Published in final edited form as:

Magn Reson Med. 2013 July ; 70(1): 40–52. doi:10.1002/mrm.24426.

Reconstruction of undersampled radial PatLoc imaging using Total Generalized Variation

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Abstract

In the case of radial imaging with nonlinear spatial encoding fields, a prominent star-shaped artifact has been observed if a spin distribution is encoded with an undersampled trajectory. This work presents a new iterative reconstruction method based on the total generalized variation (TGV), which reduces this artifact. For this approach, a sampling operator (as well as its adjoint) is needed that maps data from PatLoc k -space to the final image space. It is shown that this can be realized as a Type-3 non-uniform FFT, which is implemented by a combination of a Type-1 and Type-2 non-uniform FFT. Using this operator, it is also possible to implement an iterative conjugate gradient (CG) SENSE based method for PatLoc reconstruction, which leads to a significant reduction of computation time in comparison to conventional PatLoc image reconstruction methods. Results from numerical simulations and in-vivo PatLoc measurements with as few as 16 radial projections are presented, which demonstrate significant improvements in image quality with the TGV based approach.

Keywords

PatLoc Imaging; Nonlinear encoding fields; Parallel Imaging; Radial Sampling; Constrained Image Reconstruction; Total Generalized Variation

Introduction

Signal encoding with nonlinear magnetic fields has recently raised increased interest in MRI research, notably PatLoc imaging (1–3) or O-Space imaging (4–6). These approaches allow to overcome physiological limitations of the gradient system, make more efficient use of the

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gradients, provide the possibility to design optimal gradient shapes which complement particular receiver array coils for parallel imaging and improves applications that involve nonlinear phase preparation (7). It was recently demonstrated that radial sampling can be generalized to imaging with nonlinear non-bijective spatial encoding magnetic fields (NB-SEMs) (8). Radial trajectories offer a number of important benefits like low sensitivity to object motion, inherent autocalibration for parallel imaging due to oversampling of k -space or robustness to mild undersampling. While these advantages are general and independent of the SEMs used, there are additional benefits that arise specifically in the context of PatLoc imaging. It was shown in (8) that the point spread function (PSF) of radial PatLoc imaging is much more localized in comparison to the Cartesian case. For quadrupolar SEM-encoding, this significantly reduces the spreading of destructive Gibbs ringing artifacts originating from signal relocalization at the center of the FOV, where the encoding fields become flat.

One of the most interesting properties of radial sampling is the potential for accelerated imaging. When using conventional linear gradients, radial undersampling leads to characteristic streaking artifacts. Strategies to reconstruct highly undersampled radial data sets were recently introduced by exploiting temporal redundancies in dynamic data (9–11), in the context of compressed sensing (12), parallel imaging (13, 14) and variationally constrained image reconstruction (15–17).

A general theory of linear PatLoc reconstruction was presented in (2). This approach is based on solving an equation system for the PatLoc reconstruction matrix. The drawback of this method is that it is often extremely challenging from a computational point of view. An accelerated direct approach for radial sampling was recently presented in (8), but it could only be applied for fully sampled acquisitions, and a prominent star-shaped artifact has been observed if a spin distribution is encoded with pure quadrupolar SEMs and an undersampled trajectory. This artifact is particularly problematic if the images are intermediately reconstructed onto the Fourier-domain of the PatLoc k -space (termed PatLoc-encoding space in (2)).

In this work, we show that undersampled radial PatLoc data sets can be reconstructed with the use of variational $L1$ constrained reconstruction, which has already proven effective in accelerated radial imaging with linear encoding fields (15–17). As variational constraints are based on image models, they have to be applied to reconstructed images in image space. The main reason for this is the necessary evaluations of image derivatives, which can only be performed after rewarping the reconstructions from encoding space (2) to image space. Therefore, it will be shown that the PatLoc forward operator can be described by means of a non-uniform Fourier transform (NUFFT). With this approach, a sampling operator (and its adjoint) can be defined, that maps data directly from radial PatLoc k -space to image space. With this operator, it is also possible to implement an adapted iterative SENSE method (18) for PatLoc reconstruction, which leads to a tremendous reduction of computation time in comparison to conventional PatLoc image reconstruction methods.

THEORY

Review of image reconstruction for PatLoc

The relationship between MR measurement data g to the underlying image u can be described as

$$\mathcal{F}(u)=g. \quad [1]$$

In equation 1, $\mathcal{F}$ is the so called forward or measurement operator. In conventional MRI with linear gradient fields, this operator consists of a multiplication with the corresponding coil sensitivities followed by a Fourier transform and Cartesian sampling of the $\mathbf{k}$ -space.

In PatLoc, this operator is generalized to account for NB-SEMs. Given the coil sensitivities $c_1, \dots, c_N$, the Cartesian PatLoc sampling operator reads as (2)

$$\mathcal{F}u=(\mathcal{F}_1u, \dots, \mathcal{F}_Nu), \quad \mathcal{F}_nu(\mathbf{k})=\frac{1}{2\pi} \int_{\mathbf{R}^2} c_n(\mathbf{x})u(\mathbf{x})e^{-i\Psi(\mathbf{x})\cdot\mathbf{k}} d\mathbf{x}. \quad [2]$$

In this equation, the components of the encoding mapping $\Psi : \Omega_{\Psi} \rightarrow \mathbf{R}^2$ (with $\mathbf{R}$ denoting the real numbers) are proportional to the SEM sensitivities, and the $\mathbf{k}$ -space variable $\mathbf{k}$ is proportional to the integrated coil currents. Whereas in standard imaging, the encoding mapping simply describes the identity, i.e., $\Psi : (x_1, x_2) \mapsto (x_1, x_2)$, we focus in this paper on arbitrary mappings Ψ . This could, for instance, be a set of orthogonal quadrupolar SEMs with $\Psi_1(x_1, x_2)=x_1^2 - x_2^2$ and $\Psi_2(x_1, x_2)=2x_1x_2$ with an additional rotation by 22.5 degrees, which correspond to those of a custom built gradient insert (19), and are shown in Fig. 1. Let us stress, however, that multi-dimensional trajectories (3) are not considered; it is assumed that only two NB-SEMs are involved.

On the other hand, it is not necessary to perform Cartesian sampling of the $\mathbf{k}$ -space: For linear encoding gradients, this leads to the well known problem of image reconstruction from non-Cartesian sampling trajectories. The usual solution of this problem is regridding the data on an underlying Cartesian grid (20, 21), followed by an inverse FFT, which is also known as the non-uniform Fourier transform (NUFFT) (22, 23). Recently, iterative reconstruction methods, e.g. by using the total variation as an image model for regularization, were also used successfully in the case of radial sampling (15). In this case, the signal equation for arbitrary sampling strategies in two dimensions and linear gradient fields can be written as

$$\mathcal{F}u=(\mathcal{F}_1u, \dots, \mathcal{F}_nu), \quad \mathcal{F}_nu(r, \phi)=\frac{1}{2\pi} \int_{\mathbf{R}^2} c_n(\mathbf{x})u(\mathbf{x})e^{-i\mathbf{x}\cdot\Phi(r, \phi)} d\mathbf{x}. \quad [3]$$

Here, we assume that the sampled $\mathbf{k}$ -space coordinates can be parametrized by the two variables r and ϕ where r is interpreted as a common time coordinate within a single trajectory and ϕ is the coordinate which determines the trajectory within the set of all

trajectories. This parametrization is denoted by $\Phi : \Omega_\Phi \rightarrow \mathbf{R}^2$. For instance, radial sampling corresponds to setting $\Omega_\Phi = [-R, R] \times [-\pi/2, \pi/2)$ and $\Phi(r, \phi) = (r \cos \phi, r \sin \phi)$.

In this paper, the two above-mentioned frameworks are combined, i.e., we consider two-dimensional PatLoc $\mathbf{k}$ -spaces with arbitrary sampling. The corresponding non-uniform PatLoc sampling operator then reads as

$$\mathcal{F}u = (\mathcal{F}_1 u, \dots, \mathcal{F}_N u), \quad \mathcal{F}_n u(r, \phi) = \frac{1}{2\pi} \int_{\mathbf{R}^2} c_n(\mathbf{x}) u(\mathbf{x}) e^{-i\Psi(\mathbf{x}) \cdot \Phi(r, \phi)} d\mathbf{x}. \quad [4]$$

It covers, in particular, standard Cartesian sampling (2) as well as radial sampling (8) in PatLoc imaging.

The usage of arbitrary encoding fields as well as arbitrary $\mathbf{k}$ -space sampling strategies have the consequence that gridding generally yields a distorted, intensity modulated and aliased representation of the image. In (8) an efficient direct reconstruction algorithm is presented that works very well for quadrupolar fields and densely sampled radial $\mathbf{k}$ -space data. However, when the acquisition is accelerated by reducing the number of radial spokes, a prominent star-shaped artifact appears. An alternative to direct image reconstruction is to indirectly estimate an image u that fits the measured $\mathbf{k}$ -space data g . This can be achieved by minimizing the L^2 norm of the residuum:

$$\min_u \| \mathcal{F}(u) - g \|_2^2. \quad [5]$$

A solution for u is usually obtained by iterative approaches like the conjugate-gradient (CG) method, which is the basis for the widespread iterative SENSE method (18), and which has already been adapted successfully to PatLoc imaging (3, 24). In (8), it was shown that the iterative CG-method indeed reduces the star-shaped artifact, without however, eliminating it. In this paper, we show that iterative image reconstruction in combination with an appropriate regularization term effectively suppresses the artifact. Another problem with the CG-method is that evaluation of $\mathcal{F}$ and its adjoint operation is extremely slow if not accelerated, as done in iterative SENSE. In the next section we present an algorithm that speeds up the CG-method by several orders of magnitude.

Efficient evaluation of the PatLoc sampling operator

The approach in Equation 5 is very general and can, in principle, be used with any $\mathbf{k}$ -space trajectory and arbitrary SEM fields by adjusting the forward sampling operator $\mathcal{F}$ according to Eq. 4. In view of Eq. 3, i.e., encoding with linear gradients and non-uniform sampling, it is common to choose $\mathcal{F}$ as the multiplication with the coil sensitivity profile and a 2D non-uniform FFT (Type-2 NUFFT) (22) mapping data on a Cartesian grid to the Fourier transform evaluated at the sampling trajectories. However, for regularly sampled PatLoc imaging with two SEMs according to Eq. 2, i.e., encoding with non-linear gradients and evaluation of the data on a Cartesian grid (1, 2), the PatLoc sampling operator is not a conventional Fourier transform anymore.

For the efficient numerical realization of a discrete version, it is useful to first consider the continuous setting. There, the appropriate generalization of matrices as well as matrix-vector multiplication is given by the concept of linear operators between function spaces and the application of such an operator to a function, respectively. Also, matrix-matrix multiplication is reflected by the composition of linear operators which we will denote by $\circ$. Our aim is to decompose each component $\mathcal{F}_n$ of the PatLoc operator according to Eq. 4 into simpler linear operators which, eventually, can be evaluated in a fast manner. We need some Hilbert space theory for this purpose. First, introduce the Hilbert space of complex square-integrable functions on a set Ω with standard scalar product which is given by

$$L^2(\Omega) = \{f: \Omega \rightarrow \mathbb{C} \mid \int_{\Omega} |f|^2 dx < \infty\}, \quad \langle f_1, f_2 \rangle = \int_{\Omega} f_1 \overline{f_2} dx$$

(with $\mathbb{C}$ denoting the complex numbers). These spaces allow, in particular, the generalization of the adjoint of a matrix to linear operators. For $A: L^2(\Omega_1) \rightarrow L^2(\Omega_2)$ linear and continuous, the Hilbert space adjoint operator $A^*: L^2(\Omega_2) \rightarrow L^2(\Omega_1)$ is the linear and continuous operator which satisfies

$$\langle f_1, A^* f_2 \rangle = \langle A f_1, f_2 \rangle \quad \text{for all } f_1 \in L^2(\Omega_1) \text{ and } f_2 \in L^2(\Omega_2).$$

This requirement uniquely determines A^* . A well-known example for the application of the Hilbert space framework is given by the Fourier transform. One can show that the 2D Fourier transform operator

$$\text{FT}(v_n)(\mathbf{k}) = \hat{v}_n(\mathbf{k}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} v_n(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{k}} d\mathbf{x}$$

maps $\text{FT}: L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ isometrically, i.e.,

$$\int_{\mathbb{R}^2} |\hat{v}_n|^2 d\mathbf{k} = \int_{\mathbb{R}^2} |v_n|^2 d\mathbf{x} \quad \text{for all } v_n \in L^2(\mathbb{R}^2).$$

Hilbert space theory then tells us that the inverse coincides with the adjoint, i.e., $\text{FT}^{-1} = \text{FT}^*$. We are moreover interested in the reparametrization operator associated with PatLoc encoding, i.e.,

$$R_{\Psi}: L^2(\mathbb{R}^2) \rightarrow L^2(\Omega_{\Psi}), \quad (R_{\Psi} v_n)(\mathbf{x}) = v_n(\Psi(\mathbf{x})), \quad [6]$$

and, especially, in its Hilbert space adjoint. As we will see in the following, this corresponds to the so-called “warping”. Denoting by J_{Ψ} the Jacobian, i.e., $J_{\Psi}(\mathbf{x}) = |\det \nabla \Psi(\mathbf{x})|$, the latter can be seen to read as

$$(R_{\Psi}^* u_n)(y) = \sum_{\mathbf{x} \in \Psi^{-1}(\{y\})} J_{\Psi}(\mathbf{x})^{-1} u_n(\mathbf{x}) \quad [7]$$

for each point $y \in \mathbf{R}^2$ in the transformed space, since, by the change of coordinate formula, we have

$$\begin{aligned} \langle v_n, R_\Psi^* u_n \rangle &= \langle R_\Psi v, u_n \rangle = \int_{\Omega_\Psi} v_n(\Psi(x)) \overline{u_n(x)} dx \\ &= \int_{\Omega_\Psi} J_\Psi(x) v_n(\Psi(x)) J_\Psi(x)^{-1} u_n(x) dx = \int_{\mathbf{R}^2} v_n(y) \sum_{x \in \Psi^{-1}(\{y\})} \overline{J_\Psi(x)^{-1} u_n(x)} dy \end{aligned}$$

for each modulated Cartesian image $u_n \in L^2(\Omega_\Psi)$ and each $v_n \in L^2(\mathbf{R}^2)$. Thus, in case of a bijective Ψ , the adjoint of a reparametrization operator corresponds to the inverse reparametrization of Ψ weighted with the Jacobian. If Ψ is non-bijective, for instance, for orthogonal quadrupolar SEMs, the data associated with all points Ω_Ψ which are mapped, via Ψ , to a single $y \in \mathbf{R}^2$ are summed up. This phenomenon is called “warping” in (1, 2). Adapting this terminology, we will also phrase $v_n \in L^2(\mathbf{R}^2)$ “warped” image.

Note that with these terms, the non-uniform Fourier transform is then nothing else than the composition of the Fourier transform and a reparametrization operator. We need the following two:

$$\text{NFT}_{-\Psi} = R_{-\Psi} \circ \text{FT}, \quad \text{NFT}_\Phi = R_\Phi \circ \text{FT}. \quad [8]$$

As we will see, $\text{NFT}_{-\Psi}$ accounts for the non-linear SEM fields while NFT_Φ accounts for the non-Cartesian sampling.

The crucial observation is now that the Hilbert space adjoint $\text{NFT}_{-\Psi}^*$ can be written as follows:

$$(\text{NFT}_{-\Psi}^* u_n)(\mathbf{k}) = \frac{1}{2\pi} \int_{\mathbf{R}^2} u_n(x) e^{-i\Psi(x) \cdot \mathbf{k}} dx$$

since we have

$$\begin{aligned} \langle \hat{v}_n, \text{NFT}_{-\Psi}^* u_n \rangle &= \langle \text{NFT}_{-\Psi} \hat{v}_n, u_n \rangle = \frac{1}{2\pi} \int_{\Omega_\Psi} \int_{\mathbf{R}^2} \hat{v}_n(\mathbf{k}) e^{i\mathbf{k} \cdot \Psi(x)} d\mathbf{k} \overline{u_n(x)} dx \\ &= \int_{\mathbf{R}^2} \frac{1}{2\pi} \int_{\Omega_\Psi} \overline{u_n(x) e^{-i\mathbf{k} \cdot \Psi(x)}} dx \hat{v}_n(\mathbf{k}) d\mathbf{k} \end{aligned}$$

for each modulated Cartesian image $u_n \in L^2(\Omega_\Psi)$ and Fourier transformed “warped” image $\hat{v}_n \in L^2(\mathbf{R}^2)$. In order to obtain $\mathcal{F}_n$ in the PatLoc operator in Eq. 4, we have to apply $\text{NFT}_{-\Psi}^*$ to $u_n = c_n u$ and evaluate the result at $\Phi(r, \phi)$ instead of k . The modulation operation can be expressed by

$$M_n: L^2(\Omega_\Psi) \rightarrow L^2(\Omega_\Psi), \quad M_n u = c_n u,$$

while the evaluation at $\Phi(r, \phi)$ is given by the reparametrization operator R_Φ according to Eq. 6. Hence,

$$\mathcal{F}_n = R_\Phi \circ \text{NFT}_{-\Psi}^* \circ M_n.$$

and writing $R_\Phi = R_\Phi \circ \text{FT} \circ \text{FT}^{-1}$, we finally arrive at the decomposition

$$\mathcal{F}_n = \text{NFT}_\Phi \circ \text{FT}^{-1} \circ \text{NFT}_{-\Psi}^* \circ M_n \quad [9]$$

which can, after discretization, efficiently and accurately be evaluated with the help of the non-uniform fast Fourier transforms.

Before discussing discretization issues, let us remark that with the help of the identities $\text{FT}^* = \text{FT}^{-1}$, $\text{FT}^2 = R_{\text{id}}$ and Eq. 8, $\text{NFT}_{-\Psi}^*$ can also be interpreted as

$$\text{NFT}_{-\Psi}^* = \text{FT}^* \circ R_{-\Psi}^* = \text{FT}^{-1} \circ R_{\text{id}} \circ R_\Psi^* = \text{FT} \circ R_\Psi^*,$$

showing that “warping”, i.e., the inverse Jacobian-weighted reparametrization R_Ψ^* according to Eq. 7, is implicitly contained in the adjoint. This leads to the decomposition

$$\mathcal{F}_n = \text{NFT}_\Phi \circ R_\Psi^* \circ M_n, \quad [10]$$

i.e., modulation followed by “warping” and the non-uniform Fourier transform with respect to Φ . In (8), a two-step inversion of $\mathcal{F}$ which bases on this decomposition is mentioned: First, the least-squares problem with respect to NFT_Φ is solved with a CG method to yield “warped” data. Second, the operator $u \mapsto (R_\Psi^* M_1 u, \dots, R_\Psi^* M_N u)$ is inverted explicitly, which is possible thanks to its simple structure. We will refer to this method as “CG on warped space”, see the Materials and Methods section for details. In contrast to that, the other approaches presented in this paper consider each $\mathcal{F}_n$ as a joint operator and only decompose it to perform fast evaluation in terms of the non-uniform fast Fourier transform.

The discretization of each $\mathcal{F}_n$ is now done by discretizing each of its components in Eq. 9. Therefore, for the sake of readability, we will drop, in what follows, the index n from u_n , v_n and $\hat{v}_n$. First, on a $K \times K$ Cartesian grid, $K = 2$ a power of 2, the discrete version of the 2D Fourier transform and its inverse read as

$$(\text{DFT}v)_{k_1, k_2} = \frac{1}{K} \sum_{x_1, x_2 = -\frac{K}{2}}^{\frac{K}{2}-1} v_{x_1, x_2} e^{-\frac{2\pi i}{K}(x_1 k_1 + x_2 k_2)},$$

$$(\text{DFT}^{-1}\hat{v})_{x_1, x_2} = \frac{1}{K} \sum_{k_1, k_2 = -\frac{K}{2}}^{\frac{K}{2}-1} \hat{v}_{k_1, k_2} e^{\frac{2\pi i}{K}(x_1 k_1 + x_2 k_2)}$$

for $k_1, k_2, x_1, x_2 \in \left\{-\frac{K}{2}, \dots, \frac{K}{2} - 1\right\}$. It is known that both operators can be applied with $\mathcal{O}((K \log K)^2)$ -complexity via the fast Fourier transform (FFT) and inverse fast Fourier transform (IFFT) (25). Next, we discretize NFT_{Ψ} and its adjoint. The PatLoc image domain Ω_{Ψ} is then represented by a $L \times L$ grid on which the fields $\Psi = (\Psi_1, \Psi_2)$ are given and scaled in a compatible way, i.e., $(\Psi_1)_{l_1, l_2}, (\Psi_2)_{l_1, l_2} \in \left[-\frac{K}{2}, \frac{K}{2}\right)$ for each $0 \leq l_1, l_2 < L$. The discrete version of NFT_{Ψ}^* then read as

$$(\text{DNFT}_{\Psi}^* u)_{k_1, k_2} = \frac{1}{K} \sum_{l_1, l_2=0}^{L-1} u_{l_1, l_2} e^{-\frac{2\pi i}{K}((\Psi_1)_{l_1, l_2} k_1 + (\Psi_2)_{l_1, l_2} k_2)}$$

for $l_1, l_2 \in \{0, \dots, L-1\}$. This operator can efficiently be approximated by a type-1 non-uniform fast Fourier transform (Type-1 NUFFT) with complexity $\mathcal{O}((K \log K)^2 + L^2)$, see (22, 23). Finally, for the discretization of NFT_{Φ}^* let $J-1$ and Φ evaluated at J sampling

points of Ω_{Φ} be given, i.e., $(\Phi_r)_j, (\Phi_{\phi})_j \in \left[-\frac{K}{2}, \frac{K}{2}\right)$ for $j = 0, \dots, J-1$. This leads to the discretization

$$(\text{DNFT}_{\Phi} v)_j = \frac{1}{K} \sum_{x_1, x_2 = -\frac{K}{2}}^{\frac{K}{2}-1} v_{x_1, x_2} e^{-\frac{2\pi i}{K}(x_1(\Phi_r)_j + x_2(\Phi_{\phi})_j)}$$

for $j \in \{0, \dots, J-1\}$. An efficient approximation of this operator is realized by the Type-2 NUFFT, the computational complexity corresponds to $\mathcal{O}((K \log K)^2 + J)$. Finally, the multiplication M_n with discrete coil sensitivities can be done in a straightforward manner with $\mathcal{O}(L^2)$ complexity. This results in channelwise discrete operators according to

$$\text{D}\mathcal{F}_n = \text{DNFT}_{\Phi} \circ \text{DFT}^{-1} \circ \text{DNFT}_{\Psi}^* \circ M_n. \quad [11]$$

Finally, these operators are assembled channelwise to $\text{D}\mathcal{F} = (\text{D}\mathcal{F}_1, \dots, \text{D}\mathcal{F}_N)$ in order to give the full discrete non-uniform PatLoc operator. Let us remark that the above combination of the Type-1 and Type-2 NUFFT operators in particular leads to an approximation of the Type-3 NUFFT which realizes the Fourier transform when the points are non-uniform in the spatial and the frequency domain (26–28). It can therefore be regarded as an easy way to “emulate” the Type-3 NUFFT in terms of the Type-1 and Type-2 NUFFT, we will term this approximation “1+2 NUFFT” for the rest of the paper. The steps involved in the evaluation of the discrete operator (equation Eq. 11) as well as the relations to the reparametrization and the NUFFT operators are illustrated in Fig. 2.

In our implementations, we choose $2L \leq K \leq 4L$, so the overall complexity of the discrete operator can be estimated by $\mathcal{O}(N((L \log L)^2 + J))$. The same holds true for its adjoint which is also utilized in the numerical solution strategies. In contrast to this, a straightforward

discretization of Eq. 4 leads to a complexity of $\mathcal{O}(N^2L^2)$ which is observed to be less efficient in actual implementations.

Constrained image reconstruction with TGV²

Due to ill-conditioning, the solution of equation 5 is usually stabilized with a regularization term $\mathcal{R}$:

$$\min_u \| \mathcal{F}(u) - g \|_2^2 + \lambda \mathcal{R}(u). \quad [12]$$

This term is a penalty that introduces a priori information about the structure of the imaged object, and $\lambda > 0$ is a regularization parameter. An often used approach is Tikhonov regularization (29). Recently, L^1 -norm based regularization methods have gained increasing interest and total variation (TV) (15,30) or wavelets (12,31) were already applied successfully to MR image reconstruction. A constraint that is based on the second order total generalized variation (TGV²) was used in this work, as it was already used successfully for radial imaging with linear encoding fields (16, 17). It is given by:

$$\mathcal{R}(u) = \min_w \| \nabla u - w \|_1 + 2 \| \varepsilon w \|_1, \quad [13]$$

where $\varepsilon w = \frac{1}{2} (\nabla w + \nabla w^T)$ is the symmetrized gradient of a complex-valued vector field w . While this approach is comparable to TV in terms of noise removal and edge preservation, it is a more generalized image model because it is capable of measuring image characteristics up to a certain order of differentiation. Therefore it can also be applied when the reconstructed image includes regions with smooth signal changes, a situation that violates the assumption of piecewise constancy in conventional TV and leads to the introduction of artificial staircasing artifacts. A detailed description of the TGV² penalty is given in (32), and the application for MRI as well as a detailed comparison to conventional TV is presented in (16). In particular, an algorithm for the numerical minimization of Eq. 12 for general $\mathcal{F}$ can be found in this paper. In the next section, a brief description of the implementation of this algorithm with respect to PatLoc is given.

MATERIALS AND METHODS

Image reconstruction methods

In the experiments of this work, four different reconstruction strategies were used, which are described in this section. All calculations were performed on a conventional desktop PC (Intel Core i7 870 2.93 GHz and 12GB of random access memory). Three of them, which are introduced in this paper, are illustrated in Fig. 3.

Iterative inversion of the encoding matrix with the conjugate gradient method

The most general way of image reconstruction for PatLoc is to invert the encoding matrix. The theoretical basis of this approach is described in (2), and it was also applied for 4D-RIO

trajectories (3) and for radial sampling (8). As described in the theory section, the forward operator $\mathcal{F}$ only consists of linear operations and can therefore be written in matrix form. It is then usually called the encoding matrix. The task of image reconstruction is now the inversion of this matrix. As this matrix can become very large the conjugate gradient (CG) method is usually used instead of a direct inversion. In principle, this can be compared to the well known iterative SENSE method (18). An advantage of this approach is the implicit regularization. Stopping after a certain number of CG iterations allows to trade off between bias and spatial resolution (too few) and noise amplification (too many iterations). The main drawback of this reconstruction strategy is that the encoding matrix can often become too large for storage in a conventional desktop PC. For 128×128 data points with 8 coils and reconstruction on a 128×128 grid, the size of the matrix is ≈ 17 GB.

In the more realistic case of a 256×256 acquisition reconstructed to a 512×512 grid (the grid size that used in the in-vivo experiments of this work) the size matrix size even rises to 2.2 TB. This problem can be overcome, for instance, by recomputing the matrix entries every time a matrix/vector multiplication is performed. This leads, however, to a tremendous increase of reconstruction time.

CG-SENSE on the warped space using Type 1 and Type 2 NUFFT operators

As it was already pointed out in the Theory section and also mentioned in (8), the forward operators $\mathcal{F}_n$ obey Eq. 10 which leads to the signal equation

$$g_n = (\text{NFT}_\Phi \circ R_\Psi^* \circ M_n)u, \quad n=1, \dots, N. \quad [14]$$

Introducing the warped modulated images $v_n = R_\Psi^* M_n u$, this equation becomes

$$g_n = \text{NFT}_\Phi v_n, \quad n=1, \dots, N \quad [15]$$

which means that each acquired signal g_n is sampled from v_n as in conventional encoding with linear gradients. After discretization of equation 15, the images v_n can therefore be reconstructed with the conventional CG-SENSE (18) method, using min-max interpolation Type 1 (regridding) and Type 2 (inverse gridding) NUFFT operators (22). The output of this CG-SENSE algorithm can, in the case of two quadrupolar SEMs, easily be converted to two un-aliased images in the warped coordinate system (2), which are then reparametrized back to conventional Cartesian coordinates as the final step of the reconstruction pipeline. As fast NUFFT operators are used during the CG iterations, this approach allows a major speedup of reconstruction time in comparison to the inversion of the encoding matrix.

CG-SENSE using the NUFFT 1+2 operator without regularization

As the regularization with TGV^2 depends on Cartesian derivatives, see Eq. 13, TGV^2 constrained reconstruction cannot be performed on the “warped space”. We will therefore use the full operator involving the 1+2 NUFFT according to Eq. 10 in the reconstruction procedure.

To evaluate the effects of this sampling operator on the reconstructed images, and to separate them from those of the TGV^2 regularization, a CG-SENSE method without regularization was implemented for the proposed NUFFT 1+2 operator. Since the “warping” operation is implicitly integrated in the operator, the output of this algorithm is un-aliased image in a conventional Cartesian coordinate system.

TGV²-regularized reconstruction with a primal-dual-method using the NUFFT 1+2 operator

Due to the TGV^2 penalty, equation 12 is a non-smooth optimization problem, and it was shown in (16) that it can be solved with a projected primal-dual (PD) extragradient method (33,34). The gradient operators ∇ and ε were implemented using finite differences. Details about the algorithm can be found in (16). In addition, code for TGV^2 based image reconstruction can be downloaded from the webpage of our institution¹. In each iteration of the PD algorithm, the NUFFT 1+2 operator is used to map the data between PatLoc k -space (for the evaluation of the data fidelity) and Cartesian image space (for the evaluation of the derivatives). The output of this algorithm is an un-aliased, regularized image in a conventional Cartesian coordinate system.

Simulations

The properties of the different reconstruction methods were first investigated using simulations, which were carried out with Matlab (R2011a, The MathWorks Inc., Natick, MA, USA).

The first experiment was an analysis of the PSF. A zero-matrix (size $N \times N = 256 \times 256$) was created with a single impulse of value 1 at the central position $[\frac{N}{2}, \frac{N}{2}]$. Another run of calculations was performed for an impulse at a more peripheral location $[\frac{N}{2} + \frac{N}{8}, \frac{N}{2} + \frac{N}{8}]$. A radial sampling trajectory was designed with 256 and 32 radial projections, each consisting of 256 data points. Coil sensitivity profiles were obtained from measured sensitivities of an 8 channel head coil and were processed based on the method presented in (35). Orthogonal quadrupolar SEMs were used, as in line with (2). Based on this, the forward sampling operator $\mathcal{F}$ was constructed as described in the theory section, and synthetic radial PatLoc k -space data were generated using this operator. Images were reconstructed on a 256×256 grid. As the concept of PSFs can only be applied for linear reconstruction methods, results of this experiment are only shown for reconstructions without TGV^2 regularization (inversion of the encoding matrix, CG on the warped space and CG with the NUFFT 1+2 operator). Due to the extensive requirements in terms of computation time and memory, the inversion of the encoding matrix was only performed for this experiment.

For the next experiment, the well known Shepp-Logan phantom (matrix 256×256) was used. The original phantom only consists of regions with constant signal intensities, which is a best case scenario for bounded variation methods and might lead to overly-optimistic results for the proposed TGV^2 approach. Therefore an additional smooth signal modulation was applied to the phantom to make the reconstruction more realistic. The modified phantom is shown in Fig. 4. Synthetic k -space data were generated in the same way as for

¹<http://www.imt.tugraz.at/research/mr-image-reconstruction>

the PSF experiments, using 128, 64, 32 and 16 radial projections, each with 256 data points. Due to the Nyquist criterion for radial sampling which states that $\frac{\pi}{2}n$ projections (402 for $n = 256$) have to be measured to obtain a fully sampled data set (36) for the reconstruction of an $n \times n$ matrix, this corresponds to undersampling of approximately 3, 6, 12 and 25. The behavior with respect to different levels of SNR was then investigated by adding Gaussian distributed noise to the real and imaginary parts of the k -space data from each receiver channel. Images were reconstructed using CG on the warped space, CG with the NUFFT 1+2, and with the TGV² regularized PD approach.

In-vivo experiments

Further experiments were performed with in-vivo measurements on a 3T Tim Trio system (Siemens Healthcare, Erlangen, Germany) with a modified encoding hardware for brain imaging using PatLoc coils that generate two orthogonal quadrupolar SEMs (37). Ethics approval and written consent were obtained prior to the experiment and measurements were carried out according to safety considerations for in-vivo PatLoc imaging (38). A conventional 8 channel receive coil was used, and sensitivity profiles were post-processed based on (35). Spin-echo images with TR=500ms and TE=11ms were acquired with a conventional radial sequence, where the generated currents were applied to the PatLoc coils instead of the standard gradient coils. 410 Projections with 256 samples along each readout were acquired with alternating directions between successive projections to distribute the effects of trajectory inaccuracies more evenly. A gold standard reference reconstruction of the fully sampled data was performed with iterative inversion of the encoding matrix and is shown in Fig.4. An undersampled dataset was obtained by taking a reduced number of projections from the full dataset and images were reconstructed on a 512×512 grid. Identical to the case of the simulations using the Shepp-Logan phantom, CG on the warped space, CG with the NUFFT 1+2 and PD TGV² were used for image reconstruction.

RESULTS

Figure 5 shows the results of the PSF analysis for 256 and 32 projections. It can immediately be observed that resolution is improved when moving away from the center of the FOV, which is a fundamental feature of PatLoc imaging with quadrupolar SEMs (1), and is of course independent of used image reconstruction strategy. The evaluation of the non-uniform DFT serves as a gold standard against which the other approaches can be compared. As described in (8), a signal at the center leads to pronounced Gibbs ringing which results in streaks originating from the center and going to the edges of the image. While an iterative reconstruction already mitigates this effect in comparison to a direct method (8), this effect is still amplified significantly in the case of CG on the warped space when reconstructing from 32 projections. In contrast, the PSFs are similar for 256 and 32 projections in the case of the NUFFT 1+2 operator, indicating improved performance and robustness for undersampled data. It can also be observed that in contrast to the gold standard evaluation of the DFT, PSFs include 8 additional streaks originating from the center of the FOV. This artifact is introduced because of the intermediate regridding to a Cartesian k -space, an effect that is known from Cartesian PatLoc imaging (2). If the signal is moved away from the center, the performance of the two methods is very similar. Note, however, that for CG on

the warped space, not only the off-center peak is reconstructed, but also a smaller peak located opposite to the center. This is due to the “aliasing” introduced by the non-bijective SEM fields which map the two peak positions to the same point in the “warped” space (2, 8). Both methods resolve the aliasing by incorporating, in terms of the forward operator, information from the coil sensitivity profiles in the inversion process. As one can see, the CG NUFFT 1+2 method is also performing better in suppressing this artificial peak at the cost of slightly more Gibbs artifacts near the boundary of the FOV.

The experiments with the modified Shepp-Logan phantom are shown in Fig. 6 for the ideal noiseless case and four different levels of undersampling. CG on the warped space allows reconstructions of moderately undersampled data with 128 projections without introducing streaking artifacts. In the case of higher undersampling, artifacts as described in (8) start to appear in the peripheral parts of the FOV and get increasingly worse as the number of projections is reduced. Additionally, all reconstructions show artificial circular Gibbs ringing structures at the center of the image (8). The circular structures are caused by the side-lobes of the central PSF that dominates the PSF from neighboring pixels. The dominance is caused by the weak encoding fields at the center of the FOV with the consequence of prominent signal accumulation at the center and far-reaching signal contamination which compromises image quality in the nearby region. This completely obscures details at the center of the image, e.g. the two small circular structures at the center of the Shepp-Logan phantom. Both artifacts are mitigated with the use of the NUFFT 1+2 operator, although residual artifacts can still be observed in the cases of high undersampling. With the TGV² constrained method, artifact free images can be obtained even in the extreme case of 16 projections.

Fig. 7 displays the results of the experiments where complex Gaussian noise was added to the raw data of each channel such that the ratios of the standard deviations of the noise to the norm of the data were 1/20 and 1/10.

Finally, results from undersampled in-vivo radial PatLoc measurements are shown in Fig. 8. Images were reconstructed using 103, 52, 35 and 26 radial projections, corresponding to undersampling factors of ≈ 3.9 , ≈ 7.7 , ≈ 11.5 and ≈ 15.5 in comparison to a fully sampled radial scan (Fig. 4). The results are consistent with the simulations and show pronounced streaks for CG on the warped space at higher accelerations and corruption of the central part of the FOV due to signal relocalization. Both effects are reduced significantly when using the NUFFT 1+2 operator, and again, reconstructions without residual aliasing can only be obtained with TGV² regularization at the highest undersampling rates. It can also be observed that the radius of the artifact at the center of the FOV can be reduced with TGV² in comparison to the two CG based methods.

Table 1 shows a comparison of the reconstruction times for CG on the warped space and CG with the NUFFT 1+2 operator. Results are shown for simulations with 128 and 16 projections (Fig. 6) and the in-vivo measurements with 103 and 26 projections (Fig. 8). Note that the two data sets are reconstructed to different image grid sizes. Computation times are given for one CG iteration and the separate reparametrization step of CG on the warped space that is performed after the final CG iteration. As described in the theory section, this reparametrization is inherent in the CG NUFFT 1+2 method. To reduce the effect of timing

inaccuracies, each experiment was repeated 100 times and the mean values of the computation times are given. The standard deviations of the 100 repetitions were between 0.03s (warped space CG iterations, in-vivo data set, 103 projections) and 0.15s (NUFFT 1+2 CG iterations, in-vivo data set, 26 projections).

DISCUSSION

The results demonstrate that radial imaging with two quadrupolar encoding fields and TGV² regularization yields reconstructions without residual aliasing artifacts, even at very high reduction factors. It should be noted here that in addition to the undersampling this is a very challenging imaging situation, as the SEMs do not provide any encoding at the center of the FOV. In the simulations, artifact-free reconstructions were obtained with as few as 16 projections, corresponding to an effective acceleration of more than 25 compared to a fully sampled radial trajectory. It must be noted that even with the additional smooth signal changes, the Shepp-Logan phantom is still well suited for bounded variation methods like TGV², which is the reason why it was possible to obtain higher acceleration factors than in the in-vivo case of Figure 8. No staircasing artifacts are visible in any of the reconstructions, in line with the results of (16). Although not eliminated, the radius of the ringing artifact due to signal relocalization at the center of the FOV is reduced significantly with the use of the NUFFT 1+2 operator and the TGV² regularized reconstruction (Figs. 6, 7, 8). This shows that in the context of PatLoc, TGV² is not only useful to eliminate artifacts resulting from undersampling, but also to enhance other properties of the PSF, thereby improving the overall quality of the reconstructed images.

For all CG based methods, 25 iterations were used, which already serves as implicit regularization. In the case of TGV², the value of the regularization parameter λ in equation 12 was set to $4 \cdot 10^{-5}$ for all experiments. This choice was made based on previous experiments with linear encoding fields (16). In line with the approach of that work, we did not perform individual tuning of the parameter for the different data sets. In principle, the regularization parameter should be chosen based on the noise level of the data. It is likely that the image quality of the TGV² could still be improved with individual parameter optimization. The reason why this was not done in this work is that such a tuning of the parameter is often impractical to conduct during practical application. Therefore our choice of a “default” parameter illustrates the practical applicability of TGV² based image reconstruction to a wide range of data sets.

In this study, the formulation of the PatLoc sampling operator using a NUFFT was based on the assumption of two independent SEMs and the experiments considered two quadrupolar fields, as proposed in the context of PatLoc imaging. In principle, the number of SEMs for image encoding defines the dimensionality of the NUFFT that is used to construct the sampling operator. We expect that the NUFFT 1+2 operator, as well as the TGV² constrained reconstruction, can also be extended to a higher number as well as more general types of SEMs, e.g. two pairs of third-order multipolar SEMs. Potential applications are O-space imaging, where two linear and one quadratic SEM are used (4–6), or 4D-RIO trajectories, utilizing two linear and two quadratic SEMs (3). Of course, the higher dimensionality of the NUFFT will increase the computational load as well as the memory

requirements to evaluate the forward operator, and it remains to be seen if such an approach can be implemented efficiently in order to be of actual use. However, the proposed framework allows the use of arbitrary SEMs, e.g. linear combinations of linear and quadratic fields, and it is the topic of future investigations to explore the encoding potential of two SEMs with such a structure, to which the proposed 2D NUFFT operator can be applied directly.

As illustrated by the computation times in Table 1, the drawback of the NUFFT 1+2 operator in comparison to CG on the warped space is the higher computational load. The reason is that the reparametrization step that transforms the reconstructed images from the warped coordinate system to final image space has to be performed in each iteration. In contrast, in the CG on the warped space method the reparametrization is only performed as the final step after the last iteration. Note that in this case the computation times of the CG iterations depend on the number of projections in the data set, not on the final reconstruction grid size. In contrast, the reparametrization step only depends on the grid size. As 25 CG iterations were performed in all experiments, this corresponds to absolute reconstruction times of approximately 12s for CG on the warped space against 65s for NUFFT 1+2 CG in the case of the phantom reconstruction from 128 projections. However, both approaches reduce the computation time by several orders of magnitude when compared with the iterative inversion of the encoding matrix. For comparison, a reconstruction of this 128 projections phantom data set took 5 hours with this approach. It is expected that a more efficient numerical implementation of the Type-3 NUFFT will lead to a further reduction of computation times in comparison to the NUFFT 1+2 approach, which is also a topic of future work.

CONCLUSIONS

It is demonstrated that nonlinear reconstruction methods based on the total generalized variation lead to pronounced improvements in image quality of undersampled radial acquisitions with nonlinear encoding fields, especially under difficult encoding conditions, including regions with vanishing SEM-gradients. The use of the NUFFT 1+2 sampling operator already enhances properties of the PSF when used in conventional CG based image reconstruction. This leads to a reduction of artifacts due to signal relocalization at the center of the FOV, where the PatLoc encoding gradients flatten out. In addition, the NUFFT based methods allow significant reductions in computation time when compared to the conventional conjugate gradient based inversion of the encoding matrix.

Acknowledgments

Grant sponsor: Austrian Science Fund; Grant number SFB F3209-18.

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Figure 1.

Nonlinear non-bijective spatial encoding magnetic fields $\Psi_1(x_1, x_2) = x_1^2 - x_2^2$ and $\Psi_2(x_1, x_2) = 2x_1x_2$ with an additional rotation by 22.5 degrees, approximations thereof were used in the experiments of this work.

Figure 2.

Flowchart of the discrete forward sampling operator $D\mathcal{F}_n$ for channel n according to Eq. 11. It is composed of multiplication with the coil sensitivity profile M_n and a Type-3 NUFFT which is “emulated” by the NUFFT 1+2 operator. The latter consists of the adjoint discrete non-uniform Fourier transform associated with quadrupolar SEM fields $DNFT_{-\Psi}^*$ (Type-1 NUFFT), the inverse discrete Fourier transform DFT^{-1} (IFFT) and the discrete non-uniform Fourier transform associated with radial sampling $DNFT_{\Phi}$ (Type-2 NUFFT).

Figure 3.

The three reconstruction methods that are introduced in this work: CG-SENSE on the warped space: Output of the CG algorithm is one un-aliased image in the warped coordinate system for each SEM. These are the reparametrized in a separate step, to yield a single image in Cartesian coordinates (top); CG-SENSE using NUFFT 1+2: Output of the CG algorithm is a single image in Cartesian coordinates, the reparametrization is included in the sampling operator (middle); Primal Dual TGV² using NUFFT 1+2: Output of the PD algorithm is a single, regularized image in Cartesian coordinates. TGV² regularization is applied in image space, in Cartesian coordinates (bottom).

Modified Shepp Logan phantom

In vivo 410 projections

Figure 4.

Shepp Logan phantom with additional smooth signal modulation that was used to generate synthetic undersampled radial PatLoc sampling data (left). Fully sampled in vivo radial PatLoc spin-echo measurement of the brain with 410 projections, reconstructed with iterative inversion of the encoding matrix (right).

Figure 5.

PSFs with signals placed at the center of the FOV (first, third and fifth row) and in the periphery (second, fourth and sixth row) from radial PatLoc data with 256 (left) and 32 (right) projections (matrix size 256×256). Iterative inversion of the encoding matrix (top rows), CG on the warped space (middle rows) and with the proposed NUFFFT 1+2 operator (bottom rows) are shown.

Figure 6. Results from CG on warped space (left), CG NUFFFT 1+2 (middle) and TGV² (right) constrained reconstructions of ideal noiseless radial PatLoc data generated from the modified shepp logan phantom (256×256) with 128, 64, 32 and 16 projections.

Figure 7. Results from CG on warped space (left), CG NUFFT 1+2 (middle) and TGV^2 (right) constrained reconstructions with two different levels of SNR. Reconstructions from 128 and 32 projections are shown.

Figure 8.

In-vivo radial PatLoc brain measurements with 103, 52, 35 and 26 projections. CG on warped space (left), CG with NUFFT 1+2 operator (middle) and TGV^2 (right) reconstructions are displayed.

Table 1

Comparison of the reconstruction times for the CG on the warped space method and CG with the NUFFT 1+2 operator. Results are shown for simulations with the Shepp-Logan phantom and the in-vivo measurements, each with different numbers of projections and with a different reconstruction grid size. Computation times are given for one CG iteration and the separate reparametrization step of the CG on the warped space method that is performed after the final CG iteration. In the CG NUFFT 1+2 method, the reparametrization is inherent. For comparison, a reconstruction of the 128 projections phantom data set took approximately 5 hours with conventional iterative inversion of the encoding matrix.

Data set	Projections	Grid	CG ref.	Repar. ref.	CG NUFFT 1+2
Shepp-Logan	128	256×256	0.4 s	1.9 s	2.6 s
Shepp-Logan	16	256×256	0.3 s	1.9 s	2.4 s
In-vivo	103	512×512	0.4 s	3.4 s	3.7 s
In-vivo	26	512×512	0.3 s	3.2 s	3.7 s