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Masses of Fluid for Cylindrical Tanks in Rock With Partial Uplift of Bottom Plate

This study proposes the use of a slice model consisting of a set of thin rectangular tanks for evaluating the masses of fluid contributing to the rocking motion of cylindrical tanks; the effective mass of fluid for rocking motion, that for rocking–bulging interaction, effective moment inertia of fluid for rocking motion and its centroid. They are mathematically or numerically quantified, normalized, tabulated, and depicted as functions of the aspect of tanks for different values of the ratio of the uplift width of the tank bottom plate to the diameter of tank for the designer's convenience. [DOI: 10.1115/1.4032784]

Introduction

For unanchored tanks in seismically active areas, the tank rocking motion is inevitable and sufficient design for the uplift part of the tank bottom plate is indispensable. Although the tank rocking motion has attracted many researchers [1,2, among others], the dynamical mechanism and physical quantities contributing to it have not been fully understood. Even in the latest seismic design code [3], uplift of the tank bottom plate is determined by a diagram derived from a parametric study with the finite element model tanks as functions of the overturning moment for different values of the aspect of tanks. Complexity of the tank rocking motion, which discontinuously appears during an earthquake, may put the tank's rocking motion beyond analytical treatment.

In analyzing the liquid containing upright, circular cylindrical tank excited by an angular or rocking base motion about a horizontal axis at the tank center, Veletsos and Tang presented the response of the liquid governed by Laplace's equation in cylindrical coordinates and its solution given by the sum of an impulsive component and a convective component. They derived liquid pressure to the tank, the effective mass [4], and the effective mass moment of inertia [5]. In an unanchored tank model, since the uplifting area of the base plate seldom exceeds 10% of the total base area, Malhotra and Veletsos neglected the effective mass moment inertia from the equilibrium of moments (see Eq. (5b) in Ref. [1]).

In contrast, presumption of an angular or rocking base motion about a horizontal axis at the tank bottom edge differentiates the present work from the previous work. References [6] and [7] first proposed the use of a dynamical model consisting of a spring-mass-rigid-body combined system for evaluating the rocking motion of cylindrical tanks about a horizontal axis at the tank bottom edge subjected to horizontal ground acceleration. By analogy with the equivalent mass fastened rigidly to the wall of the container [8], the effective mass of fluid for rocking motion and that for rocking–bulging interaction were predicted and quantified empirically [7]. Since they play important roles in the tank rocking motion and reach a half of the total mass of the fluid as shown latter, their rigorous quantification is indispensable for accurate evaluation of the tank rocking motion. However due to mathematical difficulty in solving a boundary-value problem with the asymmetric boundary conditions in cylindrical coordinates that is caused by the crescentlike uplift of the tank bottom plate, their rigorous definitions have not been provided to date.

In contrast, Ref. [9] introduced the use of a slice model consisting of a set of thin rectangular tanks for evaluating the fluid pressure on the cylindrical tank which has the crescentlike uplift part in the bottom plate. Each slice model is laid radially across the tank center. The benefit of the use of the slice model is that a boundary-value problem with asymmetric boundary conditions in cylindrical coordinates that is caused by the crescentlike uplift of the tank bottom plate is converted into a boundary-value problem with corresponding boundary conditions in Cartesian coordinates that is determined by the location of the slice model in the cylindrical tank, because Cartesian coordinates is capable of solving a boundary-value problem with complex boundary conditions.

This paper tries to expand its application to evaluating the effective mass of fluid for rocking motion and that for rocking–bulging interaction of cylindrical tanks which has the crescentlike uplift part in the bottom plate, because they are indispensable in evaluating the tank rocking motion with the spring-mass-rigid-body combined system. In defining these masses, this paper proposes the use of the slice model placed perpendicular to a rotational axis of the tank rocking motion at the tank bottom edge. Since this arrangement enables the slice model to evaluate both effective mass of fluid for rocking motion and that for bulging motion simultaneously, it is indispensable to evaluate the effective mass of fluid for rocking–bulging interaction by following the manner proposed in Ref. [10].

For the designer's convenience, values of the effective mass of fluid for rocking motion, that for rocking–bulging interaction, effective moment inertia of fluid for rocking motion and its centroid are mathematically or numerically quantified, normalized by the total mass of fluid, tabulated, and depicted for the tank commonly used geometry as functions of the aspect of tanks for different values of the ratio of the uplift width of the tank bottom plate to the diameter of tank. Here, this study ignores the effects of sloshing on the tank rocking motion based on the following several reasons. First, the tank bulging motion may dominate the tank rocking motion, because both lateral force and overturning moment induced by the tank bulging motion are relatively large comparing to that induced by the tank sloshing motion [1,4, among others]. Second, in analyzing the rocking response of liquid storage tanks, Veletsos and Tang defined the convective component of fluid to satisfy the linearized boundary condition at free surface that includes disturbance induced by the impulsive component of fluid (see Eq. (43c) in Ref. [4]). However, a previous shaker table test whose driving frequency was identical with the natural frequency of the tank bulging motion revealed that the tank rocking motion did not stimulate any significant sloshing waves except very small and negligible fluctuation of fluid at free surface like a ripple [6]. Therefore, the previous shaker table test may qualitatively conclude that the convective component of fluid

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Fig. 1 Cylindrical tank, slice model, and their coordinates

may not be developed in principle by the tank rocking motion or may be negligible if it is developed.

Fluid Pressure on Each Slice Model Due to Tank Rocking Motion

Consider a cylindrical tank whose shell is undeformable and depth of fluid and diameter are h and $2l$, respectively (see Fig. 1).

Fig. 2 (a) Slice model which has an unlifted bottom part and (b) slice model whose bottom part undergoes uplift

The tank rotates around the Y -axis with the angular velocity $\dot{\theta}_0(t)$ pivoting at its left bottom edge, at the origin O of the global Cartesian coordinates X - Y - Z , and its bottom plate has a crescentlike uplift part in consequence. The value of $2\delta l$ gives the width of the uplift part of the tank bottom plate (see hatched area of Figs. 1 and 2), while $l - \delta l$ gives the diameter of the circular unlifted part of the tank bottom plate. The circular unlifted part was first assumed by Clough [2]. A thin rectangular tank, so-called the slice model, is set inside the cylindrical tank at a distance a from the X -axis and perpendicular to the Y -axis (see Fig. 1). To simplify calculation presented later, let the element coordinates x - z to the slice model, which is parallel to the X - and Z -axes of the global coordinates, whose origin O' is set at a left bottom edge of the slice model.

Figures 2(a) and 2(b) show a plane view of the tank bottom plate where the hatched part shows the crescentlike uplift part of the tank bottom plate. In these figures, the bottom part of the slice model is shown as a segment $O'-N$. From these figures, a boundary condition that should be applied to the bottom part of the slice model changes in accordance with the location of the slice model in the cylindrical tank. If the slice model is located between $Y = 0$ and $Y = \pm l - \delta l$, the boundary condition for the bottom part of the slice should contain the unlifted part of the tank bottom plate in between uplift parts of the tank bottom plate. In contrast, if the slice model is located between $Y = \pm l - \delta l$ and $Y = \pm l$, the boundary condition applied to the bottom part consists of only uplift part of the tank bottom plate. To simplify calculation, the bottom of the cylindrical shell is assumed to uplift rectilinearly

Fig. 3 (a) Boundary conditions for the slice model which has an unlifted bottom part and (b) boundary conditions for the slice model whose bottom part undergoes uplift

and deformation of the tank bottom plate can be approximated by a linear function at any sections. However, the slope of the linear function changes at i - j section shown in Fig. 2(b), where i and j represent the vertex of the circular unuplift part of the tank bottom plate and that of the circular tank bottom plate on the X - Y plane, respectively.

First consider the slice model located between $Y = 0$ and $Y = \pm l - \delta l$ (see Fig. 2(a)). The fluid is assumed to be inviscid, incompressible, and irrotational. The fluid motion may, then, be completely defined by a velocity potential function $\Phi(x, z, t)$. The Laplace equation in Cartesian coordinate gives the continuity condition

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(x, z, t) = 0 \quad (1)$$

Figure 3(a) shows the boundary conditions of the side walls and bottom part of the slice model. Since the bottom part contains both uplift part and unuplift part of the tank bottom plate, the boundary condition applied to the bottom part of the slice model changes from section to section. All boundary conditions applied to the slice model concerned are

$$\text{Free surface: } \Phi(x, h, t) = 0 \quad (0 \leq x \leq 2\sqrt{l^2 - a^2}) \quad (2a)$$

$$\text{Left wall: } \frac{\partial \Phi}{\partial x}(0, z, t) = -\dot{\theta}_0(t)z \quad (0 \leq z \leq h) \quad (2b)$$

$$\text{Right wall: } \frac{\partial \Phi}{\partial x}(2\sqrt{l^2 - a^2}, z, t) = -\dot{\theta}_0(t)z \quad (0 \leq z \leq h) \quad (2c)$$

$$\begin{aligned} \text{Bottom: } \frac{\partial \Phi}{\partial z}(x, 0, t) &= \frac{(l - \sqrt{l^2 - a^2})\dot{\theta}_0(t)}{\delta l + \sqrt{(1 - \delta^2)l^2 - a^2} - \sqrt{l^2 - a^2}} \\ &\times \left(x - \sqrt{l^2 - a^2} + \sqrt{(1 - \delta^2)l^2 - a^2} + \delta l\right) \quad (O' \leq x \leq L) \\ &= 0 \quad (L \leq x \leq M) \\ &= \frac{(l - \sqrt{l^2 - a^2})\dot{\theta}_0(t)}{\delta l - \sqrt{(1 - \delta^2)l^2 - a^2} + \sqrt{l^2 - a^2}} \\ &\times \left(x - \sqrt{l^2 - a^2} - \sqrt{(1 - \delta^2)l^2 - a^2} + \delta l\right) \quad (M \leq x \leq N) \end{aligned} \quad (2d)$$

where $O'(0)$, $L(\sqrt{l^2 - a^2} - \sqrt{(1 - \delta^2)l^2 - a^2} - \delta l)$, $M(\sqrt{l^2 - a^2} + \sqrt{(1 - \delta^2)l^2 - a^2} - \delta l)$, and $N(2\sqrt{l^2 - a^2})$ are the x -values of the element coordinate of the slice model and give geometry of the slice model and boundaries between uplift section and unuplift section of the bottom part of the slice model.

To obtain a solution of the Laplace equation that satisfies all boundary conditions, Fourier transformation is applied. Fourier inverse transformation is then applied to it and finds a solution of the velocity potential function of the slice model. Its derivative with respect to time gives the fluid pressure at an arbitrary point inside the slice model of interest. The Appendix shows its derivation process

$$\begin{aligned} P(x, z, t) &= -\rho \frac{\partial \Phi(x, z, t)}{\partial t} = -\rho \ddot{\theta}_0(t) \Phi(x, z) = -\rho \ddot{\theta}_0(t) \left[\frac{1}{2\sqrt{l^2 - a^2}} \left\{ \frac{e}{2} L^2 + fL + \frac{q}{2} N^2 - \frac{q}{2} M^2 + rN - rM \right\} (z - h) \right. \\ &+ \frac{1}{\sqrt{l^2 - a^2}} \sum_{n=1}^{\infty} \left[\left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2 - a^2}} - 1 \right\} \right. \right. \\ &+ f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \\ &+ r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \left. \cdot \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \left[- \left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2 - a^2}} - 1 \right\} \right. \right. \right. \right. \\ &+ f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \\ &+ r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \left. \right] \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} - \{ 1 - (-1)^n \} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left(\frac{h}{\cosh \frac{n\pi h}{2\sqrt{l^2 - a^2}}} - \frac{2\sqrt{l^2 - a^2}}{n\pi} \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} \right) \left. \right] \\ &\cdot \cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \{ 1 - (-1)^n \} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left(z - \frac{2\sqrt{l^2 - a^2}}{n\pi} \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} \right) \left. \right] \cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} \left. \right] \end{aligned} \quad (3)$$

where $e = (l - \sqrt{l^2 - a^2})/(\delta l + \sqrt{(1 - \delta^2)l^2 - a^2} - \sqrt{l^2 - a^2})$, $f = l - \sqrt{l^2 - a^2}$, $q = (l - \sqrt{l^2 - a^2})/(\delta l - \sqrt{(1 - \delta^2)l^2 - a^2} + \sqrt{l^2 - a^2})$, $r = \{(l - \sqrt{l^2 - a^2})(\delta l - \sqrt{l^2 - a^2})\}/(\delta l - \sqrt{(1 - \delta^2)l^2 - a^2} + \sqrt{l^2 - a^2})$.

Here, if the slice model is located at the tank center ($a = 0$) and the tank bottom plate fully uplifts ($\delta = 1$), Eq. (3) naturally becomes Eq. (21) in Ref. [11]. Since the analytical accuracy of Eq. (21) in Ref. [11] was confirmed by comparing it with the numerical results performed by the finite element

analysis, their consistent corroborates the analytical accuracy of Eq. (3).

Next, consider the slice model located between $Y = \pm l - \delta l$ and $Y = \pm l$ (see Fig. 2(b)). Figure 3(b) shows the boundary conditions applied to the side walls and bottom part of the slice model. Since the slope of the uplift part of the tank bottom plate changes at i - j section defined on Fig. 2(b), the boundary condition applied to the bottom part changes accordingly. All boundary conditions applied to the slice model concerned are

$$\text{Free surface: } \Phi(x, h, t) = 0 \quad (0 \leq x \leq 2\sqrt{l^2 - a^2}) \quad (4a)$$

$$\text{Left wall: } \frac{\partial \Phi}{\partial x}(0, z, t) = -\dot{\theta}_0(t)z \quad (0 \leq z \leq h) \quad (4b)$$

$$\text{Right wall: } \frac{\partial \Phi}{\partial x}(2\sqrt{l^2 - a^2}, z, t) = -\dot{\theta}_0(t)z \quad (0 \leq z \leq h) \quad (4c)$$

$$\begin{aligned} \text{Bottom: } \frac{\partial \Phi}{\partial z}(x, 0, t) &= \frac{(a-l+\delta l) - (l-\sqrt{l^2-a^2})}{\sqrt{l^2-a^2}+a-l} \dot{\theta}_0(t)x + (l-\sqrt{l^2-a^2})\dot{\theta}_0(t) \quad (O' \leq x \leq K) \\ &= \frac{(l-\sqrt{l^2-a^2}) - (a-l+\delta l)}{\sqrt{l^2-a^2}-a+l} \dot{\theta}_0(t)x + \left[(l+\sqrt{l^2-a^2}) - 2\sqrt{l^2-a^2} \frac{(l-\sqrt{l^2-a^2}) - (a-l+\delta l)}{\sqrt{l^2-a^2}-a+l} \right] \dot{\theta}_0(t) \quad (K \leq x \leq N) \end{aligned} \quad (4d)$$

where $O'(0)$, $K(\sqrt{l^2-a^2}+a-l)$, $N(2\sqrt{l^2-a^2})$ are the x -values of the element coordinate of the slice model and give geometry of the slice model and boundaries where the slope of the uplift section of the tank bottom plate changes. Following the foregoing approach, the fluid pressure of the slice model whose bottom part undergoes uplift at an arbitrary point inside the slice model of interest is derived as

$$\begin{aligned} P(x, z, t) &= -\rho \frac{\partial \Phi(x, z, t)}{\partial t} = -\rho \ddot{\theta}_0(t) \Phi(x, z) = -\rho \ddot{\theta}_0(t) \left[\frac{1}{2\sqrt{l^2-a^2}} \left\{ \frac{e}{2} K^2 + fK + \frac{q}{2} N^2 - \frac{q}{2} K^2 + rN - rK \right\} (z-h) \right. \\ &\quad + \frac{1}{\sqrt{l^2-a^2}} \sum_{n=1}^{\infty} \left[\left[e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 K \sin \frac{n\pi K}{2\sqrt{l^2-a^2}} + e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi K}{2\sqrt{l^2-a^2}} - 1 \right\} + f \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi K}{2\sqrt{l^2-a^2}} \right\} \right. \right. \\ &\quad \left. \left. - q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ K \sin \frac{n\pi K}{2\sqrt{l^2-a^2}} \right\} + q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi K}{2\sqrt{l^2-a^2}} \right\} + r \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi K}{2\sqrt{l^2-a^2}} \right\} \right] \cdot \sinh \frac{n\pi z}{2\sqrt{l^2-a^2}} \right. \\ &\quad \left. + \left[- \left[e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 K \sin \frac{n\pi K}{2\sqrt{l^2-a^2}} + e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi K}{2\sqrt{l^2-a^2}} - 1 \right\} + f \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi K}{2\sqrt{l^2-a^2}} \right\} - q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \right. \right. \right. \\ &\quad \left. \left. \times \left\{ K \sin \frac{n\pi K}{2\sqrt{l^2-a^2}} \right\} + q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi K}{2\sqrt{l^2-a^2}} \right\} + r \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi K}{2\sqrt{l^2-a^2}} \right\} \right] \tanh \frac{n\pi h}{2\sqrt{l^2-a^2}} \right. \\ &\quad \left. \left. - \{1 - (-1)^n\} \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left(\frac{h}{\cosh \frac{n\pi h}{2\sqrt{l^2-a^2}}} - \frac{2\sqrt{l^2-a^2}}{n\pi} \tanh \frac{n\pi h}{2\sqrt{l^2-a^2}} \right) \right] \cdot \cosh \frac{n\pi z}{2\sqrt{l^2-a^2}} + \{1 - (-1)^n\} \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \right. \\ &\quad \left. \left(z - \frac{2\sqrt{l^2-a^2}}{n\pi} \sinh \frac{n\pi z}{2\sqrt{l^2-a^2}} \right) \right] \cos \frac{n\pi x}{2\sqrt{l^2-a^2}} \end{aligned} \quad (5)$$

where $e = (a-l+\delta l/\delta) - (l-\sqrt{l^2-a^2})/\sqrt{l^2-a^2}+a-l$, $f = l-\sqrt{l^2-a^2}$, $q = (l-\sqrt{l^2-a^2}) - ((a-l+\delta l)/\delta)/\sqrt{l^2-a^2}-a+l$, $r = [(l+\sqrt{l^2-a^2}) - 2\sqrt{l^2-a^2} - (l-\sqrt{l^2-a^2}) - ((a-l+\delta l)/\delta)/\sqrt{l^2-a^2}-a+l]$.

Effective Density of Fluid for Rocking Motion of Each Slice Model

In this study, the effective mass of fluid for the tank motion of interest may be considered to be the mass of fluid undergoing the inertia force in unison with the tank motion of interest.

Accordingly, the effective density of fluid for the tank motion of interest is defined as the nominal density of the small volume of fluid at an arbitrary point inside the tank that contributes to constituting the effective mass of fluid for the tank motion of interest. Reference [12] revealed that the pressure gradient of fluid in a given direction gave the effective density of fluid moving in unison with the tank in the given direction at an arbitrary point inside the tank. For evaluating the effective density of fluid for rocking motion of the slice model of interest, consider the equilibrium of forces in the rotational direction acting on a small volume (see Fig. 4).

From the equilibrium of forces, the effective density of fluid for rocking motion at an arbitrary point inside the slice model is given as a function of the pressure gradient in the rotational direction.

$$\rho_{\text{eff}}(r, \theta) = -\frac{1}{r^2 \ddot{\theta}_0(t)} \cdot \frac{\partial P(r, \theta, t)}{\partial \theta} \quad (6)$$

Employing Eqs. (3) and (5) and coordinate transformation, the effective density of fluid for rocking motion at an arbitrary point inside the slice model of interest is given as

$$\rho_{\text{eff}}(x, z) = f_A(x, z)\rho \quad \text{or} \quad f_B(x, z)\rho \quad (7)$$

where $f_A(x, z)$ and $f_B(x, z)$ are the ratios of the effective density of fluid for rocking motion to the original density of fluid at an arbitrary point inside each slice model. Here, $f_A(x, z)$ is that for the slice model located between $Y = 0$ and $Y = \pm l - \delta l$ (see Fig. 2(a)), while $f_B(x, z)$ is that for the slice model located between $Y = \pm l - \delta l$ and $Y = \pm l$ (see Fig. 2(b)). Employing $b = l - \sqrt{l^2 - a^2}$, $f_A(x, z)$ is given as

Fig. 4 Equilibrium of forces acting on a small volume in the slice model in rocking motion

$$\begin{aligned} f_A(x, z) = & -\frac{z}{(x+b)^2 + z^2} \left[-\frac{1}{2(l^2 - a^2)} \sum_{n=1}^{\infty} \left[\left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2 - a^2}} - 1 \right\} \right. \right. \right. \\ & + f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \\ & \left. \left. + r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \right] \cdot \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \left[-\left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2 - a^2}} - 1 \right\} \right. \right. \right. \\ & + f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \\ & \left. \left. + r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \right] \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} - \{ 1 - (-1)^n \} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left(\frac{h}{\cosh \frac{n\pi h}{2\sqrt{l^2 - a^2}}} - \frac{2\sqrt{l^2 - a^2}}{n\pi} \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} \right) \right] \\ & \cdot \cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \{ 1 - (-1)^n \} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left(z - \frac{2\sqrt{l^2 - a^2}}{n\pi} \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} \right) \left[\sin \frac{n\pi x}{2\sqrt{l^2 - a^2}} \right] \\ & + \frac{x+b}{(x+b)^2 + z^2} \left[\frac{1}{2\sqrt{l^2 - a^2}} \left\{ \frac{e}{2} L^2 + fL + \frac{q}{2} N^2 - \frac{q}{2} M^2 + rN - rM \right\} \right. \\ & + \frac{1}{\sqrt{l^2 - a^2}} \sum_{n=1}^{\infty} \left[\frac{n\pi z}{2\sqrt{l^2 - a^2}} \left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2 - a^2}} - 1 \right\} \right. \right. \\ & + f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \\ & \left. \left. + r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \right] \cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \frac{n\pi z}{2\sqrt{l^2 - a^2}} \left[-\left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \right. \right. \\ & \left\{ \cos \frac{n\pi L}{2\sqrt{l^2 - a^2}} - 1 \right\} + f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \\ & \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} + r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \left] \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} - \{ 1 - (-1)^n \} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \right. \\ & \left. \left(\frac{h}{\cosh \frac{n\pi h}{2\sqrt{l^2 - a^2}}} - \frac{2\sqrt{l^2 - a^2}}{n\pi} \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} \right) \right] \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \{ 1 - (-1)^n \} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left(1 - \cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} \right) \left[\cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} \right] \right] \end{aligned} \quad (8a)$$

while $f_B(x, z)$ is given as

$$\begin{aligned}
 f_B(x, z) = & -\frac{z}{(x+b)^2 + z^2} \left[-\frac{1}{2(l^2 - a^2)} \sum_{n=1}^{\infty} \left[\left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 K \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi K}{2\sqrt{l^2 - a^2}} - 1 \right\} \right. \right. \\
 & + f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ K \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} \\
 & + r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} \left. \right] \cdot \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \left[-\left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 K \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \right. \right. \\
 & \left\{ \cos \frac{n\pi K}{2\sqrt{l^2 - a^2}} - 1 \right\} + f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ K \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \\
 & \left\{ \cos(n\pi) - \cos \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} + r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} \left. \right] \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} - \{1 - (-1)^n\} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \\
 & \left(\frac{h}{\cosh \frac{n\pi h}{2\sqrt{l^2 - a^2}}} - \frac{2\sqrt{l^2 - a^2}}{n\pi} \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} \right) \left. \right] \cdot \cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \{1 - (-1)^n\} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \\
 & \left(z - \frac{2\sqrt{l^2 - a^2}}{n\pi} \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} \right) \left. \right] \sin \frac{n\pi x}{2\sqrt{l^2 - a^2}} \left. \right] + \frac{x+b}{(x+b)^2 + z^2} \left[\frac{1}{2\sqrt{l^2 - a^2}} \left\{ \frac{e}{2} K^2 + fK + \frac{q}{2} N^2 - \frac{q}{2} K^2 + rN - rK \right\} \right. \\
 & + \frac{1}{\sqrt{l^2 - a^2}} \sum_{n=1}^{\infty} \left[\frac{n\pi z}{2\sqrt{l^2 - a^2}} \left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 K \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi K}{2\sqrt{l^2 - a^2}} - 1 \right\} \right. \right. \\
 & + f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ K \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} \\
 & + r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} \left. \right] \cdot \cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + \frac{n\pi z}{2\sqrt{l^2 - a^2}} \left[-\left[e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 K \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right. \right. \\
 & + e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi K}{2\sqrt{l^2 - a^2}} - 1 \right\} + f \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} - q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ K \sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} \\
 & + q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} + r \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi K}{2\sqrt{l^2 - a^2}} \right\} \left. \right] \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} \\
 & - \{1 - (-1)^n\} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left(\frac{h}{\cosh \frac{n\pi h}{2\sqrt{l^2 - a^2}}} - \frac{2\sqrt{l^2 - a^2}}{n\pi} \tanh \frac{n\pi h}{2\sqrt{l^2 - a^2}} \right) \left. \right] \cdot \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} \\
 & + \{1 - (-1)^n\} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left(1 - \cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} \right) \left. \right] \cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} \left. \right] \left. \right] \quad (8b)
 \end{aligned}$$

Table 1 Values of the ratio of the effective mass of fluid for rocking motion to the total mass of fluid fulfilled in the tank

	$h/2l$												
	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
$\delta = 1.0$	0.8955	0.8609	0.8310	0.8050	0.7850	0.7730	0.7623	0.7540	0.7490	0.7470	0.7440	0.7450	0.7450
$\delta = 0.1$	0.1482	0.1801	0.2150	0.2510	0.2840	0.3170	0.3490	0.3770	0.3970	0.4250	0.4410	0.4600	0.4810
$\delta = 0.09$	0.1360	0.1678	0.2030	0.2390	0.2760	0.3080	0.3380	0.3690	0.3900	0.4150	0.4380	0.4580	0.4720
$\delta = 0.08$	0.1243	0.1569	0.1920	0.2290	0.2640	0.3000	0.3300	0.3610	0.3830	0.4090	0.4290	0.4500	0.4670
$\delta = 0.07$	0.1130	0.1447	0.1810	0.2190	0.2540	0.2890	0.3210	0.3530	0.3760	0.4030	0.4230	0.4440	0.4650
$\delta = 0.06$	0.1009	0.1337	0.1700	0.2080	0.2440	0.2800	0.3130	0.3450	0.3690	0.3970	0.4180	0.4400	0.4580
$\delta = 0.05$	0.0893	0.1216	0.1590	0.1970	0.2350	0.2720	0.3050	0.3380	0.3630	0.3910	0.4150	0.4350	0.4560
$\delta = 0.04$	0.0780	0.1107	0.1480	0.1880	0.2250	0.2640	0.2970	0.3310	0.3590	0.3860	0.4090	0.4300	0.4520
$\delta = 0.03$	0.0664	0.0989	0.1370	0.1770	0.2160	0.2540	0.2890	0.3240	0.3530	0.3790	0.4040	0.4270	0.4480
$\delta = 0.02$	0.0554	0.0882	0.1260	0.1660	0.2070	0.2460	0.2820	0.3170	0.3470	0.3730	0.3990	0.4230	0.4440
$\delta = 0.01$	0.0450	0.0770	0.1160	0.1580	0.2010	0.2390	0.2770	0.3110	0.3420	0.3690	0.3970	0.4190	0.4420

Fig. 5 Effective mass of fluid for rocking motion

Fig. 6 Effective moment inertia of fluid for rocking motion

Effective Mass of Fluid for Rocking Motion of Cylindrical Tanks

Consider a small volume of fluid inside the slice model whose fluid density is specified by Eq. (7). The effective mass of fluid for rocking motion of cylindrical tank M_r is given by summing up the mass of all small volumes inside all slice models

$$M_r = f_r M_{\text{Total}} \quad (9a)$$

$$f_r = \frac{1}{l^2 h \pi} \left[\int_0^h \int_0^{2\sqrt{l^2 - a^2}} \int_{-(l-\delta l)}^{l-\delta l} f_A(x, z) da dx dz + 2 \int_0^h \int_0^{2\sqrt{l^2 - a^2}} \int_{l-\delta l}^l f_B(x, z) da dx dz \right] \quad (9b)$$

Here, f_r is a ratio of the effective mass of fluid for rocking motion of cylindrical tank M_r to the total mass of the fluid fulfilled in the cylindrical tank M_{Total} .

Values of f_r are evaluated numerically, tabulated and depicted for the tank commonly used geometry as functions of the aspect of the tank $h/2l$ for different values of δ (see Table 1 and Fig. 5). In evaluating fluid pressure and effective masses of fluid, the first five terms give a sufficiently converged value. According to Ref. [2], the value of δ is up to 6–7% for broad tanks. Assuming that the value of δ varies from 1% to 10%, the value of f_r is computed. The value of f_r increases as the tank becomes taller, while it increases as the uplift part extends (see Fig. 5). In tall cylindrical tanks, since it is about a half of the total mass of the fluid fulfilled in the tanks, ignoring its effects on the tank rocking motion in analyzing the seismic response of cylindrical tanks may yield erroneous results.

Effective Moment Inertia of Fluid for Rocking Motion of Cylindrical Tanks

In analyzing the rocking motion of cylindrical tanks with the spring-mass-rigid-body combined system [13, for instance], the effective moment inertia of fluid for rocking motion of cylindrical tanks about its centroid is indispensable. Consider the effective mass of small volume $dm_{\text{eff}} = \rho_{\text{eff}} dx dz$ at an arbitrary point inside each slice model, the horizontal and vertical distances toward the centroid of the effective mass of fluid for rocking motion from the rotational axis of the tank rocking motion of cylindrical tank, $R_{r,x}$ and $R_{r,z}$, are readily calculated as

$$R_{r,x} = \frac{\sum (x + (l - \sqrt{l^2 - a^2})) dm_{\text{eff}}}{\sum dm_{\text{eff}}} = d_{r,x} \cdot 2l \quad (10a)$$

$$R_{r,z} = \frac{\sum z dm_{\text{eff}}}{\sum dm_{\text{eff}}} = d_{r,z} \cdot h \quad (10b)$$

where $d_{r,x}$ and $d_{r,z}$ are the horizontal and vertical distance toward the centroid normalized by the diameter and depth of fluid (see Fig. 6). Values of $d_{r,x}$ and $d_{r,z}$ are evaluated numerically, tabulated and depicted for the tank commonly used geometry as functions of the aspect of the tank $h/2l$ for different values of δ (see Tables 2 and 3, and Figs. 7(a) and 7(b)).

On the contrary, employing the distance between centroid of the small volume and that of the effective mass of fluid for rocking motion of cylindrical tank rr (see Fig. 6), the effective moment inertia of fluid for rocking motion of cylindrical tank I_{eff} about its centroid is readily calculated as

$$I_{\text{eff}} = sI = \rho \left[\int_0^h \int_0^{2\sqrt{l^2 - a^2}} \int_{-(l-\delta l)}^{l-\delta l} f_A(x, z) rr^2 da dx dz + 2 \int_0^h \int_0^{2\sqrt{l^2 - a^2}} \int_{l-\delta l}^l f_B(x, z) rr^2 da dx dz \right] \quad (11a)$$

Table 2 Values of the ratio of horizontal distance toward centroid to the diameter of tank

	$h/2l$												
	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$\delta = 1.0$	0.5164	0.5159	0.5141	0.5116	0.5087	0.5056	0.5026	0.4998	0.4973	0.4950	0.4931	0.4915	0.4902
$\delta = 0.1$	0.6162	0.5543	0.5086	0.4774	0.4566	0.4434	0.4356	0.4313	0.4289	0.4291	0.4293	0.4310	0.4331
$\delta = 0.09$	0.6151	0.5508	0.5041	0.4726	0.4525	0.4394	0.4318	0.4283	0.4263	0.4264	0.4278	0.4296	0.4310
$\delta = 0.08$	0.6131	0.5463	0.4989	0.4676	0.4475	0.4356	0.4283	0.4254	0.4236	0.4242	0.4253	0.4273	0.4293
$\delta = 0.07$	0.6103	0.5412	0.4930	0.4621	0.4420	0.4309	0.4246	0.4222	0.4213	0.4219	0.4233	0.4255	0.4283
$\delta = 0.06$	0.6072	0.5348	0.4862	0.4558	0.4368	0.4264	0.4209	0.4191	0.4183	0.4195	0.4212	0.4238	0.4263
$\delta = 0.05$	0.6023	0.5271	0.4783	0.4485	0.4309	0.4216	0.4170	0.4159	0.4156	0.4172	0.4198	0.4220	0.4250
$\delta = 0.04$	0.5952	0.5174	0.4692	0.4408	0.4246	0.4171	0.4131	0.4128	0.4134	0.4155	0.4178	0.4204	0.4234
$\delta = 0.03$	0.5856	0.5054	0.4584	0.4320	0.4183	0.4119	0.4094	0.4097	0.4108	0.4129	0.4159	0.4190	0.4220
$\delta = 0.02$	0.5700	0.4898	0.4459	0.4233	0.4122	0.4073	0.4058	0.4068	0.4083	0.4109	0.4140	0.4175	0.4207
$\delta = 0.01$	0.5456	0.4696	0.4325	0.4146	0.4062	0.4029	0.4027	0.4040	0.4064	0.4093	0.4128	0.4161	0.4198

Table 3 Values of the ratio of vertical distance toward centroid to the diameter of tank

	$h/2l$												
	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$\delta = 1.0$	0.4733	0.4630	0.4520	0.4454	0.4408	0.4377	0.4360	0.4354	0.4359	0.4366	0.4378	0.4396	0.4411
$\delta = 0.1$	0.4947	0.4921	0.4873	0.4860	0.4857	0.4860	0.4869	0.4892	0.4910	0.4933	0.4958	0.4978	0.5006
$\delta = 0.09$	0.4965	0.4940	0.4895	0.4886	0.4882	0.4884	0.4900	0.4915	0.4935	0.4956	0.4982	0.5000	0.5035
$\delta = 0.08$	0.4986	0.4962	0.4921	0.4914	0.4911	0.4915	0.4928	0.4943	0.4962	0.4983	0.5006	0.5025	0.5059
$\delta = 0.07$	0.5011	0.4991	0.4949	0.4943	0.4941	0.4946	0.4954	0.4969	0.4988	0.5010	0.5033	0.5051	0.5083
$\delta = 0.06$	0.5046	0.5022	0.4980	0.4976	0.4976	0.4978	0.4985	0.4996	0.5015	0.5037	0.5058	0.5087	0.5111
$\delta = 0.05$	0.5090	0.5059	0.5020	0.5014	0.5010	0.5011	0.5015	0.5025	0.5044	0.5066	0.5081	0.5107	0.5122
$\delta = 0.04$	0.5144	0.5103	0.5062	0.5053	0.5049	0.5043	0.5045	0.5056	0.5067	0.5085	0.5105	0.5131	0.5146
$\delta = 0.03$	0.5220	0.5161	0.5110	0.5096	0.5081	0.5079	0.5076	0.5084	0.5097	0.5109	0.5132	0.5155	0.5170
$\delta = 0.02$	0.5322	0.5231	0.5164	0.5136	0.5120	0.5108	0.5106	0.5111	0.5124	0.5138	0.5159	0.5179	0.5195
$\delta = 0.01$	0.5448	0.5308	0.5214	0.5177	0.5151	0.5136	0.5134	0.5137	0.5146	0.5158	0.5179	0.5193	0.5215

$$rr = \sqrt{\left(R_{r,x} - \left(l - \sqrt{l^2 - a^2}\right) - x' - \frac{dx}{2}\right)^2 + \left(R_{r,z} - z' - \frac{dz}{2}\right)^2} \quad (11b)$$

$$I = \rho \int_0^h \int_0^{2\sqrt{l^2 - a^2}} \int_{-l}^l rr^2 dadxdz \quad (11c)$$

where s is a ratio of the effective moment inertia of fluid for rocking motion of cylindrical tanks normalized by the moment inertia of the total mass of the fluid fulfilled in the cylindrical tank I . Values of s are numerically evaluated, tabulated and depicted for the tank commonly used geometry as functions of the aspect of the tank $h/2l$ for different values of δ (see Table 4 and Fig. 8). The

value of s increases as the tank becomes taller, while it increases as the uplift part extends. In tall cylindrical tanks, since it is about two-thirds of the moment inertia of the total mass of the fluid fulfilled in the tanks, ignoring its effects on the tank rocking motion in analyzing the seismic response of cylindrical tanks may yield erroneous results.

Fluid Pressure on Each Slice Model Due to Tank Bulging Motion

The tank bulging motion has been thoroughly investigated by many researchers and the accompanying fluid pressure on the cylindrical tanks was given in the open literature [8,14, among others]. Employing the tank bulging motion, this section demonstrates evaluation accuracy of the effective mass of fluid of the

Fig. 7 (a) Horizontal distance toward the centroid of effective mass of fluid for rocking motion and (b) vertical distance toward the centroid of effective mass of fluid for rocking motion

Fig. 8 Effective moment inertia of fluid for rocking motion

Fig. 9 Boundary conditions of slice model for specifying the tank bulging motion

Table 4 Values of the ratio of the effective moment inertia of fluid for rocking motion to the moment inertia of total mass

	$h/2l$												
	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$\delta = 1.0$	0.5833	0.6252	0.6549	0.6685	0.6853	0.7029	0.7235	0.7474	0.7760	0.8056	0.8376	0.8728	0.9079
$\delta = 0.1$	0.1208	0.1363	0.1577	0.1859	0.2206	0.2625	0.3112	0.3652	0.4180	0.4800	0.5364	0.5969	0.6651
$\delta = 0.09$	0.1111	0.1266	0.1481	0.1762	0.2126	0.2542	0.3030	0.3581	0.4123	0.4717	0.5362	0.5974	0.6569
$\delta = 0.08$	0.1014	0.1171	0.1382	0.1672	0.2032	0.2475	0.2959	0.3516	0.4031	0.4674	0.5284	0.5893	0.6545
$\delta = 0.07$	0.0915	0.1070	0.1290	0.1584	0.1943	0.2386	0.2883	0.3448	0.4015	0.4624	0.5248	0.5865	0.6563
$\delta = 0.06$	0.0817	0.0971	0.1193	0.1489	0.1860	0.2309	0.2814	0.3383	0.3951	0.4577	0.5195	0.5864	0.6498
$\delta = 0.05$	0.0718	0.0873	0.1098	0.1398	0.1776	0.2233	0.2724	0.3323	0.3903	0.4536	0.5190	0.5824	0.6485
$\delta = 0.04$	0.0618	0.0777	0.1003	0.1310	0.1692	0.2166	0.2656	0.3265	0.3865	0.4505	0.5145	0.5792	0.6471
$\delta = 0.03$	0.0502	0.0677	0.0908	0.1219	0.1613	0.2086	0.2614	0.3206	0.3820	0.4439	0.5117	0.5792	0.6414
$\delta = 0.02$	0.0419	0.0578	0.0816	0.1137	0.1543	0.2017	0.2555	0.3151	0.3768	0.4405	0.5087	0.5772	0.6437
$\delta = 0.01$	0.0323	0.0485	0.0729	0.1061	0.1473	0.1956	0.2505	0.3096	0.3734	0.4375	0.5072	0.5717	0.6445

cylindrical tank by the slice model. Let a slice model set at a distance a from the X -axis and perpendicular to the Y -axis inside the cylindrical tank as shown in Fig. 1. Solve the Laplace equation given by Eq. (1) with the boundary conditions specifying the tank bulging motion given as follows. Figure 9 shows boundary conditions at the side walls and the bottom of the slice model according to the location of the slice model inside the cylindrical tank

$$\text{Free surface: } \Phi(x, h, t) = 0 \quad (0 \leq x \leq 2\sqrt{l^2 - a^2}) \quad (12a)$$

$$\text{Left wall: } \frac{\partial \Phi(0, z, t)}{\partial x} = u_0 \quad (0 \leq z \leq h) \quad (12b)$$

$$\text{Right wall: } \frac{\partial \Phi(2\sqrt{l^2 - a^2}, z, t)}{\partial x} = u_0 \quad (0 \leq z \leq h) \quad (12c)$$

$$\text{Bottom: } \frac{\partial \Phi(x, 0, t)}{\partial z} = 0 \quad (0 \leq x \leq 2\sqrt{l^2 - a^2}) \quad (12d)$$

Employing the variables separation procedure, the velocity potential function of the slice model is readily calculated. Its derivative with respect to time gives the fluid pressure at an arbitrary point inside the slice model of interest

$$P(x, z, t) = -\rho \frac{\partial \Phi(x, z, t)}{\partial t} = \rho \frac{\partial u_0}{\partial t} \left[\sum_{n=1}^{\infty} \left[\frac{\cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} 4\sqrt{l^2 - a^2}}{\cosh \frac{n\pi h}{2\sqrt{l^2 - a^2}} (n\pi)^2} \times \{(-1)^n - 1\} \right] \cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} + (x - \sqrt{l^2 - a^2}) \right] \quad (13)$$

Effective Density of Fluid for Bulging Motion of Each Slice Model

Unlike the tank rocking motion, the pressure gradient of fluid in the horizontal direction gives the effective density of fluid for bulging motion (see Fig. 10)

$$\rho_{\text{eff}} = -\frac{1}{u_0} \frac{\partial P(x, z, t)}{\partial x} \quad (14)$$

Employing Eq. (13), the effective density of fluid for bulging motion at an arbitrary point inside the slice model of interest is given as

$$\rho_{\text{eff}}(x, z) = f_{b1}(x, z)\rho \quad (15a)$$

Fig. 10 Equilibrium of forces acting on a small volume in the slice model in bulging motion

$$f_{b1}(x, z) = \left[\sum_{n=1}^{\infty} \frac{2\sqrt{l^2 - a^2}}{n\pi} \left[\frac{\cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} 4\sqrt{l^2 - a^2}}{\cosh \frac{n\pi h}{2\sqrt{l^2 - a^2}} (n\pi)^2} \times \{(-1)^n - 1\} \right] \sin \frac{n\pi x}{2\sqrt{l^2 - a^2}} + 1 \right] \quad (15b)$$

where $f_{b1}(x, z)$ is a ratio of effective density of fluid for bulging motion to the original density of the fluid fulfilled in the tank at an arbitrary point inside each slice model.

Fig. 11 Evaluation accuracy of effective mass of fluid for bulging motion

Table 5 Values of ratio of effective mass of fluid for rocking–bulging interaction to the total mass of fluid fulfilled in the tank

	$h/2l$												
	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
$\delta = 1.0$	0.3575	0.4351	0.4929	0.5346	0.5648	0.5879	0.6048	0.6185	0.6290	0.6394	0.6468	0.6558	0.6630
$\delta = 0.1$	0.0757	0.1041	0.1408	0.1749	0.2079	0.2418	0.2750	0.3048	0.3282	0.3564	0.3756	0.3972	0.4199
$\delta = 0.09$	0.0695	0.0978	0.1333	0.1662	0.2016	0.2342	0.2660	0.2978	0.3218	0.3486	0.3735	0.3967	0.4113
$\delta = 0.08$	0.0639	0.0924	0.1259	0.1586	0.1925	0.2284	0.2587	0.2912	0.3155	0.3425	0.3649	0.3881	0.4068
$\delta = 0.07$	0.0590	0.0845	0.1183	0.1519	0.1849	0.2192	0.2516	0.2843	0.3097	0.3367	0.3596	0.3834	0.4059
$\delta = 0.06$	0.0526	0.0780	0.1107	0.1432	0.1771	0.2120	0.2447	0.2775	0.3033	0.3310	0.3542	0.3783	0.3979
$\delta = 0.05$	0.0466	0.0715	0.1032	0.1353	0.1696	0.2048	0.2377	0.2710	0.2972	0.3257	0.3522	0.3738	0.3970
$\delta = 0.04$	0.0413	0.0655	0.0957	0.1286	0.1619	0.1988	0.2311	0.2646	0.2940	0.3222	0.3468	0.3695	0.3938
$\delta = 0.03$	0.0348	0.0577	0.0882	0.1203	0.1549	0.1906	0.2248	0.2589	0.2883	0.3151	0.3422	0.3672	0.3895
$\delta = 0.02$	0.0281	0.0502	0.0809	0.1139	0.1489	0.1844	0.2189	0.2533	0.2828	0.3103	0.3380	0.3633	0.3857
$\delta = 0.01$	0.0235	0.0448	0.0742	0.1074	0.1429	0.1786	0.2140	0.2477	0.2789	0.3071	0.3348	0.3587	0.3829

Effective Mass of Fluid for Bulging Motion of Cylindrical Tanks

Consider a small volume of fluid inside the slice model whose fluid density is specified by Eq. (15a). The effective mass of fluid for bulging motion of the cylindrical tank M_b is given by summing up the mass of all small volumes inside all slice models

$$M_b = f_b M_{\text{Total}} \quad (16a)$$

$$f_b = \frac{1}{\rho^2 h \pi} \left[\int_0^h \int_0^{2\sqrt{l^2 - a^2}} \int_{-l}^l f_{b1}(x, z) da dx dz \right] \quad (16b)$$

Here, f_b is a ratio of effective mass of fluid for bulging motion of the cylindrical tank M_b to the total mass of the fluid fulfilled in the cylindrical tank M_{Total} . Values of f_b are numerically evaluated and depicted for the tank commonly used geometry as functions of the aspect of the tank $h/2l$. Figure 11 compares the values of f_b with those in the open literature [14]. The value of f_b agrees well each other for tall tank, while its evaluation accuracy of the proposed method is slightly deteriorates for broad tanks. Because the values provided in the handbook were derived from the Bessel function given as the solution of a boundary-value problem in cylindrical coordinates that is the exact solution to the problem of interest, while the proposed method approximates them as a sum of the effective mass of fluid of each slice model in the cylindrical tanks.

Effective Mass of Fluid for Rocking–Bulging Interaction of Cylindrical Tanks

Reference [10] presented that the effective density of fluid for rocking–bulging interaction was given as the quotient of the product of the effective density of fluid for rocking motion and that for

bulging motion divided by the original density of fluid fulfilled in the tank. Following the manner presented in Ref. [10], this study defines the effective density of fluid for rocking–bulging interaction of each slice model. Summing up the effective mass of all small volumes inside all slice models, the effective mass of fluid for rocking–bulging interaction of the cylindrical tank is defined as

$$M_{rb} = f_{rb} M_{\text{Total}} \quad (17a)$$

$$f_{rb} = \frac{1}{\rho^2 h \pi} \left[\int_0^h \int_0^{2\sqrt{l^2 - a^2}} \int_{-(l-\delta l)}^{l-\delta l} f_A(x, z) f_{b1}(x, z) da dx dz + 2 \int_0^h \int_0^{2\sqrt{l^2 - a^2}} \int_{l-\delta l}^l f_B(x, z) f_{b1}(x, z) da dx dz \right] \quad (17b)$$

where f_{rb} is the ratio of the effective mass of fluid for rocking–bulging interaction of the cylindrical tank M_{rb} to the total mass of the fluid fulfilled in the cylindrical tank M_{Total} . Values of f_{rb} are numerically evaluated, tabulated, and depicted for the tank commonly used geometry as functions of the aspect of the tank $h/2l$ for different values of δ (see Table 5 and Fig. 12). The value of f_{rb} increases as the tank becomes taller, while it increases as the uplift part extends (see Fig. 12). In tall cylindrical tanks, it is about a third of total mass of the fluid fulfilled in the tanks, ignoring effects of the rocking–bulging interaction on the tank rocking motion in analyzing the seismic response of cylindrical tanks may yield erroneous results.

Conclusion

Consider that the cylindrical tank consists of a set of slice models consisting of thin rectangular tanks; this study quantifies the effective mass of fluid for rocking motion, that for rocking–bulging interaction, the effective moment inertia of fluid for rocking motion and its centroid. They are indispensable for analyzing the rocking motion of cylindrical tanks with the crescentlike uplift part of the tank bottom plate by the spring-mass-rigid-body combined system. The advantage of the use of the slice model is that it enables to solve a complex boundary-value problem in cylindrical coordinates due to crescentlike uplift of the tank bottom plate as a simple boundary-value problem in Cartesian coordinates with corresponding boundary conditions assigned by the location of the slice model in the cylindrical tank. Values of these masses are mathematically or numerically evaluated. For designer's convenience, they are normalized by the total mass of the fluid fulfilled in the cylindrical tanks, tabulated, and depicted for the tank commonly used geometry as functions of the aspect of tank for different values of the ratio of the uplift width of the tank bottom plate to the diameter of tank. These values increase as the tank becomes taller and the uplift width of the tank bottom plate extends. Since these masses reach about a half of mass of fluid fulfilled in a taller tank, ignoring their effects on the tank rocking motion in analyzing

Fig. 12 Effective mass of fluid for rocking–bulging interaction

the seismic response of cylindrical tanks may yield erroneous results.

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Nomenclature

- a = Y -value of where the slice model located
 $d_{r,x}, d_{r,z}$ = normalized centroid of the effective mass of fluid for rocking motion
 f_b, f_r, f_{rb} = ratio of the effective mass of fluid for bulging motion, rocking motion and rocking bulging interaction to the total mass of fluid fulfilled in the tank
 f_A, f_B = ratio of the effective density of fluid for rocking motion to the original density of fluid
 h = height of fluid fulfilled
 I = moment inertia of fluid
 l = radius of cylinder
 M_r = effective mass of fluid for rocking motion
 M_{rb} = effective mass of fluid for rocking-bulging interaction
 M_{Total} = total mass of fluid fulfilled in the tank
 P = fluid pressure
 $R_{r,x}, R_{r,z}$ = centroid of effective mass of fluid for rocking motion
 s = ratio of the effective moment inertia of fluid for rocking motion to the moment inertia of total mass
 t = time
 x, z = element coordinate values
 X, Y, Z = global coordinate values
 δ = ratio of the uplift width of the tank bottom plate to the diameter of tank
 $\dot{\theta}_0(t)$ = angular velocity of tank's rocking motion
 ρ = original fluid density
 ρ_{eff} = effective density of fluid
 Φ = velocity potential

Appendix

Assume that the velocity potential can be superposed as follows:

$$\Phi(x, z, t) = \Phi(x, z) \dot{\theta}_0(t) \quad (\text{A1})$$

Equations (1) and (2a)–(2d) are rewritten as follows by dividing both sides by $\dot{\theta}_0(t)$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \Phi(x, z) = 0 \quad (\text{A2})$$

$$\Phi(x, h) = 0 \quad (\text{A3a})$$

$$\frac{\partial \Phi}{\partial x}(0, z) = -z \quad (\text{A3b})$$

$$\frac{\partial \Phi}{\partial x}(2\sqrt{l^2 - a^2}, z) = -z \quad (\text{A3c})$$

$$\begin{aligned} \frac{\partial \Phi}{\partial z}(x, 0) &= \frac{(l - \sqrt{l^2 - a^2})}{\delta l + \sqrt{(1 - \delta^2)l^2 - a^2} - \sqrt{l^2 - a^2}} \\ &\quad \left(x - \sqrt{l^2 - a^2} + \sqrt{(1 - \delta^2)l^2 - a^2} + \delta l \right) \\ &= 0 \\ &= \frac{(l - \sqrt{l^2 - a^2})}{\delta l - \sqrt{(1 - \delta^2)l^2 - a^2} + \sqrt{l^2 - a^2}} \\ &\quad \left(x - \sqrt{l^2 - a^2} - \sqrt{(1 - \delta^2)l^2 - a^2} + \delta l \right) \end{aligned} \quad (\text{A3d})$$

Require the velocity potential by implementing the Fourier-cosine transformation

$$\int_0^{2\sqrt{l^2 - a^2}} \cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} \Phi(x, z) dx = \Phi_n(z) \quad (n = 1, 2, 3, \dots) \quad (\text{A4})$$

Employing the boundary conditions given by Eqs. (A3b) and (A3c), Eq. (A2) is given as follows:

$$\frac{d^2 \Phi_n(z)}{dz^2} - \left(\frac{n\pi}{2\sqrt{l^2 - a^2}} \right)^2 \Phi_n(z) = -\{1 - (-1)^n\}z \quad (\text{A5})$$

Its general solution is readily derived as

$$\begin{aligned} \Phi_n(z) &= A_n \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} + B_n \cosh \frac{n\pi z}{2\sqrt{l^2 - a^2}} \\ &+ \{1 - (-1)^n\} \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left(z - \frac{2\sqrt{l^2 - a^2}}{n\pi} \sinh \frac{n\pi z}{2\sqrt{l^2 - a^2}} \right) \end{aligned} \quad (\text{A6})$$

where A_n and B_n are integral constant. Considering the range of the boundary condition at the tank bottom, calculate Fourier-cosine transformation of Eq. (A3d)

$$\begin{aligned} \frac{d\Phi_n}{dz}(0) &= \int_0^{2\sqrt{l^2 - a^2}} \frac{\partial \Phi}{\partial z}(x, 0) \cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} dx \\ &= \int_0^L (ex + f) \cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} dx + \int_L^M \cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} \cdot 0 dx + \int_M^N (qx + r) \cos \frac{n\pi x}{2\sqrt{l^2 - a^2}} dx \\ &= e \left\{ \frac{2\sqrt{l^2 - a^2}}{n\pi} L \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} \\ &+ e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} - e \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 + f \frac{2\sqrt{l^2 - a^2}}{n\pi} \left\{ \sin \frac{n\pi L}{2\sqrt{l^2 - a^2}} \right\} \\ &+ q \left\{ \frac{2\sqrt{l^2 - a^2}}{n\pi} (2\sqrt{l^2 - a^2}) \sin \frac{n\pi(2\sqrt{l^2 - a^2})}{2\sqrt{l^2 - a^2}} \right\} - q \left\{ \frac{2\sqrt{l^2 - a^2}}{n\pi} M \sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \\ &+ q \left(\frac{2\sqrt{l^2 - a^2}}{n\pi} \right)^2 \left\{ \cos \frac{n\pi(2\sqrt{l^2 - a^2})}{2\sqrt{l^2 - a^2}} - \cos \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} + r \frac{2\sqrt{l^2 - a^2}}{n\pi} \left\{ \sin \frac{n\pi(2\sqrt{l^2 - a^2})}{2\sqrt{l^2 - a^2}} - \sin \frac{n\pi M}{2\sqrt{l^2 - a^2}} \right\} \end{aligned} \quad (\text{A7})$$

Calculate Fourier-cosine transformation of the boundary condition at fluid surface

$$\Phi_n(h) = \int_0^{2\sqrt{l^2-a^2}} \cos \frac{n\pi x}{2\sqrt{l^2-a^2}} \Phi(x, h) dx = 0 \quad (\text{A8})$$

From Eqs. (A6)–(A8), the integral constants A_n and B_n are determined as follows:

$$\begin{aligned} A_n = & e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2-a^2}} + e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2-a^2}} - 1 \right\} \\ & + f \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2-a^2}} \right\} - q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} \\ & + q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} + r \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} \end{aligned} \quad (\text{A9})$$

$$\begin{aligned} B_n = & - \left[e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2-a^2}} + e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2-a^2}} - 1 \right\} + f \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2-a^2}} \right\} \right. \\ & \left. - q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} + q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} + r \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} \right] \tanh \frac{n\pi h}{2\sqrt{l^2-a^2}} \\ & - \left\{ 1 - (-1)^n \right\} \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left(\frac{h}{\cosh \frac{n\pi h}{2\sqrt{l^2-a^2}}} - \frac{2\sqrt{l^2-a^2}}{n\pi} \tanh \frac{n\pi h}{2\sqrt{l^2-a^2}} \right) \end{aligned} \quad (\text{A10})$$

Therefore, $\Phi_n(z)$ is given as follows:

$$\begin{aligned} \Phi_n(z) = & \left[e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2-a^2}} + e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2-a^2}} - 1 \right\} + f \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2-a^2}} \right\} - q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \right. \\ & \left. \times \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} + q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} + r \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} \cdot \sinh \frac{n\pi z}{2\sqrt{l^2-a^2}} \right] \\ & + \left[- \left[e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 L \sin \frac{n\pi L}{2\sqrt{l^2-a^2}} + e \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos \frac{n\pi L}{2\sqrt{l^2-a^2}} - 1 \right\} + f \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ \sin \frac{n\pi L}{2\sqrt{l^2-a^2}} \right\} \right. \right. \\ & \left. \left. - q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ M \sin \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} + q \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^3 \left\{ \cos(n\pi) - \cos \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} \right. \right. \\ & \left. \left. + r \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left\{ -\sin \frac{n\pi M}{2\sqrt{l^2-a^2}} \right\} \right] \tanh \frac{n\pi h}{2\sqrt{l^2-a^2}} - \left\{ 1 - (-1)^n \right\} \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \right. \\ & \left. \left(\frac{h}{\cosh \frac{n\pi h}{2\sqrt{l^2-a^2}}} - \frac{2\sqrt{l^2-a^2}}{n\pi} \tanh \frac{n\pi h}{2\sqrt{l^2-a^2}} \right) \right] \cdot \cosh \frac{n\pi z}{2\sqrt{l^2-a^2}} + \left\{ 1 - (-1)^n \right\} \left(\frac{2\sqrt{l^2-a^2}}{n\pi} \right)^2 \left(z - \frac{2\sqrt{l^2-a^2}}{n\pi} \sinh \frac{n\pi z}{2\sqrt{l^2-a^2}} \right) \end{aligned} \quad (\text{A11})$$

On the other hand, the velocity potential $\Phi(x, z)$ is given by inverse Fourier-cosine transformation of Eq. (A11)

$$\Phi(x, z) = \frac{1}{2\sqrt{l^2-a^2}} \Phi_0(z) + \frac{1}{\sqrt{l^2-a^2}} \sum_{n=1}^{\infty} \Phi_n(z) \cos \frac{n\pi x}{2\sqrt{l^2-a^2}} \quad (\text{A12})$$

where $\Phi_0(z)$ is readily calculated by Eq. (A4) at $n = 0$ and is given as

$$\Phi_0(z) = \left\{ \frac{e}{2} L^2 + fL + \frac{q}{2} N^2 - \frac{q}{2} M^2 + rN - rM \right\} (z - h) \quad (\text{A13})$$

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