitation on the possibility of distinguishing certain processes on the basis of single graph realizations.

The Markov graph expression of the isolation and concurrent tie families is interesting for its archetypal simplicity: the presence of a constant magnitude star parameter with alternating sign is an instantly recognizable signature of these families, and may provide a clue to identifying them e.g. from crude initial estimates of k -star parameters (obtained, e.g., via maximum pseudo-likelihood estimation). Interestingly, this pattern also provides insight into a third seemingly distinct family, the models formed via geometrically weighted degree (GWD) terms (Hunter, 2007). GWD applies a weighted parameter to the i th degree statistic (for $i \geq 1$) given by $\theta_{GWD} \exp(\theta_s)(1 - (1 - \exp(-\theta_s))^i)$, where θ_{GWD} is a regular parameter and θ_s is a curved parameter (often called a “decay parameter”) governing the shape of the degree weighting function. Although it is often assumed that θ_s is positive, how does GWD behave when $\theta_s = 0$? Clearly, the GWD coefficients in this case are uniformly equal to θ_{GWD} , with corresponding k -star parameters $(\theta_{GWD}, -\theta_{GWD}, \theta_{GWD}, \dots)$. From Eq. 3, this is immediately recognizable as an isolate model (and hence equivalently a concurrent edge model) with parameter $\theta_I = -\theta_{GWD}$. The equivalence of the GWD model with $\theta_s = 0$ and the isolate model seems not to be widely known (though a very similar result is derived by Hunter and Handcock (2006) for an extended family containing the GWD models), and the relationship of both to the concurrent tie model seems not previously to have been noted. This provides yet another way to interpret all three model families,² and implies the practical cautionary that GWD families will become poorly identified as $\theta_s \rightarrow 0$ when isolate or concurrent edge terms are included in the model.

Beyond their formal utility, the present results also provide substantive intuition for the link between isolation avoidance and concurrency penalties.³ One schematic view of these processes is illustrated in Figure 1. We consider the network in terms of three classes of nodes—isolates, pendants, and nodes with concurrent ties (“concurrent nodes”)—at constant density. Under conditions of isolation avoidance, isolates exert an effective “pull” on edges; since these edges cannot come from pendants (each of which exerts an equally strong hold on its sole tie as the isolates’ attractive force), they must be drawn from those incident upon concurrent nodes (figure, arrow A). This transfer of edges converts both isolates (by edge

²Since the alternating k -stars of Snijders et al. (2006) are equivalent to the GWD terms up to an edge parameter (Hunter, 2007), the current results apply to models based on the former terms as well.

³A similar process operates in reverse, *mutatis mutandis*, in the case of isolation and/or concurrency enhancement.

addition) and concurrent nodes (by edge removal) into pendants (figure, arrows B and C). Now consider the case of concurrency penalty. Here, concurrent ties are “repelled” from high degree nodes; since these edges cannot be redirected to pendants (who resist becoming concurrent with equal strength), they must necessarily be allocated to the set of isolates (arrow A). This once more converts both isolates and concurrent nodes into pendants (arrows B and C). Although the force of edge reallocation is conceived of as a “pull” in the case of isolation avoidance and a “push” in the case of concurrency penalty, it can be seen as a common underlying driver of network structure. Interestingly, the link between increasing concurrency and enhancing the number of isolates was noted in an early simulation study by Morris and Kretzschmar (1995) using a dynamic network model; we have here demonstrated how this connection necessarily arises via the transfer of nodes into/out of the pendant class.

5 Conclusion

The edge/isolate and edge/concurrent tie families are motivated by very different social mechanisms, but can be seen to be formally equivalent. Both are also equivalent to a special case of the geometrically weighted degree models in which the weight decay parameter, θ_s , is equal to zero. Intuitively, these families can be seen as implementing an edge reallocation force that (in the negative parameter case) shifts edges from concurrent nodes to isolates, converting members of both groups to pendants. In the positive parameter case this force works in reverse, enhancing the counts of isolates and concurrent nodes relative to pendants. Although the isolation and concurrent tie parameters do have different effects on density, these cannot be discerned in the typical case of a free edge parameter; as such, it is not generally possible to distinguish isolation and concurrent tie effects (nor GWD effects with $\theta_s \approx 0$) on the basis of cross-sectional network data.

This surprising equivalence among model classes raises the question of what other equivalences exist among apparently distinct classes of structural biases. Such equivalences reveal hidden commonalities among the seeming diversity of potential social mechanisms, and their identification would seem to be an important target for further research.

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Figure 1.

Schematic illustrating structural mechanisms in the edge/isolate and edge/concurrent tie models. Under conditions of isolation/concurrency penalty at constant density, edges are drawn from vertices of degree > 1 and transferred to isolates (A). This results in a net transfer of isolates to the class of pendants (B) due to edge gain, as well as a transfer of concurrent vertices to the class of pendants (C) due to edge loss. “Repulsion” of edges from concurrent nodes (concurrency penalty) and “attraction” of edges to isolates lead to the same equilibrium state.