ON THE DIRECT DECOMPOSITION OF NILPOTENT EXPANDED GROUPS

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We decompose certain nilpotent expanded groups into a direct product such that the additive group of each factor is either a p -group or torsion-free.

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1. INTRODUCTION

We consider the direct decomposition of expanded groups. In group theory, a classical result tells that every finite nilpotent group is a direct product of p -groups. Although this result is usually proved using concepts pertinent to group theory, it has been generalized to other algebraic structures. For these generalizations, a condition on a general algebra that extends the group theoretic property of being nilpotent is needed. One concept that is often used for arbitrary algebraic structures comes from commutator theory [7, 14], which provides us with a binary commutator operation on the congruence lattice of every algebra. An algebra is defined to be *nilpotent* if the lower central series defined through this commutator operation reaches the trivial congruence after finitely many steps. (Some additional care in the definition is needed for those algebras with non-commutative commutator operation.) However, this concept of nilpotence does not lead us to a generalization of the direct decomposition theorem for nilpotent groups, since there are finite nilpotent expanded groups that fail to decompose into a product of algebras of prime power order. Nevertheless, there are different descriptions of nilpotent groups that provide more useful generalizations for our purpose. It is known that a finite group G is nilpotent of class k if and only if there is a polynomial p with real coefficients of degree k such that $\mathbf{F}_{\mathcal{V}(G)}(n)$, the free algebra over n generators in the variety generated by G , has precisely $2^{p(n)}$ elements. In [9], Kearnes proved that every finite nilpotent algebra A in a congruence modular variety such that $\log(|\mathbf{F}_{\mathcal{V}(A)}(n)|)$ is bounded from above by a polynomial in n is a direct product of algebras of prime power order. For finite expanded groups, this boundedness of the free spectrum can be characterized by the nonexistence of certain polynomial

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functions. We say that an n -ary polynomial p of the expanded group $\mathbf{A}$ is *absorbing* if $p(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in A^n$ with $0 \in \{x_1, \dots, x_n\}$. For $k \in \mathbb{N}$, we call $\mathbf{A}$ *supernilpotent of class k* if $\mathbf{A}$ has a k -ary nonconstant absorbing polynomial, and all l -ary absorbing polynomials, for all $l > k$, are constantly 0; a one element expanded group is defined to be supernilpotent of class 0. In Section 4, we will provide a modification of the argument that yields Eq. (1.2) of [8, p. 154] (cf. [5]) to show that a finite expanded group $\mathbf{A}$ is supernilpotent of class at most k if and only if there is $c > 0$ such that $\log(|F_{\mathcal{V}(\mathbf{A})}(n)|) \leq c n^k$ for all $n \in \mathbb{N}$. For groups, supernilpotence and nilpotence are equivalent concepts: in [1, Theorem 6.8], it is proved that every group of nilpotency class k is supernilpotent of class at most k . A converse implication holds for all expanded groups: by [4, Corollary 6.15], every supernilpotent expanded group of class k is nilpotent of class at most k . A direct argument will be provided in Section 4. Theorem 3.14 of [9] yields that every finite supernilpotent expanded group is a direct product of expanded groups of prime power order. Our goal is to provide a generalization of this result to infinite expanded groups (Theorem 2.1).

2. NOTATION AND RESULT

An algebraic structure $\mathbf{V} = \langle V, +, -, 0, f_1, f_2, \dots \rangle$ is called an *expanded group* if it has a binary basic operation $+$, a unary basic operation $-$, and a nullary basic operation 0 such that its reduct $\langle V, +, -, 0 \rangle$ is a (not necessarily abelian) group. Each congruence relation α of the expanded group $\mathbf{V}$ is determined by the set $A := 0/\alpha$ of those elements that are congruent to 0 modulo α ; the elements of the set $Id(\mathbf{V}) := \{0/\beta \mid \beta \text{ is a congruence of } \mathbf{V}\}$ are called the *ideals* of $\mathbf{V}$. With the operations $A \wedge B := A \cap B$, $A \vee B := A + B$, the set $Id(\mathbf{V})$ forms a modular lattice, called the *ideal lattice* of $\mathbf{V}$. For $A, B \in Id(\mathbf{V})$, we say $A < B$ if $A < B$ and there is no ideal C of $\mathbf{V}$ such that $A < C < B$.

In order to describe the structure of $\mathbf{V}$, we will make use of its polynomials. Here, an n -ary operation f on V is called a *polynomial of $\mathbf{V}$* if there is an $l \in \mathbb{N}$, elements $v_1, \dots, v_l \in V$ and a term t in the language of $\mathbf{V}$ such that $f(x_1, \dots, x_n) = t(v_1, \dots, v_l, x_1, \dots, x_n)$ for all $x_1, \dots, x_n \in V$. The set of all n -ary polynomials of $\mathbf{V}$ will be abbreviated by $\text{Pol}_n(\mathbf{V})$. An important structure associated with $\mathbf{V}$ is the near-ring of zero-preserving unary polynomials of $\mathbf{V}$. Let $P_0(\mathbf{V}) := \{p \in \text{Pol}_1(\mathbf{V}) \mid p(0) = 0\}$. Defining $(p + q)(v) := p(v) + q(v)$ and $(p \circ q)(v) := p(q(v))$, the structure $\langle P_0(\mathbf{V}), +, \circ \rangle$ is a *near-ring* [13]; a brief introduction to those concepts of near-ring theory that are related to our present work is also given in Section 6 of [2]. Let A, B be ideals of $\mathbf{V}$ with $A < B$. Then the group $\langle B/A, + \rangle$ becomes a $P_0(\mathbf{V})$ -module if we define $p * (b + A) := p(b) + A$ for all $p \in P_0(\mathbf{V})$, $b \in B$. The annihilator of this module is given by the ideal $\text{Ann}(B/A) := \{p \in P_0(\mathbf{V}) \mid p(B) \subseteq A\}$. Clearly, the near-ring $P_0(\mathbf{V})/\text{Ann}(B/A)$ operates on B/A by $(p + \text{Ann}(B/A)) * (b + A) := p(b) + A$. Let us now consider the case that $R := P_0(\mathbf{V})/\text{Ann}(B/A)$ is a ring. Then $M := B/A$ has no R -submodules other than 0 and M . So R is primitive on M , and hence M is a vector-space over the division ring $D := \langle \text{End}_R(M), +, \circ \rangle$. With two ideals A, B of an expanded group $\mathbf{V}$, we associate their commutator $[A, B]$, which is defined as the ideal generated by all $p(a, b)$ with $a \in A$, $b \in B$, and $p \in \text{Pol}_2(\mathbf{V})$ such that $p(x, 0) = p(0, x) = 0$ for all $x \in V$. In Section 2 of [3], it is proved that this commutator operation is exactly the term condition commutator widely

used in universal algebra [7], [12, Definition 4.150]. Now for all ideals A, B of $\mathbf{V}$ with $A \leq B$ and $[B, B] \leq A$, the algebra $R := P_0(\mathbf{V})/\text{Ann}(B/A)$ is a ring.

Let $\mathbb{P}$ be the set of prime numbers, and let A, B be ideals of $\mathbf{V}$ with $A < B$ and $[B, B] \leq A$. The *characteristic* $\pi(A, B) \in \mathbb{P} \cup \{0\}$ is defined as the characteristic of the division ring $\text{End}_{P_0(\mathbf{V})}(B/A)$. We notice that the characteristic of this division ring is also the exponent of the group $\langle B/A, + \rangle$, with the exponent of a group $\mathbf{G}$ defined to be 0 if there is no $n \in \mathbb{N}$ with $\mathbf{G} \models nx \approx 0$. For those prime sections $A < B$ with $[B, B] \not\leq A$, the characteristic $\pi(A, B)$ is not defined. Since every supernilpotent expanded group is nilpotent (cf. Lemma 4.1) and hence solvable, such prime sections will not occur inside the ideal lattice of a supernilpotent expanded group.

An expanded group $\mathbf{W}$ is called *π -monochromatic* if there is $c \in \mathbb{P} \cup \{0\}$ such that $\pi(A, B) = c$ for all ideals A, B of $\mathbf{W}$ with $A < B$. Now we are ready to state the main result.

Theorem 2.1. *Let $\mathbf{V}$ be a supernilpotent expanded group whose ideal lattice is of finite height. Then $\mathbf{V}$ is isomorphic to a direct product of finitely many π -monochromatic expanded groups.*

It is clear that for a finite supernilpotent expanded group, this theorem provides a direct decomposition into expanded groups of prime power order, so we obtain as a corollary a special case of Kearnes's decomposition theorem from [9].

Our proof will proceed as follows: we first show that the theorem is true if the ideal lattice is of height 2; in other words, if a supernilpotent expanded group has a three element chain as its ideal lattice, the two prime sections in this lattice have to be of equal characteristic. For finite Mal'cev algebras, this fact has been proved and used frequently in the work of Mayr (cf. [3, 11], [10, Lemma 3]); we extend his approach by comprising infinite expanded groups. Then we apply this result to obtain the theorem for expanded groups with ideal lattice of arbitrary finite height.

3. PRELIMINARIES ON LATTICES

Let $\mathbb{L}$ be a modular lattice, and let $S(\mathbb{L})$ be the set $\{(\alpha, \beta) \in L^2 \mid \alpha < \beta\}$. A *colouring of the prime sections of $\mathbb{L}$* is a mapping χ from $S(\mathbb{L})$ to some other set C such that for all (α, β) and $(\gamma, \delta) \in S(\mathbb{L})$ with $I[\alpha, \beta] \nearrow I[\gamma, \delta]$, we have $\chi(\alpha, \beta) = \chi(\gamma, \delta)$. Hence we require projective prime sections to have the same colour. We say that $\mathbb{L}$ satisfies the *transfer property* for this colouring if for all $\alpha, \beta, \gamma \in L$ with $\alpha < \beta < \gamma$, $\chi(\alpha, \beta) \neq \chi(\beta, \gamma)$, there is a $\delta \in L$ such that $\alpha < \delta < \gamma$ with $\delta \neq \beta$. Clearly, then $I[\alpha, \beta] \nearrow I[\delta, \gamma]$ and $I[\alpha, \delta] \nearrow I[\beta, \gamma]$ so that $\chi(\alpha, \delta) = \chi(\beta, \gamma)$ and $\chi(\delta, \gamma) = \chi(\alpha, \beta)$. The name “transfer property” has been chosen in order to remind of the “transfer principle” used, e.g., in [15, 16].

Lemma 3.1. *Let $\mathbb{L}$ be a modular lattice of finite height, let $\chi : \mathbb{L} \rightarrow C$ be a colouring of the prime sections of $\mathbb{L}$, and let $c \in C$. We assume that $\mathbb{L}$ satisfies the transfer property for this colouring. Then there exist $\rho, \sigma \in L$ such that $\rho \vee \sigma = 1$, $\rho \wedge \sigma = 0$, for all $\alpha, \beta \in L$ with $\rho \leq \alpha < \beta$, we have $\chi(\alpha, \beta) = c$, and for all $\gamma, \delta \in L$ with $\sigma \leq \gamma < \delta$, we have $\chi(\gamma, \delta) \neq c$.*

Proof. Let ρ be minimal such that all prime intervals in $I[\rho, 1]$ have colour c , and let σ be minimal such that all prime intervals in $I[\sigma, 1]$ have colour $\neq c$. Hence for the prime intervals in $I[\rho \vee \sigma, 1]$, there is no colour left, and thus $\rho \vee \sigma = 1$. We will now show $\rho \wedge \sigma = 0$. To this end, we first show that for all $\alpha, \beta \in L$ with $\alpha < \beta \leq \rho$, we have $\chi(\alpha, \beta) \neq c$. Seeking a contradiction, we take β to be maximal in $I[0, \rho]$ such that there is $\alpha \in L$ with $\alpha < \beta$ and $\chi(\alpha, \beta) = c$. If $\beta = \rho$, then we consider the interval $I[\alpha, 1]$. Since this is a modular lattice, every prime section of $I[\alpha, 1]$ either projects up to an interval in $I[\rho, 1]$ or down to an interval in $I[\alpha, \rho]$. This implies that all intervals of $I[\alpha, 1]$ are coloured by c ; since $\alpha < \rho$ this contradicts the minimality of ρ . In the case $\beta < \rho$, let γ be such that $\beta < \gamma \leq \rho$. By the maximality of β , we have $\chi(\beta, \gamma) \neq c$. Now we use the transfer property to obtain $\delta \in L$ such that $\alpha < \delta < \gamma$ and $\chi(\delta, \gamma) = c$. So γ has a subcover with the corresponding interval coloured by c , contradicting the maximality of β . This completes the proof that all prime intervals of $I[0, \rho]$ have colour $\neq c$.

Taking β to be maximal in $I[0, \sigma]$ such that there is $\alpha \in L$ with $\alpha < \beta < \sigma$ and $\chi(\alpha, \beta) \neq c$, and using the same arguments as above, we obtain that all prime intervals of $I[0, \sigma]$ must have colour c . Thus for the intervals in $I[0, \sigma \wedge \rho]$, there are no colours left, and therefore, $\sigma \wedge \rho = 0$. $\square$

4. PRELIMINARIES ON EXPANDED GROUPS

In this section, we will give a self-contained account of some known facts concerning nilpotent and supernilpotent expanded groups. We recall from the introductory section that an expanded group is supernilpotent of class at most k if for all $l > k$, all l -ary absorbing polynomials are constant. We will use the following facts on supernilpotence.

Lemma 4.1. *Let $\mathbf{V}$ be a supernilpotent expanded group of class at most k , and let A be an ideal of $\mathbf{V}$. Then we have:*

- (1) $\mathbf{V}/A$ is supernilpotent of class at most k ;
- (2) $\mathbf{V}$ is nilpotent of class at most k ;
- (3) Let $\mathbf{M}$ be the algebra $\langle A, +, \{p|_A \mid p \in \text{Pol}_1(\mathbf{V}), p(0) = 0\} \rangle$. Then $\mathbf{M}$ is supernilpotent of class at most k .

We notice that the first two properties are consequences of the results on the higher commutator operation in [4]. Property (1) follows from the property (HC6) of the higher commutator operations, and property (2) is a consequence of (HC8). We will now supply arguments that do not make use of the higher commutator operations studied in [4, 6].

Proof of Lemma 4.1. (1): Let $l > k$, and let f be an l -ary absorbing polynomial of $\mathbf{V}/A$. Then there is a polynomial g of $\mathbf{V}$ such that $g(\mathbf{x}) + A = f(\mathbf{x} + A)$ for all $\mathbf{x} \in V^l$. Now we use the difference operator defined in [1, Definition 3.3] and define $h \in \text{Pol}_l(\mathbf{V}) := D^{(l)}(g)$. By [1, Proposition 3.5], the function $D^{(l)}(g)$ is absorbing. Since $\mathbf{V}$ is supernilpotent of class less than l , we obtain $D^{(l)}(g) = 0$. In [1, Proposition 3.4], we find a description of $D^{(l)}(g)(x_1, \dots, x_l)$ as a sum of 2^l expressions of the form $g(y_1, \dots, y_l)$, where each y_i is an element of $\{0, x_i\}$. There

is one summand $g(x_1, \dots, x_l)$. In all other summands, at least one of the arguments y_i is 0. In fact, for each of these other summands, we have $g(y_1, \dots, y_l) + A = f(y_1 + A, \dots, y_l + A)$, and since f is absorbing, the last expression is equal to $0 + A$. Altogether, we obtain that for all $x_1, \dots, x_l \in V$, we have $D^{(l)}(g)(x_1, \dots, x_l) - g(x_1, \dots, x_l) \in A$. Since $D^{(l)}(g) = 0$, we obtain that $g(V^l) \subseteq A$, and hence f is the zero-function on $\mathbf{V}/A$. This completes the proof that $\mathbf{V}$ is supernilpotent of class at most k .

(2): Let the lower central series of $\mathbf{V}$ be defined by $\gamma_1(\mathbf{V}) := V$, $\gamma_{n+1}(\mathbf{V}) := [V, \gamma_n(\mathbf{V})]$, and let $\sigma_n(\mathbf{V})$ be the ideal generated by all $p(v_1, \dots, v_n)$, where $p \in \text{Pol}_n(\mathbf{V})$ is an absorbing polynomial of $\mathbf{V}$, and $v_1, \dots, v_n \in V$. We notice, but will not make use of, the fact that using the language of higher commutators, $\sigma_n(\mathbf{V})$ is the ideal $\underbrace{[V, \dots, V]}_n$ of $\mathbf{V}$. We will now prove by induction that for

all $n \in \mathbb{N}$, we have $\gamma_n(\mathbf{V}) \leq \sigma_n(\mathbf{V})$. For $n = 1$, we have $\gamma_1(\mathbf{V}) = V = \sigma_1(\mathbf{V})$. For the induction step, let $n \geq 2$, and let $y \in \gamma_n(\mathbf{V}) = [V, \gamma_{n-1}(\mathbf{V})]$. By the induction hypothesis, we have $y \in [V, \sigma_{n-1}(\mathbf{V})]$. The last ideal is generated by all $p(v, s)$ with p binary and absorbing, $v \in V$ and $s \in \sigma_{n-1}(\mathbf{V})$. For expanded groups, we can use the description of the generated ideal in [13, Theorem 7.123], and hence there are $h \in \mathbb{N}$, absorbing binary polynomials $p_1, \dots, p_h \in \text{Pol}_2(\mathbf{V})$, $v_1, \dots, v_h \in V$ and $s_1, \dots, s_h \in \sigma_{n-1}(\mathbf{V})$ such that $y = \sum_{i=1}^h p_i(v_i, s_i)$. We will prove that each summand lies in $\sigma_n(\mathbf{V})$. Let $i \in \{1, \dots, h\}$. We have $l \in \mathbb{N}$, absorbing polynomials $q_1, \dots, q_l \in \text{Pol}_{n-1}(\mathbf{V})$ and $w_1^{(1)}, \dots, w_{n-1}^{(1)}, \dots, w_1^{(l)}, \dots, w_{n-1}^{(l)} \in V$ such that $s_i = \sum_{j=1}^l q_j(w_1^{(j)}, \dots, w_{n-1}^{(j)})$. We will now prove by induction that for each $m \in \{0, \dots, l\}$, we have $p_i(v_i, \sum_{j=1}^m q_j(w_1^{(j)}, \dots, w_{n-1}^{(j)})) \in \sigma_n(\mathbf{V})$. For $m = 0$, we obtain $p_i(v_i, 0) = 0 \in \sigma_n(\mathbf{V})$. Now let $m \geq 1$. Then we consider the polynomial $r \in \text{Pol}_n(\mathbf{V})$ defined by

$$\begin{aligned} r(x_1, y_1, \dots, y_{n-1}) &:= p_i(x_1, q_m(y_1, \dots, y_{n-1}) + \sum_{j=1}^{m-1} q_j(w_1^{(j)}, \dots, w_{n-1}^{(j)})) \\ &\quad - p_i(x_1, \sum_{j=1}^{m-1} q_j(w_1^{(j)}, \dots, w_{n-1}^{(j)})). \end{aligned}$$

Since r is absorbing, we obtain $a := r(v_i, w_1^{(j)}, \dots, w_{n-1}^{(j)}) \in \sigma_n(\mathbf{V})$. By the induction hypothesis, we have $b := p_i(x_1, \sum_{j=1}^{m-1} q_j(w_1^{(j)}, \dots, w_{n-1}^{(j)})) \in \sigma_n(\mathbf{V})$. Hence $a + b \in \sigma_n(\mathbf{V})$, which completes the induction step of our induction on m . So we have proved that every summand $p_i(v_i, s_i)$ is an element of $\sigma_n(\mathbf{V})$, which implies $y \in \sigma_n(\mathbf{V})$. This completes the induction on n and proves that for all $n \in \mathbb{N}$, $\gamma_n(\mathbf{V}) \leq \sigma_n(\mathbf{V})$. Since our assumption on $\mathbf{V}$ was that $\sigma_{k+1}(\mathbf{V}) = 0$, we obtain $\gamma_{k+1}(\mathbf{V}) = 0$, implying that $\mathbf{V}$ is nilpotent of class at most k .

(3) Suppose that $\mathbf{M}$ is not supernilpotent of class $\leq k$. Then we have have an absorbing polynomial $p \in \text{Pol}_k(\mathbf{M})$ and $m_1, \dots, m_k \in M$ such that $p(m_1, \dots, m_k) \neq 0$. Since $p \in \text{Pol}_k(\mathbf{M})$, there is $p' \in \text{Pol}_k(\mathbf{V})$ with $p'|_{M^k} = p$. From the fact that p is absorbing, we know that for all $x_1, \dots, x_k \in M$ with $0 \in \{x_1, \dots, x_k\}$, we have $p'(x_1, \dots, x_k) = 0$. Using the construction $D^{(k)}$ from [1, Definition 3.3], we let $q := D^{(k)}(p')$. Then [1, Proposition 3.4] yields that q is

a absorbing polynomial of $\mathbf{V}$, and from [1, Proposition 3.5(1)], we obtain that $q(m_1, \dots, m_k) = p'(m_1, \dots, m_k) = p(m_1, \dots, m_k) \neq 0$. Thus we see that $\mathbf{V}$ is not supernilpotent of class at most k , contradicting the assumptions. $\square$

In the remainder of this section, we will give a brief self-contained account of the fact that polynomial growth of the free spectrum of a finite expanded group is equivalent to supernilpotence; this is an adaptation of the arguments in [8, Section 1.1] and [5]. We recall that for an expanded group $\mathbf{V}$ and for $n \in \mathbb{N}$, the algebra of n -ary term functions, $\text{Clo}_n(\mathbf{V})$, is isomorphic to the free algebra over n generators in the variety generated by $\mathbf{V}$. Now for each finite expanded group $\mathbf{V}$ and $n \in \mathbb{N}_0$, we let

$$\begin{aligned} a_n(\mathbf{V}) &:= \log_2(|\{p \in \text{Clo}_n(\mathbf{V}) \mid p \text{ is absorbing}\}|) \\ t_n(\mathbf{V}) &:= \log_2(|\text{Clo}_n(\mathbf{V})|). \end{aligned}$$

Lemma 4.2 (cf. [8, p. 154], [5]). *Let $\mathbf{V}$ be a finite expanded group. Then for each $n \in \mathbb{N}_0$, we have*

$$t_n(\mathbf{V}) = \sum_{i=0}^n a_i(\mathbf{V}) \binom{n}{i}.$$

Proof. For $n, k \in \mathbb{N}_0$ with $k \leq n$, let

$$\begin{aligned} A(n, k) &:= \{f \in \text{Clo}_n(\mathbf{V}) \mid \text{for all } (x_1, \dots, x_n) \in V^n : \\ &\quad 0 \in \{x_1, \dots, x_k\} \Rightarrow f(x_1, \dots, x_n) = 0\}. \end{aligned}$$

For $k = 0$, we obtain $A(n, 0) = \text{Clo}_n(\mathbf{V})$, and for $k = n$, $A(n, n)$ is the set of absorbing term function of $\mathbf{V}$. Now for $n \in \mathbb{N}$ and $k \in \{0, \dots, n-1\}$, we consider the mapping

$$\begin{aligned} \varphi : A(n, k) &\rightarrow \text{Clo}_{n-1}(\mathbf{V}), \\ f &\mapsto \varphi(f), \end{aligned}$$

where $\varphi(f)(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_k, 0, x_{k+1}, \dots, x_{n-1})$. The mapping φ is a group homomorphism. The image of φ is the set $A(n-1, k)$, and its kernel is the set $A(n, k+1)$. The homomorphism theorem yields $|A(n, k)| = |A(n-1, k)| \cdot |A(n, k+1)|$. Let $a(n, k) := \log_2(|A(n, k)|)$. Then we have

$$a(n, k) = a(n-1, k) + a(n, k+1)$$

for all $n, k \in \mathbb{N}_0$ with $k < n$. Now by induction on j , one can verify that for all $j \in \mathbb{N}_0$ and for all $n \in \mathbb{N}_0$ with $n \geq j$, one has $a(n, n-j) = \sum_{k=0}^j a(n-k, n-k) \binom{j}{k}$. Setting $j := n$, we obtain $t_n(\mathbf{V}) = a(n, 0) = \sum_{k=0}^n a(k, k) \binom{n}{k} = \sum_{k=0}^n a_k(\mathbf{V}) \binom{n}{k}$. $\square$

We will now relate term functions to polynomial functions. Let $\mathbf{V}^*$ be the algebra that we obtain from $\mathbf{V}$ by adding all elements of $\mathbf{V}$ as constant operations. Then we have $a_n(\mathbf{V}^*) = \log_2(|\{p \in \text{Pol}_n(\mathbf{V}) \mid p \text{ is absorbing}\}|)$ and

$t_n(\mathbf{V}^*) := \log_2(|\text{Pol}_n(\mathbf{V})|)$. Suppose $|V| = r$ and $V = \{v_1, \dots, v_r\}$. Since the mapping $\varphi : \text{Clo}_{n+r}(\mathbf{V}) \rightarrow \text{Pol}_n(\mathbf{V})$, $t(x_1, \dots, x_n, y_1, \dots, y_r) \mapsto t(x_1, \dots, x_n, v_1, \dots, v_r)$ is surjective, we have $t_n(\mathbf{V}^*) \leq t_{n+r}(\mathbf{V})$ for all $n \in \mathbb{N}_0$, and hence

$$t_n(\mathbf{V}) \leq t_n(\mathbf{V}^*) \leq t_{n+r}(\mathbf{V}) \quad (4.1)$$

for all $n \in \mathbb{N}_0$.

Corollary 4.3 (cf. [5]). *Let $\mathbf{V}$ be a finite expanded group, and let $r := |V|$. If $\mathbf{V}$ is not supernilpotent, then for all $n \in \mathbb{N}$ with $n \geq r$, we have $|\text{Clo}_n(\mathbf{V})| \geq 2^{2^{n-r}}$. Furthermore, for $k \in \mathbb{N}$, the following statements are equivalent:*

- (1) $\mathbf{V}$ is supernilpotent of class k .
- (2) There is a polynomial p of degree k such that $|\text{Clo}_n(\mathbf{V})| = 2^{p(n)}$ for all $n \in \mathbb{N}$.

Proof. Let $r := |V|$. First suppose that $\mathbf{V}$ is not supernilpotent. Then there are nonconstant absorbing polynomials of arbitrary large arity. Suppose $p(x_1, \dots, x_n)$ is a nonconstant absorbing polynomial, and let $v_1, \dots, v_n \in V$ such that $p(v_1, \dots, v_n) \neq 0$. Then $p(x_1, \dots, x_{n-1}, v_n)$ is an $(n-1)$ -ary nonconstant absorbing polynomial. Hence there are nonconstant absorbing polynomials of every arity $n \geq 1$, and thus $a_n(\mathbf{V}^*) \geq 1$ for all $n \in \mathbb{N}_0$. Therefore by applying Lemma 4.2 to the algebra $\mathbf{V}^*$, we obtain $t_n(\mathbf{V}^*) = \sum_{i=0}^n a_i(\mathbf{V}^*) \binom{n}{i} \geq 2^n$ for all $n \in \mathbb{N}_0$. Hence by (4.1), for $n \geq r$, we have $t_n(\mathbf{V}) \geq t_{n-r}(\mathbf{V}^*) \geq 2^{n-r}$.

To prove that (1) and (2) are equivalent, we first show (1) $\Rightarrow$ (2): If $\mathbf{V}$ is supernilpotent of class k , we have $a_i(\mathbf{V}^*) = 0$ for $i > k$, and $a_k(\mathbf{V}^*) \neq 0$. This implies that for $i > k$, we have $a_i(\mathbf{V}) = 0$. Now $g(n) := t_n(\mathbf{V}^*) = \sum_{i=0}^k a_i(\mathbf{V}^*) \binom{n}{i}$ is a polynomial of degree k in n . We will now show that $a_k(\mathbf{V}) \neq 0$. Assuming $a_k(\mathbf{V}) = 0$, we obtain that $h(n) := t_n(\mathbf{V}) = \sum_{i=0}^k a_i(\mathbf{V}) \binom{n}{i}$ is a polynomial of degree at most $k-1$ in n . Now by (4.1), we have $g(n) \leq h(n+r)$ for all $n \in \mathbb{N}_0$. Since $\deg(g) = k$, $\deg(h) \leq k-1$ and the leading coefficient of g is positive, this is impossible. Hence $a_k \neq 0$, and therefore, $t_n(\mathbf{V}) = \sum_{i=0}^k a_i(\mathbf{V}) \binom{n}{i}$ is a polynomial of degree k in n .

To prove (2) $\Rightarrow$ (1), we let p be a polynomial of degree k in n such that $p(n) = t_n(\mathbf{V})$ for all $n \in \mathbb{N}$. Then from Lemma 4.2, we see that $a_k(\mathbf{V}) > 0$ and $a_l(\mathbf{V}) = 0$ for all $l > k$. Hence $a_k(\mathbf{V}^*) \geq a_k(\mathbf{V}) > 0$. Now suppose that there is $l > k$ with $a_l(\mathbf{V}^*) > 0$. Then $t_n(\mathbf{V}^*) \geq a_l(\mathbf{V}^*) \binom{n}{l}$, and thus $t_n(\mathbf{V}) \geq \binom{n-r}{l}$ for all $n \geq r$. This contradicts the fact that $t_n(\mathbf{V})$ is a polynomial of degree k . Hence $a_k(\mathbf{V}^*) \neq 0$ and $a_l(\mathbf{V}^*) = 0$ for $l > k$, which implies that $\mathbf{V}$ is supernilpotent of class k . $\square$

5. EXPANDED GROUPS WITH IDEAL LATTICE OF HEIGHT 2

Lemma 5.1. *Let $\mathbf{R}$ be a ring with unit, and let $\mathbf{M}$ be a unitary $\mathbf{R}$ -module such that $\mathbf{M}$ has exactly three submodules; let Q be the submodule different from 0 and M . Then the exponents of the groups $\langle M/Q, + \rangle$ and $\langle Q, + \rangle$ are equal.*

Proof. Let $p, q \in \mathbb{P} \cup \{0\}$ such that p is the exponent of $\langle M/Q, + \rangle$ and q is the exponent of $\langle Q, + \rangle$. We assume $p \neq q$.

Case $p > 0$: The mapping $\lambda : M \rightarrow M$, $x \mapsto px$ is an $\mathbf{R}$ -homomorphism. We have $\text{im}(\lambda) \subseteq Q$. We will now show that λ is not injective. To this end, let $x \in M \setminus Q$.

Then $px \in Q$. We observe that every nonzero element of $\langle Q, + \rangle$ is divisible by p . Hence there is $y \in Q$ such that $py = px$, and thus $\lambda(y) = \lambda(x)$. Since $y \in Q$ and $x \notin Q$, we have $x \neq y$. Thus λ is not injective, and therefore we have $\ker(\lambda) \neq \{0\}$. By the fact that all nonzero-elements of Q are divisible by p , we have $\lambda(Q) = Q$ and therefore $\ker(\lambda) \neq Q$ and $\ker(\lambda) \neq M$. But $\ker(\lambda)$ is an ideal of M , and so $0, Q, M$ are the only choices. Hence the case $p > 0$ cannot occur.

Case $p = 0$: In this case, $q > 0$. The mapping $\lambda : M \rightarrow M, x \mapsto qx$ is an $\mathbf{R}$ -homomorphism. As the additive group of a vector space over the skew-field $D := \text{End}_R(M/Q)$ of characteristic 0, the group $\langle M/Q, + \rangle$ is torsion-free, and thus $\ker(\lambda) \neq M$. Since $Q \subseteq \ker(\lambda)$, we obtain $\ker(\lambda) = Q$. If $\text{im}(\lambda) = 0$, we have $\ker(\lambda) = M$, a contradiction. Since $0, Q$, and M are the only choices for the submodule $\text{im}(\lambda)$, we must have $Q \subseteq \text{im}(\lambda)$. Now the homomorphism theorem yields that the group $M/\ker(\lambda) = M/Q$ has a subgroup isomorphic to Q . This is not possible because M/Q is torsion-free. Thus the case $p = 0$ cannot occur.

We conclude that $p = q$. □

For $A, B \subseteq V$, we define

$$(A : B)_{\text{Pol}_1(\mathbf{V})} := \{f \in \text{Pol}_1(\mathbf{V}) \mid f(B) \subseteq A\},$$

and

$$(\sim_A : \sim_B)_{\text{Pol}_1(\mathbf{V})} := \{f \in \text{Pol}_1(\mathbf{V}) \mid \forall v, w \in V : v - w \in B \Rightarrow f(v) - f(w) \in A\}.$$

The following lemma is due to Mayr (cf. [10, Lemma 3]); the proof given here avoids the use of group rings and their modules and uses the polynomial ring $\mathbb{Z}[t]$ instead.

Lemma 5.2. *Let $\mathbf{V}$ be an expanded group whose ideal lattice is a three element chain $\{0\} < Q < V$. We assume that the exponents of the groups $\langle Q, + \rangle$ and $\langle V/Q, + \rangle$ are different, and that $[V, V] = Q$ and $[V, Q] = 0$. Then $\mathbf{V}$ is not supernilpotent.*

Proof. Let $p \in \mathbb{N}_0$ be the exponent of V/Q , and $q \in \mathbb{N}_0$ the exponent of Q . We define

$$\begin{aligned} M &:= (Q : V)_{\text{Pol}_1(\mathbf{V})} \cap (\sim_0 : \sim_Q)_{\text{Pol}_1(\mathbf{V})}, \\ Y &:= (Q : V)_{\text{Pol}_1(\mathbf{V})} \cap (\sim_0 : \sim_V)_{\text{Pol}_1(\mathbf{V})}. \end{aligned}$$

Hence M is the set of those polynomial functions with range contained in Q that are constant on each Q -coset, and Y is the set of constant functions with value in Q .

We will first produce an $f \in M$ and $u, v \in V$ such that

$$f(0) = 0 \quad \text{and} \quad f(u + v) - f(u) \neq f(v). \quad (5.1)$$

To this end, let c be a nonzero binary absorbing polynomial, and let $v_1, v_2 \in V$ be such that $c(v_1, v_2) \neq 0$. Then we define $f(x) := c(x, v_2)$. Since c is

absorbing, $f(0) = 0$. Since $[V, V] = Q$, we have $f \in (Q : V)_{\text{Pol}_1(V)}$. Furthermore, $f \in (\sim_0 : \sim_Q)_{\text{Pol}_1(V)}$. To establish this, we let $v \in V, q \in Q$ and show that $f(v + q) = f(v)$. Let $g(x, y) := c(v + x, y) - c(v, y)$. Then g is absorbing, and therefore $g(q, v_2) \in [Q, V] = \{0\}$. Thus $0 = g(q, v_2) = c(v + q, v_2) - c(v, v_2) = f(v + q) - f(v)$. This completes the proof of $f \in (\sim_0 : \sim_Q)_{\text{Pol}_1(V)}$; hence $f \in M$. Now suppose that $f(u + v) = f(v) + f(u)$ for all $u, v \in V$. Since $\langle Q, + \rangle$ is an abelian group, f is then a group endomorphism of $\langle V, + \rangle$. In addition, $f(Q) \subseteq \{0\}$, and thus $\ker(f) \supseteq Q$ and $\text{im}(f) \subseteq Q$. We will now show that in each case, we have $\ker(f) = V$.

Case $p > 0$: The group $\langle V/\ker(f), + \rangle$ has exponent dividing p , and therefore every element $y \in \text{im}(f)$ satisfies $p * y = 0$. Since $p \neq q$, we have $\text{im}(f) = Q \cap \text{im}(f) = \{0\}$, which implies $\ker(f) = V$.

Case $p = 0$: Then $q > 0$. Let $x \in V$. Since the group $\langle V/Q, + \rangle$ is the additive group of a vector space over $\mathbb{Q}$, it is divisible. Thus there is $y \in V$ with $q * y - x \in Q$. Thus we have $f(x) = f(q * y) = q * f(y) = 0$, and therefore $\ker(f) = V$.

Since $\ker(f) = V$, we obtain $0 = f(v_1) = c(v_1, v_2)$, a contradiction. Thus we have $u, v \in V$ with $f(u + v) - f(u) \neq f(v) - f(0)$, and therefore, f satisfies (5.1).

For every $w \in V$, we define an operation $\star_w$ of the ring $\mathbb{Z}[t]$ on M . The operation of t^1 on M is defined by

$$t^1 \star_w m(x) := m(x + w) \quad \text{for } x \in V, m \in M,$$

and the operation of an arbitrary polynomial $\beta = \sum_{i=0}^k \beta_i t^i$ on M is defined by

$$\beta \star_w m(x) := \sum_{i=0}^k \beta_i * m(x + i * w).$$

Now, for each $w \in V$, $M_w := \langle M, +, \star_w \rangle$ is an $\mathbb{Z}[t]$ -module. Obviously, Y is a submodule of M . We note that (5.1) can now be rewritten as

$$f(0) = 0 \quad \text{and} \quad ((t - 1) \star_v f)(u) \neq ((t - 1) \star_v f)(0).$$

So, $(t - 1) \star_v f$ is not constant, and hence

$$(t - 1) \star_v f \notin Y. \tag{5.2}$$

Next, we show that

$$\text{for all } m \in \mathbb{N}, \text{ there exists } w \in V \text{ such that } (t - 1)^m \star_w f \notin Y. \tag{5.3}$$

We distinguish three cases.

Case $p > 0, q = 0$: Let ψ be the greatest common divisor of $t^p - 1$ and $(t - 1)^m$ in $\mathbb{Q}[t]$. Since $\psi \mid t^p - 1$ and $\psi \mid (t - 1)^m$, there is an $e \in \{0, 1\}$ such that $\psi = (t - 1)^e$. Now we have $\alpha, \beta, \gamma \in \mathbb{Z}[t]$ and $a \in \mathbb{Z} \setminus \{0\}$ such that

$$\alpha(t^p - 1) + \beta(t - 1)^m = a(t - 1)^e.$$

Applying both sides to f , we obtain

$$(\alpha(t^p - 1)) \star_v f + (\beta(t - 1)^m) \star_v f = (a(t - 1)^e) \star_v f. \quad (5.4)$$

We have $(t^p - 1) \star_v f = 0$ and by (5.2), $(a(t - 1)^e) \star_v f \notin Y$. Therefore, (5.4) implies $(t - 1)^m \star_v f \notin Y$.

Case $p = 0, q > 0$: Let $n \in \mathbb{N}$ be such that $q^n \geq m$. Since $\langle V/Q, + \rangle$ is divisible, there is a $w \in V$ such that $q^n w - v \in Q$. Since Q has exponent q , we have $(q t^0) \star_v f = 0$. Since the polynomial $(t - 1)^{q^n} - (t^{q^n} - 1)$ has all its coefficients divisible by q , we obtain $(t - 1)^{q^n} \star_w f = (t^{q^n} - 1) \star_w f = (t - 1) \star_{q^n w} f = (t - 1) \star_v f \notin Y$. Since $q^n \geq m$, this implies $(t - 1)^m \star_w f \notin Y$.

Case $p > 0, q > 0$. In $\mathbb{Z}_q[t]$, the multiplicity of the root 1 in $t^p - 1$ is 1. Hence in $\mathbb{Z}_q[t]$, we have $\gcd_{\mathbb{Z}_q[t]}(t^p - 1, (t - 1)^m) = t - 1$, and thus there are polynomials $\alpha, \beta, \gamma \in \mathbb{Z}[t]$ such that

$$\alpha(t^p - 1) + \beta(t - 1)^m + \gamma(qt^0) = t - 1.$$

Hence we have

$$(\alpha(t^p - 1)) \star_v f + (\beta(t - 1)^m) \star_v f + (\gamma(qt^0)) \star_v f = (t - 1) \star_v f.$$

Since $(t^p - 1) \star_v f = (q t^0) \star_v f = 0$, we obtain $(\beta(t - 1)^m) \star_v f = (t - 1) \star_v f \notin Y$, and therefore, $(t - 1)^m \star_v f \notin Y$.

This completes the proof of (5.3). Now we prove that for all $k \in \mathbb{N}$, there exists a k -ary absorbing polynomial $h^{(k)}$ of $\mathbf{V}$ that is not constant. To this end, we let $k \in \mathbb{N}$ and using (5.3), we choose $w \in V$ with $(t - 1)^{k-1} \star_w f \notin Y$. We will now define a sequence $(h^{(n)})_{n \in \mathbb{N}}$ such that each $h^{(n)}$ is an n -ary absorbing polynomial that satisfies

$$h^{(n)}(x_1, w, \dots, w) = ((t - 1)^{n-1} \star_w f)(x_1) - ((t - 1)^{n-1} \star_w f)(0) \quad \text{for all } x_1 \in V. \quad (5.5)$$

For $n = 1$, we define $h^{(1)} := f$. For $n \in \{2, \dots, m\}$, we define $h^{(n)}$ by

$$\begin{aligned} h^{(n)}(x_1, \dots, x_n) &:= h^{(n-1)}(x_1 + x_n, x_2, \dots, x_{n-1}) - h^{(n-1)}(x_1, x_2, \dots, x_{n-1}) \\ &\quad + h^{(n-1)}(0, x_2, \dots, x_{n-1}) - h^{(n-1)}(x_n, x_2, \dots, x_{n-1}). \end{aligned} \quad (5.6)$$

We will now prove by induction that each $h^{(n)}$ is an absorbing n -ary polynomial that satisfies (5.5). For $n = 1$, (5.1) yields that f is absorbing, and satisfies (5.5). For the induction step, let $n \geq 2$. We first show that $h^{(n)}$ is absorbing. If one of the arguments at places 2 to $n - 1$ is set to 0, then by the induction hypothesis each of the four summands in (5.6) evaluates to 0. If $x_1 = 0$ or $x_n = 0$, then $h^{(n)}(x_1, \dots, x_n) = 0$ follows from (5.6) by easy calculations. To prove (5.5), we set $c := ((t - 1)^{n-2} \star_w f)(0)$ and compute, using the induction hypothesis in the second equality,

$$\begin{aligned} h^{(n)}(x_1, w, \dots, w, w) &= h^{(n-1)}(x_1 + w, w, \dots, w) - h^{(n-1)}(x_1, w, \dots, w) \\ &\quad + h^{(n-1)}(0, w, \dots, w) - h^{(n-1)}(w, w, \dots, w) \end{aligned}$$

$$\begin{aligned}
&= ((t-1)^{n-2} \star_w f)(x_1 + w) - c + c - ((t-1)^{n-2} \star_w f)(x_1) \\
&\quad + ((t-1)^{n-2} \star_w f)(0) - c + c - ((t-1)^{n-2} \star_w f)(w) \\
&= ((t-1) \star_w ((t-1)^{n-2} \star_w f))(x_1) \\
&\quad - ((t-1) \star_w ((t-1)^{n-2} \star_w f))(0) \\
&= ((t-1)^{n-1} \star_w f)(x_1) - ((t-1)^{n-1} \star_w f)(0).
\end{aligned}$$

This completes the induction step.

Hence $h^{(k)}$ is a k -ary absorbing polynomial with

$$h^{(k)}(x_1, w, \dots, w) = ((t-1)^{k-1} \star_w f)(x_1) - ((t-1)^{k-1} \star_w f)(0)$$

for all $x_1 \in V$. Since $(t-1)^{k-1} \star_w f \notin Y$, the function $x_1 \mapsto h^{(k)}(x_1, w, \dots, w)$ is not constant.

Therefore, for each $k \in \mathbb{N}$, $\mathbf{V}$ has a k -ary absorbing nonconstant polynomial, which implies that $\mathbf{V}$ is not supernilpotent. $\square$

Lemma 5.3. *Let $\mathbf{V}$ be a supernilpotent expanded group, and let π be the colouring of the prime sections of its ideal lattice $Id(\mathbf{V})$ introduced in Section 2. Then the lattice $Id(\mathbf{V})$ satisfies the transfer property for the colouring π .*

Proof. We let A, B, C be ideals of $\mathbf{V}$ such that in $Id(\mathbf{V})$, we have $A < B < C$, the exponent of B/A is q , and the exponent of C/B is p with $p, q \in \mathbb{P} \cup \{0\}$, $p \neq q$. Since we can factor by A and $\mathbf{V}/A$ is supernilpotent by Lemma 4.1, we can assume without loss of generality that $A = 0$. Now we consider the $P_0(\mathbf{V})$ -module $\mathbf{C} = \langle C, +, \{p|_C \mid p \in P_0(\mathbf{V})\} \rangle$. We notice that the ideals of $\mathbf{C}$ are exactly the ideals of $\mathbf{V}$ that are contained in C . By Lemma 4.1, $\mathbf{C}$ is supernilpotent and thus, again by this lemma, also nilpotent, and therefore, for the commutator operation $[\cdot, \cdot]_C$ of this module, we have $[C, C]_C < C$ and $[C, B]_C < B$. If $[C, C]_C = 0$, then C becomes a ring module over the ring $\mathbf{R} := P_0(\mathbf{V})/\text{Ann}(C)$. From Lemma 5.1, we obtain an $\mathbf{R}$ -submodule D with $D \neq B$ and $0 < D < C$. This D is an ideal of $\mathbf{V}$, which completes the proof of the transfer property in the case $[C, C]_C = 0$. If $[C, C]_C = B$, then $[C, B]_C = 0$ because $\mathbf{C}$ is nilpotent. Since $\mathbf{C}$ is supernilpotent, Lemma 5.2 tells that $\mathbf{C}$ has an ideal D with $D \neq B$ and $0 < D < C$. Since the ideals of $\mathbf{C}$ are exactly the ideals of $\mathbf{V}$ that are contained in C , D is an ideal of $\mathbf{V}$, which completes the proof of the transfer property. $\square$

6. PROOF OF THE MAIN THEOREM

Proof of Theorem 2.1. We proceed by induction on the height of the ideal lattice of $\mathbf{V}$. The result is obvious if $|V| = 1$. Lemma 5.3 tells that for the colouring π of the prime sections of the ideal lattice of $\mathbf{V}$, $Id(\mathbf{V})$ satisfies the transfer property. Let $p \in \mathbb{P} \cup \{0\}$ be a colour that actually appears in the colouring of $Id(\mathbf{V})$. From Lemma 3.1, we obtain ideals R, S of $\mathbf{V}$ such that $R + S = V$, $R \cap S = 0$, all prime sections in $I[R, V]$ are coloured by p , and all prime sections in $I[S, V]$ do not have colour p . Since the colour p actually appears, we have $S \neq 0$. Clearly,

V is isomorphic to the direct product $V/R \times V/S$. The expanded group V/R is π -monochromatic. By the induction hypothesis, V/S is isomorphic to a finite direct product of π -monochromatic expanded groups. $\square$

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