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Increasing Computational Efficiency of Cochlear Models Using Boundary Layers

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Abstract

Our goal is to develop methods to improve the efficiency of computational models of the cochlea for applications that require the solution accurately only within a basal region of interest, specifically by decreasing the number of spatial sections needed for simulation of the problem with good accuracy. We design algebraic spatial and parametric transformations to computational models of the cochlea. These transformations are applied after the basal region of interest and allow for spatial preservation, driven by the natural characteristics of approximate spatial causality of cochlear models. The project is of foundational nature and hence the goal is to design, characterize and develop an understanding and framework rather than optimization and globalization. Our scope is as follows: designing the transformations; understanding the mechanisms by which computational load is decreased for each transformation; development of performance criteria; characterization of the results of applying each transformation to a specific physical model and discretization and solution schemes. In this manuscript, we introduce one of the proposed methods (complex spatial transformation) for a case study physical model that is a linear, passive, transmission line model in which the various abstraction layers (electric parameters, filter parameters, wave parameters) are clearer than other models. This is conducted in the frequency domain for multiple frequencies using a second order finite difference scheme for discretization and direct elimination for solving the discrete system of equations. The performance is evaluated using two developed simulative criteria for each of the transformations. In conclusion, the developed methods serve to increase efficiency of a computational traveling wave cochlear model when spatial preservation can hold, while maintaining good correspondence with the solution of interest and good accuracy, for applications in which the interest is in the solution to a model in the basal region.

INTRODUCTION

The computational efficiency, and sometimes computational infeasibility, of simulating multi-element, 3D nonlinear mechanical cochlear models in the time domain is a major impediment to fully utilizing these models. This limitation creates a trade-off between accuracy and complexity and size. Computational efficiency is essential for the construction of cochlear mechanical models, whether using iterative optimization procedures or simulating iteratively through multiple ranges of multiple parameters to obtain realistic matches to data. Computational efficiency is also necessary to understand the role of

different structures and their properties on simulated data—for example, to understand how varying mechanical properties of the basilar membrane affect cochlear tuning.

Previous studies that have attempted to improve the efficiency of cochlear models include [2], [4], [1], which use different approaches than presented here. We consider cases where the whole cochlea is simulated accurately even when only the basal portion of the output is used for analysis. This would be the case, for example, when comparing to click-response data from the very base, or when using a scaling symmetry assumption.

Our goal then is to build methods to increase the computational efficiency of cochlear models when responses are desired over a restricted spatial range, $x < x_0$. The scope of this project is to design, test, characterize, and build an understanding rather than to optimize and extend the methods to other problems. In this manuscript, we focus on introducing the problem with its assumptions and setting the framework. We provide only one example, and do not detail our performance criteria and design-evaluate-characterize loop here.

METHODS

Assumptions

We aim to decrease the total number of longitudinal spatial sections required for simulation of the discretized model, which is generally determined by both the cochlear length and the spatial resolution-accuracy relationship. Since the interest is in the solution for the region of interest $x < x_0$, we hypothesize that altering the form of the model in the region $x > x_0$ using our proposed transformations may allow for preserving the solution in the region of interest while improving computational efficiency. This hypothesis is appropriate because the base of the cochlea is approximately spatially causal—in other words, the response at any point along the cochlea depends primarily on that point and the points basal to it. This hypothesis is fundamental to our approach and hence the methods are suitable for cochlear models that are consistent with the hypothesis — including models that do not incorporate emissions, models that incorporate stimulus frequency otoacoustic emissions that arise from regions basal to the point of interest, but not including models that incorporate click evoked emissions.

Approach

We keep the properties of the cochlear model as is for the region of interest (RI), $x < x_0$, and transform the region after that in an attempt to decrease the total number of longitudinal sections while retaining accuracy in RI. Cochlear models can be represented as $f(t, \vec{X}, \vec{\Theta})$ where $\vec{X}$ is the set of spatial coordinates and $\vec{\Theta}$ is the set of parameters. Hence, the transformation after the RI may be applied to the spatial coordinates and/or to the parameter set. We have designed and studied methods using parametric damping and parametric squeezing transformations, as well as spatial squeezing and complex spatial transformations. In this manuscript, we introduce only the complex spatial transformation.

The methods have two aspects: the transformed layer (TL) which is part of the cochlear length after the RI; and the computational box which may have a computational end, x_{comp} before the actual apex x_{apex} .

Mechanism

As mentioned earlier, the total number of longitudinal spatial sections depends on both the simulated length and the step size required by the resolution-accuracy relationship. Hence, our transformations may decrease the number of spatial sections (and hence improve computational efficiency) through the following mechanisms (illustrated schematically for Zweig's passive differential pressure [8] in Fig. 1).

- Mechanism A: The solution (differential pressure) is absorbed faster in the transformed model than the regular model. Therefore, the solution will have decayed entirely before the apex, allowing for imposition of a computational end before the actual apex $x_{\text{comp}} < x_{\text{apex}}$, and hence the total cochlear length required for simulation is reduced. If using a uniform step size, fewer total spatial sections would be needed for simulation with this mechanism. However, we note that this is only the case as long as the reduction in total simulated length is not outweighed by a reduction in the accuracy-spatial resolution relationship (accuracy of the lowest accuracy frequency in the RI, and resolution in the TL). Using this mechanism, we would only simulate the length minimally necessary to get the results that really pertain to the uses of the simulations, and not for $x > x_0 + \varepsilon$, ($\varepsilon = x_{\text{comp}} - x_0$).
- Mechanism B: The solution to the transformed model does not decay much faster than in the original, therefore the transformation does not change the effective apex, or cochlear length needed for simulation. However, the transformation provides an improved accuracy-spatial resolution relationship due to decreased solution/parameter gradients. Thereby, use of the transformed model would achieve the same accuracy (in the RI) as that of the regular model but by using larger step sizes in the TL. Here, the total number of sections required for simulation for the same accuracy in the RI would be decreased if one allows for a nonuniform step size for multi-frequency component stimuli.

Implementation

The proposed methods would be most valuable for a 3D nonlinear time domain implementation of a cochlear model. However, given the nature of this project, we use a case study model. The case study model [8] is approximately spatially causal for all x and is well-approximated using the first-order Wentzel-Kramers-Brillouin (WKB) approximation. The Zweig model is a 1D, longwave, linear, passive, classical transmission-line model.

For simplicity of application, performance evaluation and characterization, the discretization scheme used is a finite difference scheme with second order accuracy even at the boundaries, and direct elimination using LU decomposition is used to solve the resultant system of equations. We note, however, that these transformations would ideally be used with more efficient schemes, such as spectral methods. Also for simplicity, we first test the methods in the frequency domain to deal with ODEs rather than PDEs. To account for this, we examine the solution using the methods, for multiple frequencies within the natural range of CF for which the solution is well approximated using WKB. As a future step, we may implement the model in the time domain with a filtered impulse input to better show results for the major applications of interest.

The spatial step size is chosen to be uniform for simplicity. Depending on the mechanism of achieving the goal, a uniform or nonuniform step size would ideally be used. The frequency domain governing equation and solutions are exemplified using the spatial component of the differential pressure, P .

COMPLEX SPATIAL TRANSFORMATION

Transformation Overview

In this transformation, after x_0 , we extend the spatial coordinate x into the complex domain. The transformation is purely spatial but the function parameters are chosen based on the wavenumber. This transformation is motivated by [3]. We will use the following notation for the independent variables for the spatial transformations.

$$\text{Independent variables: } \begin{cases} x & \text{distance from stapes in original model} \\ \tilde{x} & \text{transformed} \\ u & \text{simulated (with uniform mesh)} \end{cases}$$

Transformation—The transformation can be denoted by $x\tilde{x}(\tilde{x})$ or an equivalent $dx\tilde{dx}(\tilde{x})$. We modify the governing equation based on the spatial transformation to write a modified governing equation with $\tilde{x}$ as the independent variable. The solution as a function of the corresponding independent variable is unchanged for the transformed model when compared to the original model. This is a feature of spatial (and not parametric) transformations. The transformation function is chosen based on the relationship with the simulated variable.

Simulation—We discretize the modified and original governing equations in terms of a simulative independent variable, u . To do so, we choose $x = u$ (and hence bookkeeping is not necessary and we can present them interchangeably) and $\tilde{dx} = S(u)du$ with the transformation function $S(u)$ constructed based on the effective wavenumber criteria. We use a uniform mesh for u , which is equivalent to using a uniform mesh for x and a nonuniform mesh for $\tilde{x}$. The equivalent modified governing equation is written in terms of u , and discretized to be equivalent with the regular governing equation at the RI-TL interface.

Implementation Details

For simplicity in this example, we set $S(u)$ to be purely imaginary. The transformation is purely spatial and there are no modifications made to the wavenumber, but we may think of an equivalent parametric transformation (effective wavenumber or wavenumber as a function of u rather than $\tilde{x}$) to guide the choice of $S(u)$.

The solution originally had the increase in oscillation rate before the increase in decay rate. We wish to increase the decay rate but limit an increase in the oscillation rate *while the amplitude of the solution is still large* to avoid the additional discretization error.

We choose the following transformation function,

$$S(x)=1-ia(x-x_0)^b, \quad (1)$$

with $a = 0$ for $x \leq x_0$, $a = 5$ for $x > x_0$, and $b = 3$.

Performance

For transformation construction, assuming the WKB approximation still holds, one can estimate the expected x_{comp} and choose the transformation function parameters accordingly and then test that errors are negligible for that parameter set. Performance criteria in Table 1 are the 2-norm of relative deviation between the various solutions in $x < x_0$. These tabulated criteria are for Fig. 2 for which the input frequency = 2.4 kHz. The subscript $\hat{y}$ refers to the transformed model solution. We refer to the discrete model solutions (exemplified here with a spatial step size $h = 2.9 \cdot 10^{-3}$) as functions of the discrete space index, j ; and refer to the approximation of the continuous model solution (using an appropriate $h = 2.9 \cdot 10^{-4}$) as functions of the continuous space variable, x or $\hat{x}$. The tabulated values C_1^* and C_m^* are the ‘continuous’ and discrete relative deviation error norms respectively, and C_2 and $\hat{C}_2$ are the relative residual error norms (discretization errors). C_1^* shows that in the continuous domain $P(x)$ and $\hat{P}(\hat{x})$ have decent correspondence within the RI. As indicated by C_2 and $\hat{C}_2$, in the discrete domain, for a step size $h=2.9 \cdot 10^{-3}$, $P[j]$ and $\hat{P}[\hat{j}]$ have similar accuracy when compared with their respective ‘continuous’ solutions $P(x)$ and $\hat{P}(\hat{x})$ (approximated by $h = 2.9 \cdot 10^{-4}$). The cumulative measure C_m^* shows that for $h = 2.9 \cdot 10^{-3}$, the discrete solutions, $P[j]$ and $\hat{P}[\hat{j}]$, are comparable.

While Fig. 2 (middle and bottom right) is only for a single frequency, it is illustrative of the qualitative pattern of transformation effect for frequencies within the CF range for which the solution is well approximated by WKB. Outside of this range, the deviation error increases (though the solution follows the same qualitative pattern). We attribute this increase to the process of constructing the transformation function which we based on wavenumber from an approximate solution. Specifically, as is apparent from C_1^* , the deviation in $x < x_0$ between the regular and transformed solutions are not attributed to discretization errors and instead, we anticipate, they are due to large gradients of the *effective* material properties, and may be measurable using a cumulative WKB deviation measure. We hypothesize that allowing $\text{Re}\{S(u) = 1\}$ could help overcome this problem.

Beyond construction and performance criteria that may be investigated for the case study model, further characterizations are ideally performed for a 3D nonlinear time domain simulation (our target models) to be of value. These include run time reduction and feasibility of combination with previous methods of improving efficiency.

Characterization

Mechanism—As shown in Fig. 2 for 2.4 kHz, the pressure in the transformed cochlea decays before it does in the regular cochlea allowing for imposition of a computational boundary at the artificial apex; also it decays such that its gradients are smaller than that of the regular cochlea. Hence, this category of transformations can allow for a decrease in number of spatial sections using both mechanisms A and B presented earlier.

Frequency range—The complex spatial transformation with the above specific constructed transformation function is appropriate for the Zweig model for multi-frequency stimuli with frequency components that fall within the range of frequencies for which the WKB method provides an accurate approximate solution. Within the appropriate frequency range, the transformation function is appropriate with some deviation error. Outside of this range, on the lowest frequency side of the CF range, spatial causality across the RI-TL interface is lost and hence, deviation error increases. This is not to be confused with a loss of accuracy. We hypothesize that this problem can be lessened by constructing a different transformation function.

Extendibility—The construction and hence extendibility of this transformation depends on the shape of the wavenumber(s). Hence, for some models, either the transformation function needs to be modified based on the wavenumber(s), or the same transformation function can be used after allowing a small layer (before applying the transformation) for converging the model wavenumber to a form similar in shape to Fig. 2 (top left). For example, the extra layer may be used to turn off an amplifier for an active model in which the constructed transformation function is inappropriate.

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FIGURE 1. Schematic representation of two transformation mechanisms

Top: Mechanism A (more basal absorption). Bottom: Mechanism B (reduced spatial gradients). Solid line: no transformation; Dashed line: with transformation. P is the differential pressure across the basilar membrane

FIGURE 2. Complex spatial transformation

Left top: Wavenumber as a function of the simulated variable before transformation. Left bottom: Wavenumber as a function of the simulated variable after transformation. Right top: spatial extension into the complex domain occurs after the beginning of the boundary layer (at 0.5 cm) as indicated by the dashed vertical line. Right middle: $|P(u)|$ for original and transformed models. Right bottom: Relative magnitude of difference vector between original and transformed model solutions. The input frequency was 2.4 kHz; the step size was $2.9 \cdot 10^{-3}$ cm.

TABLE 1

Performance evaluation for the complex spatial transformation using an input frequency of 2.4 kHz. The number of sections is given in parentheses.

$C_1^* (n = 7000)$	$C_2 (n = 700)$	$\hat{C}_2 (n = 700)$	$C_m^* (n = 700)$
0.05	0.01	0.01	0.05