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Extended narrow escape problem: Boundary homogenization-based analysis

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Abstract

Diffusion of particles in confined domains with absorbing spots on the otherwise reflecting boundaries is ubiquitous in nature and technology. Because of nonuniform boundary conditions, the problem of finding the mean first passage time (MFPT) of the particle to one of the spots is extremely complicated. We show how the difficulties can be overcome by means of boundary homogenization when the domain is a circular disk whose boundary contains n nonoverlapping identical absorbing arcs, which may occupy an arbitrary fraction of the boundary. We find the MFPT as a function of the fraction of the boundary occupied by the arcs (i) for n evenly spaced arcs and (ii) for two arcs arbitrarily located on the boundary. As the arc length tends to zero, our approximate solution reduces to the known asymptotic formula for the MFPT rigorously derived in studies devoted of the narrow escape problem.

I. INTRODUCTION

The narrow escape problem deals with a particle diffusing in a confined domain which contains a small absorbing spot on its otherwise reflecting boundary. This problem arises when analyzing diffusive transport in compartmentalized systems. Examples include receptor trafficking on nerve cell synaptic membranes [1], diffusion of ions and molecules in confined cellular domains, such as dendritic spines [2], channel-facilitated membrane transport [3], ligand equilibration between the tubular and vesicular parts of the endosome in the endosomal sorting [4], transport in porous media in which diffusing particles go from one compartment to the other through small openings in the compartment walls [5], to mention just a few.

The quantity of main interest in the narrow escape problem is the mean first passage time (MFPT) of the particle to the spot. Because of nonuniform boundary conditions (part of the boundary surface is absorbing while the rest is reflecting) it is rather difficult to find an analytical solution for the MFPT. The MFPT diverges, as the spot size tends to zero. Asymptotic behavior of the MFPT in this limiting case has been found [1,6]. The results for two- and three-dimensional domains are quite different. The leading term of the MFPT to a circular spot of radius a on the surface of a sphere of radius R , $a \ll R$, is given by $\pi R^3/(3Da)$, where D is the particle diffusion coefficient. In two dimensions the leading term of the

MFPT to an absorbing arc of length l on the boundary of a circular disk of radius R , $l \ll R$, is $(R^2/D)\ln(R/l)$.

In the present paper we study a particle diffusing in a circular disk with nonoverlapping identical absorbing arcs on the boundary and extend earlier rigorous results [1,6] in several respects. (i) We give an approximate formula for the MFPT to the only absorbing arc of arbitrary length, $0 < l < 2\pi R$, Eq. (4) with $n=1$. The asymptotic behavior derived in [1,6] is recovered from the formula as $l \rightarrow 0$. (ii) The formula with $n=1$ is a special case of a more general result, Eq. (4), which gives the MFPT to one of n nonoverlapping identical arcs of arbitrary length, $0 < l < 2\pi R/n$, evenly spaced along the disk boundary [Fig. 1(a)]. (iii) Another result of the present paper is the formula for the MFPT to one of the two nonoverlapping arcs of length l , $0 < l < \pi R$, arbitrarily located on the disk boundary [Fig. 1(b)], given in Eq. (7). The formulas are tested numerically by comparison with the MFPT found by solving the two-dimensional (2D) diffusion equation with nonuniform boundary conditions (BC) by the finite difference method. Comparison shows that the formulas predict the MFPT with relative error less than 2%.

II. n EVENLY SPACED ABSORBING ARCS ON THE DISK BOUNDARY

The fraction of the boundary occupied by the arcs, $\sigma = n l / (2\pi R)$, is an important parameter of the problem, $0 < \sigma < 1$. The MFPT of interest is nothing else than the mean lifetime of the particle. When the entire boundary is perfectly absorbing, $\sigma = 1$, the kinetics of the particle trapping by the boundary is multiexponential. One might expect that the kinetics becomes single-exponential as $\sigma \rightarrow 0$. In fact this is not the case. The formula, Eq. (4), shows that the boundary becomes perfectly absorbing and, hence, the kinetics is multiexponential, at arbitrarily small σ as $n \rightarrow \infty$. We analyze the question of single-exponential kinetics in the presence of n arcs on the disk boundary and show that the kinetics is single-exponential when σ satisfies $\sigma < 10^{-2n}$.

A key step of our approach to the problem is a replacement of the nonuniform BC at the disk boundary by an effective uniform radiation BC with properly chosen trapping rate κ . Such a replacement, called boundary homogenization (BH), is based on the fact that sufficiently far away from a patchy nonuniform boundary the fields of the steady-state fluxes and concentrations become uniform in the lateral direction [7]. As a consequence, one cannot distinguish between uniform and nonuniform boundary only by analyzing the flux, which is the main quantity of interest in this problem. BH is applicable when the domain near the boundary is relatively small, namely the domain where the fields are nonuniform because they keep memory of the real boundary conditions, [5(b)]. Therefore, it is far from obvious that BH is applicable to our problem. Nevertheless, we will see that BH-based formulas for the MFPT works surprisingly well in spite of the absence of the hierarchy of the length scales mentioned above. Detailed discussion of different questions related to boundary homogenization can be found in [5(b),7].

Since BH replaces the real nonuniform boundary by an effective uniform one, the problem becomes essentially one-dimensional. The mean lifetime of a particle that starts at distance r from the center of the disk of radius R with uniform partially absorbing boundary, $\tau(r)$,

satisfies the equation $d[r d\tau(r)/dr]/dr = -r/D$ with boundary conditions $d\tau(r)/dr|_{r=R} = -\kappa\tau(R)/D$ and $d\tau(r)/dr|_{r=0} = 0$ [8]. Solving this equation one finds that $\tau(r)$ is the sum of the MFPT to the disk boundary, $\tau_{MFPT}(r \rightarrow R) = (R^2 - r^2)/(4D)$, and the mean lifetime of the particle that starts from the boundary, $\tau(R) = R/(2\kappa)$,

$$\tau(r) = \frac{R^2 - r^2}{4D} + \frac{R}{2\kappa} \quad (1)$$

Averaging $\tau(r)$ over the starting point, assuming that the starting point is uniformly distributed over the disk, we obtain the averaged lifetime, $\langle\tau\rangle = (2/R^2) \int_0^R \tau(r) r dr$. Using R^2/D as a scale for time we can write

$$\langle\tau\rangle \frac{D}{R^2} = \frac{1}{8} + \frac{D}{2\kappa R}. \quad (2)$$

In this approach all information about the patchy boundary is hidden in the trapping rate κ . Trapping in the disk with absorbing arcs is identical to that in the cylindrical tube whose walls are covered by absorbing strips. With this in mind we use the formula for κ obtained in [7(c)] for a planar surface periodically covered by parallel identical absorbing strips. Denoting the strip width and the period by l and L , respectively, we can write this formula as $\kappa = (D/L)f(\sigma)$, where $\sigma = l/L$ is the strip surface fraction and $f(\sigma) = \pi/[(1-\sigma^2)\ln(2.6+0.7/\sigma)]$ [7(c)]. Then we write the effective trapping rate of the disk boundary as that of a periodically stripped surface of period $L = 2\pi R/n$. This leads to

$$\kappa = \kappa(\sigma|n) = \frac{nD}{2R(1-\sigma)^2 \ln(2.6+0.7/\sigma)}. \quad (3)$$

Substituting this $\kappa(\sigma|n)$ into Eq. (2) we arrive at

$$\langle\tau(\sigma|n)\rangle \frac{D}{R^2} = \frac{1}{8} + \frac{(1-\sigma)^2}{n} \ln\left(2.6 + \frac{0.7}{\sigma}\right), \quad (4)$$

which is one of the major results of this paper.

When $\sigma \rightarrow 0$ Eq. (4) takes the form

$$\langle\tau(\sigma|n)\rangle \frac{D}{R^2} = \frac{1}{n} \ln\left(\frac{1}{\sigma}\right). \quad (5)$$

For $n = 1$ this leads to $\langle\tau(\sigma|1)\rangle D/R^2 = \ln(1/\sigma)$, which is the result obtained in [1,6]. Thus, the BH-based approach correctly reproduces the leading term of the small- σ expansion of the MFPT, which was rigorously derived in [1,6]. The mean lifetime $\langle\tau(\sigma|n)\rangle$, Eq. (4), takes

its minimum value, $R^2/(8D)$, which corresponds to perfectly absorbing boundary, in two limiting cases: (i) when $\sigma=1$ and the entire boundary is absorbing and (ii) when $n \rightarrow \infty$ at fixed σ so that the boundary behaves as perfectly absorbing in spite of the fact that absorbing arcs occupy only its fraction σ that can be arbitrarily small. Considered as a function of n at fixed σ , $\langle \tau(\sigma|n) \rangle$ monotonically decreases with n from its maximum value $\langle \tau(\sigma|1) \rangle$, to its minimum value $R^2/(8D)$, which is independent of σ .

Numerical tests show excellent agreement between the mean particle lifetimes predicted by Eq. (4) and found numerically by solving the 2D diffusion equation with nonuniform BC by the finite difference method. The results are shown in Figs. 2 and 3. In Fig. 2 we compare $\langle \tau(\sigma|1) \rangle$ (solid line) with the mean lifetime found numerically (circles) for the case of one absorbing arc of length $l = 2\pi R\sigma$. One can see perfect agreement between the analytical and numerical results over the entire range of σ , $0 < \sigma < 1$ (the relative error is below 2%). While the agreement is expected for σ close to zero and unity, the fact that the formula works so well when the absorbing arc occupies about one half of the boundary is surprising. In Fig. 3 we test the dependence of $\langle \tau(\sigma|n) \rangle$ on the number of absorbing arcs, n , for two values of σ , $\sigma = 0.05/\pi$ and $0.5/\pi$. Again, the solid lines are theoretical predictions given by Eq. (4), while symbols are numerical results. One can see very good agreement between the two (the relative error is within 2%).

Next we establish the conditions under which the kinetics of the particle trapping by the arcs is single-exponential. The kinetics is single exponential when the mean lifetime of the particle is much larger than the characteristic equilibration time in the absence of the arcs. This happens when σ is sufficiently small and n is not too large so that $\kappa(\sigma|n)$, Eq. (3), is small enough. To specify the range of validity of the single-exponential kinetics at $n = 1$ we computed the particle survival probability at different σ , $S(t, \sigma|1)$. The results are shown in Fig. 4. When the kinetics is single-exponential, the survival probabilities found at different σ are indistinguishable if time t is measured in units of $\langle \tau(\sigma|1) \rangle$, $S(t, \sigma|1) = \exp[-t/\langle \tau(\sigma|1) \rangle]$. As shown in Fig. 4 this happens for $\sigma < 10^{-2}$. Thus we take 10^{-2} as an upper boundary for σ that determines the range of applicability of the single-exponential kinetics. For $\sigma < 10^{-2}$ the mean lifetime in Eq. (5) takes its asymptotic form $\langle \tau(\sigma|1) \rangle = (R^2/D) \ln(1/\sigma)$ obtained in [1,6]. When there are more than one absorbing arcs on the disk boundary the range of σ , for which the kinetics is single-exponential, decreases. In the case of n evenly spaced arcs it can be shown that the kinetics is single-exponential when σ is smaller than 10^{-2n} . For so small values of σ the mean particle lifetime can be written as

$$\langle \tau(\sigma|n) \rangle = \frac{R^2}{nD} \ln \left(\frac{1}{\sigma} \right) \approx \frac{\langle \tau(\sigma/n|1) \rangle}{n}, \quad \sigma < 10^{-2n}. \quad (6)$$

The second equality implies that different arcs work as independent traps for the particle so that $S(t, \sigma|n) = [S(t, \sigma/n|1)]^n$.

III. TWO ABSORBING ARCS ARBITRARILY LOCATED ON THE DISK BOUNDARY

Finally we studied the mean lifetime of the particle in the presence of two identical absorbing arcs, which can be arbitrarily located on the boundary with the only constraint that the arcs do not overlap [Fig. 1(b)]. Our goal is to learn how the mean lifetime depends on the interarc distance at different values of the arc length, l , and, hence, different $\sigma = l/(\pi R)$. We define the interarc distance, s , as the minimum distance between the arc centers measured along the disk boundary [Fig. 1(b)], $l \leq s \leq \pi R$. Our numerical results are presented in Fig. 5 (symbols). They show that at fixed σ the mean lifetime of the particle, $\langle \tau(s, \sigma|2) \rangle$, monotonically decreases with s from its maximum value at $s=l$ when the two arcs are in contact, $\langle \tau(s, \sigma|2) \rangle_{\max} = \langle \tau(\sigma|1) \rangle$, to its minimum value at $s=\pi R$ when the arcs centers are on the opposite sides of the disk diameter, $\langle \tau(s, \sigma|2) \rangle_{\min} = \langle \tau(\sigma|2) \rangle$.

We find that the numerical results are well described by the following formula:

$$\langle \tau(s, \sigma|2) \rangle \frac{D}{R^2} = \frac{1}{8} + \frac{1}{2} \left[(1 - \sigma)^2 + \left(1 - \frac{s}{\pi R} \right)^2 \right] \times \ln \left(2.6 + \frac{0.7\pi R}{s} \right), \quad l \leq s \leq \pi R, \quad (7)$$

which interpolates $\langle \tau(s, \sigma|2) \rangle$ between its maximum and minimum values. Solid lines in Fig. 5 are the s -dependences of the mean lifetime predicted by the formula for different σ . The relative error of the predictions is less than 2%. So high accuracy of the interpolating formula, Eq. (7), is surprising. When $\langle \tau(s, \sigma|2) \rangle$, Eq. (7), is compared with the formula for $\langle \tau \rangle$ in Eq. (2), it is apparent that the effective trapping rate that characterizes the homogenized boundary in this case, $\kappa(s, \sigma|2)$, is given by

$$\kappa(s, \sigma|2) = \frac{D}{R \left[(1 - \sigma)^2 + \left(1 - \frac{s}{\pi R} \right)^2 \right] \ln \left(2.6 + \frac{0.7\pi R}{s} \right)}, \quad l \leq s \leq \pi R \quad (8)$$

This trapping rate considered as a function of the interarc distance monotonically increases with s from $\kappa(\sigma|1) = \kappa(l, \sigma|2)$ at $s=l$ to $\kappa(\sigma|2) = \kappa(\pi R, \sigma|2)$ at $s=\pi R$.

IV. CONCLUDING REMARKS

In conclusion, main results of the present paper are given in Eqs. (4) and (7). The former shows how the MFPT to one of n nonoverlapping evenly spaced identical arcs depends on the number of arcs, n , and the fraction of the boundary occupied by the arcs, σ . The latter shows how the MFPT to one of two nonoverlapping unevenly spaced identical arcs depends on σ and the interarc distance, s . These results are obtained using the BH-based approach. An important advantage of this approach is that it works not only when $\sigma \rightarrow 0$, but also over the entire range of σ , $0 < \sigma < 1$. This method, when applicable, allows one to solve problems, which are too complicated to be solved by conventional methods, like the problem discussed above. Of course, BH is useful not only in two dimensions. The efficiency of this approach

in different three-dimensional problems has been demonstrated [5(b),5(c),9]. Here for the first time we demonstrate the usefulness of the approach in two dimensions.

Our results complement recent results on the narrow escape problem obtained by other researchers [10–12]. Nevertheless, there are many questions which still remain unsolved, for example, the question of how the spot clustering affects the MFPT. Study of this question in three dimensions was initiated recently [10(a),12(b)]. We believe that BH is a useful tool to analyze this question. The point is that in three dimensions BH has been studied in detail [7], including homogenization of the boundaries randomly covered by absorbing circular disks of different radii [7(b)], as well as the case when the disks are partially absorbing [7(a)]. The strategy of the analysis might be as follows. First, replace clusters of traps by effective partially absorbing spots. Second, replace the boundary covered by these partially absorbing spots by an effective uniformly absorbing boundary, which is characterized by a trapping rate properly chosen by means of BH. Third, find the MFPT to the uniformly absorbing domain boundary. This problem is much simpler than the initial one and even may have a general analytical solution for domains of relatively simple shapes.

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FIG. 1.

(a) Circular disk of radius R containing $n=6$ identical evenly spaced absorbing arcs of length l on its boundary. (b) Circular disk of radius R containing two identical absorbing arcs separated by distance s on its boundary.

FIG. 2.

Mean particle lifetime in the presence of one absorbing arc on the disk boundary, $\langle \tau(\sigma|1) \rangle$, measured in units of R^2/D , as a function of the fraction of the boundary occupied by the arc, σ . Solid line is the theoretical dependence, Eq. (4), circles are numerical results.

FIG. 3.

Dependence of the mean particle lifetime, $\langle \tau(\sigma|n) \rangle$, measured in units of R^2/D , on the number of absorbing arcs, n , at two values of the boundary fraction occupied by the arcs, $\sigma = 0.05/\pi$ and $0.5/\pi$. Solid lines are the theoretical dependences, Eq. (4), symbols are numerical results.

FIG. 4.

The particle survival probability in the presence of one absorbing arc on the disk boundary, $S(t, \sigma|1)$, found numerically as a function of the normalized time at different values of the fraction of the boundary occupied by the arc, $\sigma=0.0025/\pi$, $0.025/\pi$, $0.25/\pi$, and $2.5/\pi$. The survival probabilities are practically indistinguishable at $\sigma = 0.0025/\pi$ and $0.025/\pi$.

FIG. 5.

The mean particle lifetime, $\langle \tau(s, \sigma|2) \rangle$, measured in units of R^2/D , in the presence of two absorbing arcs of length $l = \pi R \sigma$ as a function of the interarc distance s for different values of σ : $0.001/\pi$, $0.01/\pi$, $0.05/\pi$, $0.25/\pi$, $0.5/\pi$ (from top to bottom). Solid lines are the dependencies predicted by Eq. (7), symbols are numerical results.