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A unified Fourier theory for time-of-flight PET data

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Abstract

Fully 3D time-of-flight (TOF) PET scanners offer the potential of previously unachievable image quality in clinical PET imaging. TOF measurements add another degree of redundancy for cylindrical PET scanners and make photon-limited TOF-PET imaging more robust than non-TOF PET imaging. The data space for 3D TOF-PET data is five-dimensional with two degrees of redundancy. Previously, consistency equations were used to characterize the redundancy of TOF-PET data. In this paper, we first derive two Fourier consistency equations and Fourier-John equation for 3D TOF PET based on the generalized projection-slice theorem; the three partial differential equations (PDEs) are the dual of the sinogram consistency equations and John's equation. We then solve the three PDEs using the method of characteristics. The two degrees of entangled redundancy of the TOF-PET data can be explicitly elicited and exploited by the solutions of the PDEs along the characteristic curves, which gives a complete understanding of the rich structure of the 3D X-ray transform with TOF measurement. Fourier rebinning equations and other mapping equations among different types of PET data are special cases of the general solutions. We also obtain new Fourier rebinning and consistency equations (FORCEs) from other special cases of the general solutions, and thus we obtain a complete scheme to convert among different types of PET data: 3D TOF, 3D non-TOF, 2D TOF and 2D non-TOF data. The new FORCEs can be used as new Fourier-based rebinning algorithms for TOF-PET data reduction, inverse rebinnings for designing fast projectors, or consistency conditions for estimating missing data. Further, we give a geometric interpretation of the general solutions—the two families of characteristic curves can be obtained by respectively changing the azimuthal and co-polar angles of the biorthogonal coordinates in Fourier space. We conclude the unified Fourier theory by showing that the Fourier consistency equations are *necessary* and *sufficient* for 3D X-ray transform with TOF measurement. Finally, we give numerical examples of inverse rebinning for a 3D TOF PET and Fourier-based rebinning for a 2D TOF PET using the FORCEs to show the efficacy of the unified Fourier solutions.

Keywords

Fourier consistency equations; Fourier-John equation; partial differential equation (PDE); consistency equations; John's equation; time-of-flight (TOF); positron emission tomography (PET)

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1. Introduction

Time-of-flight PET scanners with 3D data acquisitions offer the potential of previously unachievable image quality in clinical PET imaging. In TOF-PET systems, the difference between the arrival times of the two coincident photons is also measured for each annihilation event. At the extreme, where the TOF information could be measured with enough accuracy, image reconstruction could be simply derived by histogramming the events in the image space with proper corrections for scatters, randoms, attenuation and normalization. At the present time and in the foreseeable future, image reconstruction is still necessary due to the uncertainty associated with the TOF measurements. Even imperfect, this TOF information confines the annihilation positions into a finite region and reduces noise propagation along a line-of-response (LOR). Incorporating TOF information into image reconstructions can increase the contrast-to-noise ratio (CNR) of the reconstructed images, where the CNR improvement is determined by the object size and the TOF timing resolution (Snyder *et al* 1981, Tomitani 1981, Moses 2003, Karp *et al* 2008, Conti 2009).

For 3D cylindrical non-TOF PET, there is one degree of redundancy, the oblique sinograms are redundant in the sense that the 3D image can be exactly reconstructed slice-by-slice only using 2D direct sinograms. Here, the 2D direct sinograms are the subset of the 3D PET data that have zero ring difference (i.e., both photons are received by the same axial detector slice). However, the oblique sinograms help to improve the CNR in the reconstructed images. Fourier rebinning was developed to rebin 3D non-TOF Data to 2D non-TOF data for data reduction (Defrise *et al* 1997, Liu *et al* 1999). For 3D non-TOF PET, the redundancy can also be characterized by a partial differential equation (PDE) known as John's equation (Defrise and Liu 1999, Defrise *et al* 2013).

For 3D cylindrical TOF PET, TOF information is redundant in the sense that exact reconstructions can be performed after summing all TOF bins (which is equivalent to non-TOF data). The TOF information adds another degree of redundancy and it helps to improve the CNR in the reconstructed images. The data space for 3D TOF-PET data is five dimensional while the object space is three dimensional—3D TOF-PET data have two degrees of redundancy. A generalized projection-slice theorem was derived for 3D TOF PET (Cho *et al* 2008), from which Fourier-based rebinning methods were developed to convert among different PET data types (Defrise *et al* 2005, Cho *et al* 2009). John's equation for 3D TOF was also derived to characterize the redundancy (Defrise *et al* 2008), in addition, two PDEs, known as consistency equations, were derived to explore the structure of TOF-PET data (Defrise *et al* 2013).

The redundancy can be expressed in terms of either the Fourier-based rebinning equations or the consistency equations; however, the connection between them is unclear. Whether the two consistency equations are sufficient or not for 3D X-ray transform with TOF measurement has been to this point an open problem. A conjecture states that the two consistency equations are also sufficient under appropriate conditions (Defrise *et al* 2005). The goal of this paper is to explore the rich and entangled structure of 3D TOF-PET data, to discover the connection between Fourier-based rebinning equations and the consistency equations, and to expand and unify these methods within the Fourier framework. First, we

derive two Fourier-space consistency equations and the Fourier-John equation, which are the dual equations of sinogram consistency equations and John's equation. We then solve these equations using the method of characteristics; we show that the sufficiency of the consistency equations for 3D X-ray transform with TOF measurement. This paper is organized as follows. In section 2, we derive two Fourier consistency equations (FCEs) and Fourier-John equation (FJE) from the generalized projection-slice theorem for 3D TOF-PET data. In section 3, we present the general solutions to the two FCEs and FJE, and then we give the Fourier rebinning and consistency equations (FORCES) as special cases of the general solutions. We also present a unified scheme for converting among different types of PET data. Further, we prove that the two consistency equations are *necessary* and *sufficient* for 3D X-ray transform with TOF measurement. In section 4, we give a geometric interpretation for the two FCEs and FJE. In section 5, we present numerical examples for both 3D and 2D TOF PET. Finally, we give discussion and conclude the paper in sections 6 and 7.

2. Fourier consistency equations

2.1. TOF-PET data and Fourier transform

As shown in figure 1, the TOF-PET data are generally parameterized as

$$p(s, t, z; \phi, \theta) = \int_{-\infty}^{\infty} dl h(t-l) f(s\hat{u} + l\hat{n} + z\hat{e}_z), \quad (1)$$

with three unit vectors

$$\begin{aligned} \hat{u} &= [\cos \phi, \sin \phi, 0]^T, \\ \hat{n} &= [-\cos \theta \sin \phi, \cos \theta \cos \phi, \sin \theta]^T, \\ \hat{e}_z &= [0, 0, 1]^T, \end{aligned} \quad (2)$$

where f is a 3D tracer distribution and h is a TOF profile, t is the TOF parameter, s and ϕ are the usual sinogram coordinates, z is the axial coordinate of the midpoint of the LOR, and θ is the co-polar angle between the LOR and a transaxial plane. The TOF profile is modeled as a Gaussian distribution with standard deviation σ ,

$$h(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{t^2}{2\sigma^2}\right). \quad (3)$$

The TOF parameter t is related with the TOF time difference T between the two arrival time of the two gammas by $t = cT/2$ with c denotes the speed of light. The standard

deviation $\sigma = cT_{FWHM} / (4\sqrt{2\log 2})$, and T_{FWHM} is the full width at half maximum (FWHM) of the measured time difference, which is on the order of 500 ps in current clinical scanners (Karp *et al* 2008, Surti *et al* 2007, Zaidi *et al* 2011). A LaBr₃-based PET scanner developed at Penn has a timing resolution of 375 ps (Daube-Witherspoon *et al* 2010), and a newly emerged scanner, Philips Vereos PET/CT based on digital photon counting technology, has a timing resolution of 345 ps.

We introduce the Fourier transform of p with respect to the variables s , t and z

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \iiint_{\mathbb{R}^3} ds dt dz p(s, t, z; \phi, \theta) e^{-j\omega_s s - j\omega_t t - j\omega_z z}, \quad (4)$$

where ω_s , ω_t and ω_z are the frequency variables corresponding s , t and z , respectively. Putting (1) into (4) and integrating the TOF variable t , we obtain

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \hat{h}(\omega_t) \iiint_{\mathbb{R}^3} ds dl dz f(s\hat{u} + l\hat{n} + z\hat{e}_z) e^{-j\omega_s s - j\omega_t l - j\omega_z z}, \quad (5)$$

where $\hat{h}(\omega_t) = e^{-\sigma^2 \omega_t^2 / 2}$ is the Fourier transform of the TOF profile $h(t)$. The argument of f in (5) is $\vec{x} = \mathbf{A} \vec{r}$, with $\vec{r} = [s, l, z]^T$ and affine matrix $\mathbf{A} = [\hat{u}, \hat{n}, \hat{e}_z]$. Applying the Fourier stretch theorem (Bracewell 2006)

$$\mathcal{F}[f(\mathbf{A} \vec{r})](\vec{\omega}) = \frac{1}{|\det \mathbf{A}|} [\mathcal{F}f](\mathbf{A}^{-T} \vec{\omega}), \quad \vec{\omega} = (\omega_s, \omega_t, \omega_z)^T \quad (6)$$

we can use $\hat{f}$ to represent the Fourier transform $\hat{p}$ in (5):

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{f}(\omega_s \hat{u}^* + \omega_t \hat{n}^* + \omega_z \hat{e}_z^*), \quad (7)$$

where

$$\begin{aligned} \hat{u}^* &= \hat{u} = [\cos \phi, \sin \phi, 0]^T, \\ \hat{n}^* &= [-\sec \theta \sin \phi, \sec \theta \cos \phi, 0]^T, \\ \hat{e}_z^* &= [\tan \theta \sin \phi, -\tan \theta \cos \phi, 1]^T. \end{aligned} \quad (8)$$

Here, $\hat{u}^*$, $\hat{n}^*$ and $\hat{e}_z^*$ are the three dual basis vectors of $\hat{u}$, $\hat{n}$ and $\hat{e}_z$, and they are the three column vectors of $\mathbf{A}^* = \mathbf{A}^{-T}$, i.e., $\mathbf{A}^* = [\hat{u}^*, \hat{n}^*, \hat{e}_z^*]$. The two sets basis vectors denoted by $\mathbf{A}$ and $\mathbf{A}^*$ form a biorthogonal system. We can also rewrite (7) into following non-vector form as

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{f}(\omega_s \cos \phi - \chi \sin \phi, \omega_s \sin \phi + \chi \cos \phi, \omega_z), \quad (9)$$

where

$$\chi = \omega_t \sec \theta - \omega_z \tan \theta. \quad (10)$$

Equation (7) and (9) are the generalized projection-slice theorem for 3D TOF-PET data, and (9) is the same as (2) given in (Cho *et al* 2009). Here we give a concise derivation to bring to light a close connection between the generalized projection-slice theorem and Fourier stretch theorem using the dual basis. An equivalent equation, in histo-image format, is given by (18) in (Li *et al* 2015a). A non-TOF version of (7) with $\omega_t = 0$ is given in (Liu *et al* 1999). Here we use (1) and (7) as the two basis parameterizations for TOF-PET data in respectively spatial and Fourier domains to develop a unified Fourier theory.

2.2. Fourier-space partial differential equations

The TOF-PET sinogram p has five dimensions (depends on five variables) and the object f has only three—the two degrees of redundancy can be expressed as two independent partial differential equations (PDEs) and referred to as consistency equations (Defrise *et al* 2008, 2013). A Gaussian TOF profile is assumed in the development of the following consistency equations; however, the redundancy in TOF-PET data exists beyond this assumption. The consistency equations can be used as a tool to explore the rich structure of TOF-PET data. Similar to the TOF-PET sinogram p , Fourier projection $\hat{p}$ also has two degrees of redundancy. We prove in appendix A that the Fourier projection $\hat{p}$ given by (7) satisfy the following two independent PDEs:

$$\frac{\partial \hat{p}}{\partial \phi} + (\sec \theta \omega_t - \tan \theta \omega_z) \frac{\partial \hat{p}}{\partial \omega_s} - \cos \theta \omega_s \frac{\partial \hat{p}}{\partial \omega_t} = \sigma^2 \cos \theta \omega_s \omega_t \hat{p}, \quad (11)$$

$$\frac{\partial \hat{p}}{\partial \theta} - (\tan \theta \omega_t - \sec \theta \omega_z) \frac{\partial \hat{p}}{\partial \omega_t} = \tan \theta \hat{p} + \sigma^2 (\tan \theta \omega_t - \sec \theta \omega_z) \omega_t \hat{p}, \quad (12)$$

where the arguments of $\hat{p}$ and its derivatives are $(\omega_s, \omega_t, \omega_z; \phi, \theta)$. Equations (11) and (12) are Fourier consistency equations (FCEs: FCE1, FCE2), which are the dual equations of the sinogram consistency equations (CEs) (4) and (2) in (Defrise *et al* 2013). When $\theta = 0$, (11) becomes the Fourier consistency equation for 2-D TOF-PET data, and its dual equation was used to determine the attenuation sinogram (Defrise *et al* 2012). There are two degrees of redundancy in TOF-PET data, which are characterized by the two consistency equations. A natural conjecture is that the two CE's are sufficient to determine the TOF-PET data, or mathematically, any function that satisfies the two CE's is in the range of X-ray transform with TOF measurement (Defrise *et al* 2008). In the next section, we will show a positive answer to this conjecture.

Adding $-(\tan \theta \omega_t - \sec \theta \omega_z)$ (11) and $(\cos \theta \omega_s)$ (12), we obtain Fourier-space John's equation (FJE)

$$\left[(\sec^2 \theta + \tan^2 \theta) \omega_t \omega_z - \sec \theta \tan \theta (\omega_t^2 + \omega_z^2) \right] \frac{\partial \hat{p}}{\partial \omega_s} - (\tan \theta \omega_t - \sec \theta \omega_z) \frac{\partial \hat{p}}{\partial \phi} + \cos \theta \omega_s \frac{\partial \hat{p}}{\partial \theta} = \sin \theta \omega_s \hat{p}. \quad (13)$$

Equation (13) is the dual equation of John's equation (6) in sinogram format (Defrise *et al* 2008). When $\omega_t = 0$, (13) becomes the Fourier-space John's equation for 3D non-TOF PET (Defrise and Liu 1999). The sinogram p has the symmetry and periodicity property $p(s, t, z, \phi, \theta) = p(-s, -t, z, \phi + \pi, -\theta)$ (Defrise *et al* 2005), and its Fourier transform $\hat{p}$ also has a symmetry and periodicity property

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \hat{p}(-\omega_s, -\omega_t, \omega_z; \phi + \pi, -\theta). \quad (14)$$

2.3. Link with sinogram consistency equation

The two sinogram consistency equations and John's equation are given in (Defrise *et al* 2013). For completeness, we show that these equations are just the inverse Fourier

transforms of (11), (12) and (13). The Fourier transform has the following spatial and frequency derivative properties:

$$\frac{\partial}{\partial r} \xleftrightarrow{\mathcal{F}} j\omega_r, \quad r \xleftrightarrow{\mathcal{F}} j \frac{\partial}{\partial \omega_r}, \quad \text{for } r=s, t, z. \quad (15)$$

Taking the inverse Fourier transform of the two CEs (11) and (12) and using (15), we obtain the two sinogram CEs:

$$\frac{\partial p}{\partial \phi} + t \cos \theta \frac{\partial p}{\partial s} - s \sec \theta \frac{\partial p}{\partial t} + s \tan \theta \frac{\partial p}{\partial z} + \sigma^2 \cos \theta \frac{\partial^2 p}{\partial s \partial t} = 0, \quad (16)$$

$$\frac{\partial p}{\partial \theta} + t \tan \theta \frac{\partial p}{\partial t} - t \sec \theta \frac{\partial p}{\partial z} + \sigma^2 \tan \theta \frac{\partial^2 p}{\partial t^2} - \sigma^2 \sec \theta \frac{\partial^2 p}{\partial t \partial z} = 0. \quad (17)$$

It is worth noting that the order of the operators should be kept when taking the inverse Fourier transform because two operators are not necessarily commutative, e.g.

$\frac{\partial}{\partial t} t \xleftrightarrow{\mathcal{F}} j\omega_t j \frac{\partial}{\partial \omega_t}$, and $\frac{\partial}{\partial t} t - t \frac{\partial}{\partial t} = 1 \neq 0$. Taking the inverse Fourier transform of (13), we obtain the sinogram John's equation

$$s \tan \theta \sec \theta \frac{\partial^2 p}{\partial t^2} - s \left(\sec^2 \theta + \tan^2 \theta \right) \frac{\partial^2 p}{\partial t \partial z} + s \tan \theta \sec \theta \frac{\partial^2 p}{\partial z^2} - \tan \theta \frac{\partial^2 p}{\partial \phi \partial t} + \sec \theta \frac{\partial^2 p}{\partial \phi \partial z} + \cos \theta \frac{\partial^2 p}{\partial \theta \partial s} - \sin \theta \frac{\partial p}{\partial s} = 0. \quad (18)$$

3. Explicit solutions to Fourier consistency equations

Equation (7) is the solution for both Fourier consistency equations (11) and (12). In this section, we use the method of characteristics (John 1982, Courant and Hilbert 1989) to obtain the general solutions for the two Fourier consistency equations and Fourier-John equation. The detailed derivations are given in appendices B and C for the readers who are familiar with differential equations or willing to know the derivations. In sections 3.1 to 3.3, we review the applications for the three general solution, and we summarize them in section 3.4.

3.1. Explicit solution to the first Fourier consistency equation

Applying the method of characteristics (John 1982, Courant and Hilbert 1989), we can write the general solution to (11) as (see appendix B for the detailed derivation)

$$\begin{aligned} \hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) &= \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \hat{p}(\omega_{s,0}, \omega_{t,0}, \omega_z; \phi_0, \theta), \\ \omega_s &= \omega_{s,0} \cos(\phi_0 - \phi) - (\omega_{t,0} \sec \theta - \omega_z \tan \theta) \sin(\phi_0 - \phi), \\ \omega_t &= \omega_{s,0} \cos \theta \sin(\phi_0 - \phi) + \omega_{t,0} \cos(\phi_0 - \phi) + \omega_z \sin \theta [1 - \cos(\phi_0 - \phi)], \end{aligned} \quad (19)$$

where $\omega_{t,0}$, $\omega_{s,0}$ and ϕ_0 are initial arguments. The characteristic curves are determined by ω_s and ω_t with parameter ϕ with constant-valued ω_z and θ . For each initial point $(\omega_{s,0}, \omega_{t,0}, \omega_z; \phi_0, \theta)$, the values are propagated according to the first equation in (19) along the characteristic curve passing this point. The solution is obtained from the union of the family

of the characteristic curves passing through different initial points (any points along the same characteristic curve are considered as one initial point). One can easily see that the general solution (19) characterizes one degree of redundancy in TOF-PET data.

We have $\omega_t = 0$ for non-TOF data, $\theta = 0$ for 2D PET Data, so we can apply (19) to convert among different types of PET data: 3D TOF data, 3D non-TOF data, 2D TOF data and 2D non-TOF data. When $\omega_{t,0} = 0$, (19) becomes

$$\begin{aligned}\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) &= \hat{h}(\omega_t) \hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, \theta), \\ \omega_s &= \omega_{s,0} \cos(\phi_0 - \phi) + \omega_z \tan \theta \sin(\phi_0 - \phi), \\ \omega_t &= \omega_{s,0} \cos \theta \sin(\phi_0 - \phi) + \omega_z \sin \theta [1 - \cos(\phi_0 - \phi)].\end{aligned}\quad (20)$$

One can use (20) to convert 3D non-TOF data to 3D TOF data, or rebin 3D TOF data to 3D non-TOF data for data reduction. One can also verify that (20) is equivalent to the mapping A in (Cho *et al* 2009), and (12) and (13) in (Defrise *et al* 2013).

When $\theta = 0$, (19) becomes

$$\begin{aligned}\hat{p}(\omega_s, \omega_t, \omega_z; \phi, 0) &= \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \hat{p}(\omega_{s,0}, \omega_{t,0}, \omega_z; \phi_0, 0), \\ \omega_s &= \omega_{s,0} \cos(\phi_0 - \phi) - \omega_{t,0} \sin(\phi_0 - \phi), \\ \omega_t &= \omega_{s,0} \sin(\phi_0 - \phi) + \omega_{t,0} \cos(\phi_0 - \phi).\end{aligned}\quad (21)$$

The above equation exploits the redundancy in 2D TOF data, and can be applied, e.g., to estimate the missing data in TOF PET due to incomplete linear- or angular-sampling. The corresponding differential equation (16) with $\theta = 0$ was used to restore fine azimuthal sampling of measured TOF projection data (Panin *et al* 2010), and was used to determine the attenuation sinogram from 2D TOF-PET data (Defrise *et al* 2012).

When $\omega_{t,0} = 0$ and $\theta = 0$, (19) becomes

$$\begin{aligned}\hat{p}(\omega_s, \omega_t, \omega_z; \phi, 0) &= \hat{h}(\omega_t) \hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0), \\ \omega_s &= \omega_{s,0} \cos(\phi_0 - \phi), \quad \omega_t = \omega_{s,0} \sin(\phi_0 - \phi).\end{aligned}\quad (22)$$

This equation can be used to convert 2D TOF data to 2D non-TOF data, or rebin 2D TOF data to 2D non-TOF data for data reduction. One can also verify that (22) is equivalent to the mapping E in (Cho *et al* 2009).

3.2. Explicit solution to the second Fourier consistency equation

Applying the method of characteristics (John 1982, Courant and Hilbert 1989), we can write the general solution to (12) as (see appendix C for detailed derivation)

$$\begin{aligned}\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) &= \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \frac{\cos \theta_0}{\cos \theta} \hat{p}(\omega_s, \omega_{t,0}, \omega_z; \phi, \theta_0), \\ \omega_t &= \omega_z \sin \theta + (\omega_{t,0} \sec \theta_0 - \omega_z \tan \theta_0) \cos \theta,\end{aligned}\quad (23)$$

where $\omega_{t,0}$ and θ_0 are initial arguments. The solution (23) characterizes the other degree of redundancy in TOF-PET data, and it can be used to convert among different types of PET data. When $\omega_{t,0} = 0$, (23) becomes

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{p}(\omega_s, 0, \omega_z; \phi, \theta_0), \quad (24)$$

$$\omega_t = \omega_z \sin \theta - \omega_s \tan \theta_0 \cos \theta.$$

Similar to (20), (24) can be used to inversely rebin 3D non-TOF data to 3D TOF data, or exactly rebin 3D TOF data to 3D non-TOF data.

When $\theta_0 = 0$, (23) becomes

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{\hat{h}(\omega_t)}{\hat{h}(\omega_{t,0})} \frac{1}{\cos \theta} \hat{p}(\omega_s, \omega_{t,0}, \omega_z; \phi, 0), \quad (25)$$

$$\omega_t = \omega_z \sin \theta + \omega_{t,0} \cos \theta.$$

Equation (25) can be used to inversely rebin 2D TOF data to 3D TOF data, or exactly rebin 3D TOF data to 2D TOF data for each axial slice at fixed radial variable s and angle ϕ . An efficient forward projector for 3D TOF PET based on (25), as an alternative to the line integral computation, was investigated by Panin and Defrise (2009). The corresponding differential format of (25) was developed for axial rebinning 3D TOF data to 2D TOF data (Defrise *et al* 2005, 2008). It worth noting that (25) becomes the single slice rebinning for TOF data (TOF-SSRB) when $\sigma \rightarrow 0$ (Mullani *et al* 1982, Moses 2003, Defrise *et al* 2005). Applying $\lim_{\sigma \rightarrow 0} \hat{h}(\omega_t) / \hat{h}(\omega_{t,0}) = 1$ and taking the inverse Fourier transform of (25) with respect to ω_s , ω_t and ω_z , we obtain $p(t, s, \phi, z, \theta) \approx p(t \cos \theta, s, \phi, z + t \sin \theta, 0)$. This equation can be compared to (8) in (Defrise *et al* 2005), which used a different parameterization.

When $\omega_{t,0} = 0$ and $\theta_0 = 0$, (23) becomes

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{p}(\omega_s, 0, \omega_z; \phi, 0), \quad \omega_t = \omega_z \sin \theta \quad (26)$$

Equation (26) can be used to exactly rebin 3D TOF data to 2D non-TOF data; however, TOF data may not be fully utilized since (26) only includes one degree of redundancy.

3.3. Explicit solution to John's equation

Similar to (19) and (23), we can write the general solution to (13) as

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{\cos \theta_0}{\cos \theta} \hat{p}(\omega_{s,0}, \omega_t, \omega_z; \phi_0, \theta_0),$$

$$\omega_s = \sqrt{\omega_{s,0}^2 + (\omega_t \sec \theta_0 - \omega_z \tan \theta_0)^2 - (\omega_t \sec \theta - \omega_z \tan \theta)^2}, \quad (27)$$

$$\phi = \phi_0 + \tan^{-1} \frac{\omega_t \sec \theta_0 - \omega_z \tan \theta_0}{\omega_{s,0}} - \tan^{-1} \frac{\omega_t \sec \theta - \omega_z \tan \theta}{\omega_s},$$

where $\omega_{s,0}$, ϕ_0 and θ_0 are initial arguments. Here, we only consider Fourier data $\hat{p}$ on the domain $\omega_s \geq 0$, and one can use symmetric property (14) for $\hat{p}$ on $\omega_s < 0$. Equation (27) can be considered as a combination of (19) and (23), and it among is another useful equation to convert different types of PET data.

When $\omega_t = 0$, (27) becomes

$$\begin{aligned}\hat{p}(\omega_s, 0, \omega_z; \phi, \theta) &= \frac{\cos \theta_0}{\cos \theta} \hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, \theta_0), \\ \omega_s &= \sqrt{\omega_{s,0}^2 + (\omega_z \tan \theta_0)^2 - (\omega_z \tan \theta)^2}, \\ \phi &= \phi_0 - \tan^{-1} \frac{\omega_z \tan \theta_0}{\omega_{s,0}} + \tan^{-1} \frac{\omega_z \tan \theta}{\omega_s}.\end{aligned}\quad (28)$$

Equation (28) exploits the redundancy in 3D non-TOF data, and it can be applied, e.g., to estimate the missing data in 3D non-TOF data.

When $\theta_0 = 0$, (27) becomes

$$\begin{aligned}\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) &= \frac{1}{\cos \theta} \hat{p}(\omega_{s,0}, \omega_t, \omega_z; \phi_0, 0), \\ \omega_s &= \sqrt{\omega_{s,0}^2 + \omega_t^2 - (\omega_t \sec \theta - \omega_z \tan \theta)^2}, \\ \phi &= \phi_0 + \tan^{-1} \frac{\omega_t}{\omega_{s,0}} - \tan^{-1} \frac{\omega_t \sec \theta - \omega_z \tan \theta}{\omega_s}.\end{aligned}\quad (29)$$

Similar to (25), (29) can be used to exactly rebin 3D TOF data to 2D TOF data, or to convert 2D TOF data to 3D TOF data. One can also verify that (29) is equivalent to the mapping B in (Cho *et al* 2009).

When $\omega_t = 0$ and $\theta_0 = 0$, (27) becomes

$$\begin{aligned}\hat{p}(\omega_s, 0, \omega_z; \phi, \theta) &= \frac{1}{\cos \theta} \hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0), \\ \omega_s &= \sqrt{\omega_{s,0}^2 - (\omega_z \tan \theta)^2}, \quad \phi = \phi_0 + \tan^{-1} \frac{\omega_z \tan \theta}{\omega_s}.\end{aligned}\quad (30)$$

By taking the Fourier transform with respect to ϕ , (30) becomes the exact Fourier rebinning for 3D non-TOF PET (Defrise *et al* 1997, Liu *et al* 1999, Defrise and Liu 1999). It is equivalent to the mapping D in (Cho *et al* 2009). By taking the first order approximation with respect to the term $\omega_z \tan \theta$, we obtain

$\hat{p}(\omega_s, 0, \omega_z; \phi, \theta) \approx \frac{1}{\cos \theta} \hat{p}(\omega_s, 0, \omega_z; \phi - \omega_z \tan \theta / \omega_s, 0)$, which can be compared to the FORE algorithm (28) in (Defrise *et al* 1997).

3.4. A unified picture for Fourier-based rebinning TOF-PET data

Based on the solutions to the two consistency equations and John's equation, we can obtain a unified picture for converting different types of PET data as shown in figure 2. There are two degrees of redundancy in 3D TOF data, and one degree of redundancy in 3D non-TOF data and 2D TOF data. We have two operations for converting the 3D TOF data either to 2D TOF data or to 3D non-TOF data due to the two degrees of redundancy in 3D TOF data. Figure 2 shows twelve basis conversions among different types of PET data. A combination of these conversions can also be applied. For instance, one can convert 3D TOF data to 2D non-TOF data by first converting to 2D TOF data using (25) or (29) and then converting to 2D non-TOF data using (22), or by first converting to 3D non-TOF data using (20) or (24) and then converting to 2D non-TOF data using (30). One can verify that the four types of combinations give the following identical method

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0),$$

$$\begin{aligned}\omega_s &= \omega_{s,0} \cos(\phi_0 - \phi), \\ \omega_t &= \omega_{s,0} \sin(\phi_0 - \phi) \cos \theta + \omega_z \sin \theta.\end{aligned}\quad (31)$$

Similar to (26), (31) can be used to exactly rebin 3D TOF data to 2D non-TOF data and the two degrees of redundancy are utilized. One can also verify (31) is equivalent to Mapping C in (Cho *et al* 2009), which we obtained from the solution of the consistency equations. Since $\hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0)$ depends only on three variables, and we write it as

$$\hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0) = \hat{f}(\omega_{s,0} \cos \phi_0, \omega_{s,0} \sin \phi_0, \omega_z). \quad (32)$$

Equation (32) is just the projection-slice theorem for 2D non-TOF PET data (Natterer 1986), but we consider it as an initial condition for the consistency equations. This initial condition is also known as the Helgason-Ludwig consistency condition, which is a global condition for the Radon transform (Helgason 1965, Ludwig 1966). One can see from (32) that 2D non-TOF Fourier data $\hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0)$ is independent of ϕ_0 when $\omega_{s,0} = 0$. Taking the inverse Fourier transform of (32) with respect to $\omega_{s,0}$ and ω_z , we obtain

$$\begin{aligned}p_{2D, \text{non-TOF}}(s, z, \phi) &\equiv \int_{-\infty}^{\infty} dt p(s, t, z; \phi, 0), \\ &= \int_{-\infty}^{\infty} dl f(s \cos \phi - l \sin \phi, s \sin \phi + l \cos \phi, z).\end{aligned}\quad (33)$$

Equation (33) is the 2D non-TOF PET data formulation, which can be considered as an initial condition in the formulation of 3D TOF-PET data. Putting the initial condition (32) into (31), we obtain

$$\begin{aligned}\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) &= \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{f}(\omega_{s,0} \cos \phi_0, \omega_{s,0} \sin \phi_0, \omega_z), \\ \omega_{s,0} \cos \phi_0 &= \omega_s \cos \phi - (\omega_t \sec \theta - \omega_z \tan \theta) \sin \phi, \\ \omega_{s,0} \sin \phi_0 &= \omega_s \sin \phi + (\omega_t \sec \theta - \omega_z \tan \theta) \cos \phi.\end{aligned}\quad (34)$$

It is obvious that (34) is the same as (7) and (9). We have the following theorems for the sufficiency of the consistency equations for 3D X-ray transform with TOF measurement.

Theorem 1 (Sufficiency of the two Fourier consistency equations)—Let

$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) \in C^\infty(\mathbb{R}^3 \times S^2)$ satisfy the two Fourier-space consistency equations (11)

and (12), and $\hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0)$ satisfies the initial condition (32) with $\hat{f} \in C^\infty(\mathbb{R}^3)$, then $\hat{p}$ can be determined in the form of (7).

Proof: By solving the two consistency equations (11) and (12) in respectively appendices B and C, we give the general solutions in (19) and (23). By combining the two special cases (22) and (25) of the two general solutions, we obtain (34) after applying the initial condition (32), which is equivalent to (7).

Theorem 2 (Sufficiency of the two consistency equations)—Let a rapidly

decreasing or compactly supported function $p(s, t, z; \phi, \theta) \in C^\infty(\mathbb{R}^3 \times S^2)$ satisfy the two consistency equations (16) and (17), and the initial condition (33) with a rapidly decreasing

or *compactly supported function* f on $\mathbb{R}^3$, then p can be determined in the form of (1), which is the 3D X-ray transform of the 3D object f with TOF measurement.

Proof: Equations (16) and (17) are the inverse Fourier transforms (11) and (12); (1) is the inverse Fourier transforms of the projection-slice theorem (7). Applying **Theorem 1**, we prove that (16) and (17) are sufficient for 3D X-ray transform with TOF measurement with initial condition (33). The proof is based on Fourier transform; a mathematically rigorous proof of the theorem is beyond the scope of this paper.

To the best of our knowledge, this is the first time that the sufficiency of the two consistency equations has been proven for 3D X-ray transform with TOF measurement. For the non-TOF case, the sufficiency was proven by John (1938). In appendix A, we show that (11) and (12) are necessary for (7), and equations (16) and (17) are necessary for (1) as well after applying the inverse Fourier transforms. So the two consistency equations are *necessary* and *sufficient* for 3D X-ray transform with TOF measurement. From **Theorem 2**, we know that there are two and only two independent consistency equations (16) and (17) for 3D X-ray transform with TOF measurement. Any additional consistency equation must be a linear combination of the two consistency equations, e.g. John's equation (18).

4. Geometric interpretation

In this section, we give a geometric interpretation for the solutions of the consistency equations. We easily see from (7) that $\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) \cos \theta / \hat{h}(\omega_t)$ is invariant if the argument of $\hat{f}$ is invariant under some coordinate transformations. Based on (7), we write the frequency vector defined by the arguments of $\hat{f}$ as

$$\vec{\xi} = [\xi_x, \xi_y, \xi_z]^T = \mathbf{A}^{-T}(\phi, \theta) [\omega_s, \omega_t, \omega_z]^T = \omega_s \hat{u}^*(\phi, \theta) + \omega_t \hat{n}^*(\phi, \theta) + \omega_z \hat{e}_z^*(\phi, \theta), \quad (35)$$

where $\mathbf{A}^{-T}(\phi, \theta) = \mathbf{A}^*(\phi, \theta) = [\hat{u}^*(\phi, \theta), \hat{n}^*(\phi, \theta), \hat{e}_z^*(\phi, \theta)]$ with $\hat{u}^*$, $\hat{n}^*$ and $\hat{e}_z^*$ are given in (8) and $\mathbf{A}(\phi, \theta) = [\hat{u}(\phi, \theta), \hat{n}(\phi, \theta), \hat{e}_z(\phi, \theta)]$ with $\hat{u}$, $\hat{n}$ and $\hat{e}_z$ are given by (2). Replacing ω_s , ω_t in (35) by the expressions of the characteristic curve for FCE1 in (19), or replacing ω_t by the expression for FCE2 in (23), one verifies that the result is independent of ϕ and θ for FCE1 and FCE2, respectively. As shown in figure 3, this means that the characteristic curves of the Fourier consistency equation are loci of data points sharing a common frequency point $\vec{\xi}$. The characteristic curves are generated by changing ϕ and θ for FCE1 and FCE2, respectively. So the affine transform for FCE1 is

$$[\omega_s, \omega_t, \omega_z]^T = \mathbf{A}^T(\phi, \theta) \mathbf{A}^{-T}(\phi_0, \theta) [\omega_{s,0}, \omega_{t,0}, \omega_z]^T, \quad (36)$$

where

$$\mathbf{A}^T(\phi, \theta) \mathbf{A}^{-T}(\phi_0, \theta) = \begin{bmatrix} \cos(\phi_0 - \phi) & -\sec \theta \sin(\phi_0 - \phi) & \tan \theta \sin(\phi_0 - \phi) \\ \cos \theta \sin(\phi_0 - \phi) & \cos(\phi_0 - \phi) & \sin \theta [1 - \cos(\phi_0 - \phi)] \\ 0 & 0 & 1 \end{bmatrix}. \quad (37)$$

The affine transform for FCE2 is

$$[\omega_s, \omega_t, \omega_z]^T = \mathbf{A}^T(\phi, \theta) \mathbf{A}^{-T}(\phi, \theta_0) [\omega_s, \omega_{t,0}, \omega_z]^T, \quad (38)$$

where

$$\mathbf{A}^T(\phi, \theta) \mathbf{A}^{-T}(\phi, \theta_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sec \theta_0 \cos \theta & \sin \theta - \tan \theta_0 \cos \theta \\ 0 & 0 & 1 \end{bmatrix}. \quad (39)$$

The Fourier-John equation is a combination of FCE1 and FCE2, and it can be interpreted by changing both ϕ and θ but keeping ω_t unchanged. The affine transform for Fourier-John equation is

$$[\omega_s, \omega_t, \omega_z]^T = \mathbf{A}^T(\phi, \theta) \mathbf{A}^{-1}(\phi_0, \theta_0) [\omega_{s,0}, \omega_t, \omega_z]^T, \quad (40)$$

where

$$\mathbf{A}^T(\phi, \theta) \mathbf{A}^{-1}(\phi_0, \theta_0) = \begin{bmatrix} \cos(\phi_0 - \phi) & -\sec \theta_0 \sin(\phi_0 - \theta) & \tan \theta_0 \sin(\phi_0 - \phi) \\ \cos \theta \sin(\phi_0 - \phi) & \sec \theta_0 \cos \theta \cos(\phi_0 - \phi) & \sin \theta - \tan \theta_0 \cos \theta \cos(\phi_0 - \phi) \\ 0 & 0 & 1 \end{bmatrix}. \quad (41)$$

We geometrically interpret the general solutions to the two Fourier consistency equations as—the two families of characteristic curves (1D manifolds) can be respectively obtained by changing the azimuthal and co-polar angles of the biorthogonal coordinates in Fourier space. Since the two FCEs are independent, the two families of characteristic curves are also independent and can be combined. By linearly combining the two families of characteristic curves, one can construct characteristic surfaces (2D manifolds) inside the 5D TOF-PET Fourier space, along which $\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) \cos \theta / \hat{h}(\omega_t)$ is constant. Specifically, for each point $(\omega_s, \omega_t, \omega_z; \phi, \theta)$ in the 5D space, there are two and only two curves from the two families of characteristic curves passing through this point. By moving one curve along the other, one can construct a 2D characteristic surface along which the scaled version of $\hat{p}$ is constant. By integrating $\hat{p}$ along the two families of characteristic curves or the characteristic surfaces with possible weightings, one can develop different Fourier-based rebinning methods using the FORCES. These rebinning methods can substantially reduce variance for noisy TOF-PET data and reduce the data storage without information loss. It is worth noting that the transfer matrix $\mathbf{A}$ can be generalized to a non-singular matrix (affine transform) to interpret TOF-PET data with other parameterizations, e.g., $\mathbf{A}$ becomes the identity matrix for histo-image parameterization (Li *et al* 2015a). The geometric interpretation of the redundancy in TOF-PET data is more general than the Fourier consistency equations since it does not require a Gaussian TOF profile. So the FORCES developed in section 3 are also valid independent of the TOF profile.

5. Numerical examples

We develop a unified picture for converting different types of PET data with twelve FORCES. The full implementations of the FORCES are beyond the scope of the paper. As for (30), both the approximate and exact implementations of Fourier rebinning for 3D non-TOF PET were given in (Defrise *et al* 1997, Liu *et al* 1999); the corresponding inverse rebinning for 3D non-TOF PET was given in (Cho *et al* 2007). The approximate implementation of (20) to rebin 3D TOF-PET to 3D non-TOF PET data was implemented elsewhere (Cho *et al* 2009, Ahn *et al* 2011, Bai *et al* 2014). The exact rebinning of 3D TOF-PET data to 2D TOF-PET data using the consistency equation (17) was given in (Defrise *et al* 2008). Here, we perform numerical simulations with a generic 3D TOF-PET system with timing resolution of 500 ps FWHM to demonstrate of the potential applications of the FORCES. We used 27 TOF bins with bin size of 24 mm, and 320 angular samplings uniformly spaced over 180°. We used the NEMA IEC phantom with hot spheres of 10, 13, 17 and 22 mm with contrast ratio of 4 : 1 and cold spheres of 28 and 37 mm. As shown in figure 4, The height of the phantom is 180 mm and the centers of the spheres are axially all in the central plane. The phantom was discretized in $240 \times 240 \times 121$ with 2 mm voxel size with an oversampling factor of 2^3 for each voxel.

5.1. Inverse rebinning from 2D non-TOF PET data to 3D TOF-PET data

To investigate a practical implementation of inverse rebinning and the exactness of the corresponding FORCE, we performed inverse rebinning from 2D non-TOF PET data to 3D TOF-PET data. We generated noise-free 2D non-TOF sinograms of the 3D NEMA phantom using a strip-integral model of 2mm for each axial slice. We then performed inverse rebinning from the noise-free 2D non-TOF data to 3D TOF data. Specifically, we performed gridded data interpolation to compute 3D TOF Fourier data $\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta)$ from the 2D non-TOF Fourier data $\hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0)$ using the following equation

$$\begin{aligned} \hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) &= \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0), \\ \omega_{s,0} &= \sqrt{(\omega_t \sec \theta - \omega_z \tan \theta)^2 + \omega_s^2}, \\ \phi_0 &= \phi + \tan^{-1} \frac{\omega_t \sec \theta - \omega_z \tan \theta}{\omega_s}. \end{aligned} \quad (42)$$

Equation (42) is equivalent to (31) since $\hat{p}(\omega_{s,0}, 0, \omega_z; \phi_0, 0)$ is Hermitian symmetric with respect to $\omega_{s,0}$ and ω_z and only the domain $\omega_{s,0} \geq 0$ is needed. We applied the B-spline interpolation method in each respective dimension of $(\omega_{s,0}, \phi_0)$ since it outperforms the bilinear method with similar computing time.

We also generated reference transverse and axial TOF-PET sinograms using a strip-integral model with a TOF profile of 75 mm FWHM. The transverse TOF sinograms are $256 \times 320 \times 27$ along s , ϕ and t with $z = 0$ and $\theta = 0$. The axial TOF sinograms are $241 \times 121 \times 27$ along z , θ and t with $s = -1$ mm and $\phi = -\pi/2$. There are 121 samplings for co-polar angle θ with $\tan \theta$ uniformly spaced over $(-1/\sqrt{3}, 1/\sqrt{3})$, which corresponds to an acceptance angle of 60°. The comparison of the inverse rebinned TOF sinograms with the corresponding reference sinograms is shown in figure 5. The normalized root-mean-square errors

(NRMSEs) of the four transverse TOF sinograms with TOF bins 0, 2, 4 and 6 are respectively 0.17%, 0.45%, 0.36% and 0.48%; the average NRMSE over 27 TOF bins is 0.34%. The NRMSEs of the four axial TOF sinograms with TOF bins 0, 2, 4 and 6 are respectively 0.59%, 0.75%, 0.54% and 0.87%; the average NRMSE over 27 TOF bins is 0.66%.

5.2. Fourier-based rebinning of 2D TOF-PET data and image reconstructions

We also performed Fourier-based rebinning from 2D TOF-PET data to 2D non-TOF PET data for the central transverse slice of the NEMA phantom. The rebinned non-TOF data were then reconstructed using OSEM with 8 subsets. We also performed non-TOF reconstruction by summing the TOF bins and TOF reconstruction using OSEM with the same number of subsets for comparison. We generated 60 noise realizations of noisy 2D TOF sinograms with 1×10^6 total counts in each realization. We then performed rebinning according to FORCE (22) to compute the 2D non-TOF sinograms for each noise realization. Specifically, we computed the non-TOF Fourier data $\hat{p}(\omega_s, 0, 0, \omega_z; \phi_0, 0)$ by taking a weighted average of the scaled TOF Fourier data $\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) / \hat{h}(\omega_t)$ over TOF bins ω_t . For optimal rebinning, we selected weight $w(\omega_t) = \hat{h}^2(\omega_t) = \exp(-\sigma^2 \omega_t^2)$ (Ahn *et al* 2011). Again, we applied the gridded B-Spline interpolation in (ω_s, ϕ) for each ω_t due to the relatively coarse sampling in ω_t . The mean and variance of the rebinned non-TOF sinograms were calculated and compared. The comparison of the rebinned sinogram and the non-TOF sinograms is shown in figure 6(a); the horizontal profiles of the mean and variance sinograms through $\phi = 45^\circ$ are shown in figure 6(b).

We then performed image reconstructions from the noisy rebinned sinograms using OSEM, which is referred as FORCE reconstructions. We also performed reconstructions from the non-TOF and TOF sinograms using the same OSEM for comparison. We ran up to 128 iterations with 8 subsets to ensure that convergence can be achieved for all the three types of reconstruction. The contrast recovery coefficients (CRCs) for the four hot rods of 10, 13, 17 and 22 mm diameter were also calculated (Daube-Witherspoon *et al* 2002, Li *et al* 2015b). Figure 7 shows a comparison of the sample, mean and variance of the three types of reconstructions (non-TOF, FORCE and TOF). The three types of reconstructed images in figures 7 have approximately matched CRC for the 10 mm rod in the mean reconstructed images with respectively iteration numbers 18, 18 and 7. The central horizontal profiles of the mean and variance images through the hot rods of 22 and 10 mm diameter are shown in figure 8 for a quantitative comparison. We see from the figure that the mean profiles are very similar; however, the variance profiles of FORCE and TOF reconstructions are much smaller than the non-TOF reconstructions. The FORCE reconstructions from the rebinned sinograms have similar variance (slightly higher variance in the 22 mm hot-rod region) compared to the corresponding TOF reconstructions.

For a more quantitative comparison, we used CRC versus the standard deviation as image quality metric to compare the three types of reconstructions. The standard deviations were calculated as the square root of the mean variance for the four hot rods. The calculated CRC versus standard deviation for the three types of reconstructions are shown in figure 9. We see

the FORCE reconstructions from the rebinned non-TOF sinogram have similar performance compared to the TOF reconstructions; and both reconstructions show much better performance than the non-TOF reconstructions.

6. Discussion

In n -dimensional imaging, an object as a function of n variables is measured as a transform of the object with m variables, e.g., the data space of X-ray transform with and without TOF measurement are respectively $m = 2n - 1$ and $m = 2n - 2$. In general the data space is larger than the object space, $m > n$. As the data-processing inequality states that no clever manipulation of the data can improve the inferences that can be made from the data (Cover and Thomas 2006), the large space of measured data cannot increase information about the object, but can reduce variance for noisy measured data. There are $m - n$ consistency equations to characterize the redundancy of measured data. For 3D TOF PET, the data space is five dimensional while the object is three dimensional—the two degrees of redundancy are characterized by the two independent consistency equations. In this paper, we derived two Fourier consistency equations (FCEs) and the Fourier-John equation (FJE) for 3D TOF PET based on the generalized projection-slice theorem, which are the dual of the sinogram consistency equations and John's equation (Defrise *et al* 2013). We then solved the three equations using the method of characteristics and obtained the Fourier rebinning and consistency equations (FORCEs). The two degrees of entangled redundancy of TOF-PET data can be explicitly elicited and exploited by the solutions of the FCEs and FJE along the characteristic curves, which give a complete understanding of the rich structure of the 3D X-ray transform with TOF measurement.

To the best of our knowledge, we showed for the first time that the two consistency equations are *necessary* and *sufficient* for 3D X-ray transform with TOF measurement. For non-TOF case, the sufficiency of John's equation for X-ray transform was proven by (John 1938). The necessary condition of the two Fourier consistency equations (FCEs) (11) and (12) was proved in appendix A. From the general solutions of the two FCEs, we obtained the generalized projection-slice theorem (7) and thus showed that the two FCEs are also sufficient. Since TOF-PET data (1) is the inverse Fourier transform of (7) and (16,17) are dual of (11,12), the two consistency equations (CEs) (11,12) are also necessary and sufficient for 3D X-ray transform with TOF measurement (1).

We obtained the general solutions (19), (23) and (27) to the two FCEs and FJE, from which we obtained another nine Fourier rebinning and consistency equations (FORCEs) as the special cases of the general solutions. Four of them were previously derived as Fourier rebinning equation and mapping equations: (30) is equivalent to the Fourier rebinning equation (Defrise *et al* 1997, Liu *et al* 1999), and (20), (29), (30), and (22) are respectively equivalent to the mapping A, B, D and E in (Cho *et al* 2009). As a byproduct, we discover the connection between the mapping equations (including Fourier rebinning equation) and consistency equations—the mapping equations are special cases of the general solutions to the Fourier consistency equations. We also obtained new FORCEs, which can be used as new Fourier-based rebinning algorithms for TOF-PET data reduction, or inverse rebinnings for designing fast projectors, or consistency conditions for estimating missing data. From the

twelve FORCES, we obtained a complete scheme to convert among different types of PET data: 3D TOF, 3D non-TOF, 2D TOF and 2D non-TOF data, as shown in figure 2.

In the implementations of Fourier-based rebinning of 3D TOF data using the FORCES (25) and (29), the Fourier transform of TOF sinograms with respect to z is required. However, the measured sinograms can be truncated axially along z for oblique sinograms ($\theta \neq 0$). The 3DRP and 3D-FRP algorithms overcome this axial truncation by initially estimating the unmeasured oblique sinograms, which is done by forward projecting an initial image reconstructed from only the direct sinograms in spatial and Fourier domains, respectively (Kinahan and Rogers 1989, Matej and Lewitt 2001). This approach was also adopted by the FOREX algorithm (Defrise *et al* 1997, Liu *et al* 1999). Instead of projecting an initial image reconstructed from the direct sinograms, inverse Fourier rebinning can be applied to estimate the unmeasured oblique sinograms. Similar to the implementation of the FOREX algorithm, the inverse Fourier rebinning using the FORCES (25) and (29) can also be applied to estimate the unmeasured oblique TOF sinograms.

7. Conclusion

Fully 3D TOF-PET data with rich structure and redundancy are governed by the consistency equations. In this paper, we gave general solutions to the Fourier consistency equations and the Fourier-John equation. The Fourier rebinning equations and mapping equation are just the special cases of the general solutions. As a byproduct, we have derived a few new Fourier rebinning and consistency equations (FORCES) in Fourier space, and thus we obtain a unified and complete picture to convert between different types of PET data (3D TOF, 3D non-TOF, 2D TOF and 2D non-TOF). We have proven that the consistency equations are *necessary* and *sufficient* for 3D X-ray transform with TOF measurement. Further, we gave a geometric interpretation for the Fourier consistency equations, which provides a deep insight into the rich and entangled structure of 3D TOF data. The FORCES can be very useful in many clinical TOF-PET imaging applications, e.g., developing fast TOF image reconstructions using Fourier-based rebinning or inverse Fourier rebinning, estimation of missing data and simultaneous emission and attenuation reconstructions from TOF-PET data without transmission scan. Finally, we presented numerical examples of inverse rebinning for a 3D TOF PET and Fourier-based rebinning for a 2D TOF PET using FORCES, and we showed that the FORCE reconstructions from the rebinned non-TOF sinograms can fully utilize TOF information and produce comparable results compared to the TOF OSEM reconstructions. The potential applications of the FORCES can only be partially demonstrated by the numerical examples.

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Appendix

A. Proof of the two Fourier consistency equations

We assume that f has compact support and continuous partial derivatives up to the second order (this condition is more restrictive than necessary), which implies that $\hat{f} \in C^\infty$ and $\hat{p} \in C^\infty$ are infinitely differentiable functions. To prove the consistency equations (11) and (12), we calculate the first-order derivatives of $\hat{p}$ with respect to φ , θ , ω_t and ω_s . We omit the arguments of f and its derivatives, and denote the gradient of f with respect to x , y and z by ∇f . Then, we have:

$$\frac{\partial \hat{p}}{\partial \phi} = \frac{\hat{h}(\omega_t)}{\cos \theta} \left[\omega_s \cos \theta \hat{n}^* \cdot \nabla \hat{f} - (\omega_t \sec \theta - \omega_z \tan \theta) \hat{u}^* \cdot \nabla \hat{f} \right], \quad (\text{A.1})$$

where we used $\partial \hat{u}^* / \partial \phi = \cos \theta \hat{n}^*$, $\partial \hat{n}^* / \partial \phi = -\sec \theta \hat{u}^*$ and $\partial \hat{e}_z^{ast} / \partial \phi = \tan \theta \hat{u}^{ast}$,

$$\frac{\partial \hat{p}}{\partial \theta} = \tan \theta \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{f} + \frac{\hat{h}(\omega_t)}{\cos \theta} (\omega_t \tan \theta - \omega_z \sec \theta) \hat{n}^* \cdot \nabla \hat{f}, \quad (\text{A.2})$$

where we used $\partial \hat{u}^* / \partial \theta = 0$, $\partial \hat{n}^* / \partial \theta = \tan \theta \hat{n}^*$ and $\partial \hat{e}_z^{ast} / \partial \theta = -\sec \theta \hat{n}^{ast}$;

$$\frac{\partial \hat{p}}{\partial \omega_t} = \frac{-\sigma^2 \omega_t \hat{h}(\omega_t)}{\cos \theta} \hat{f} + \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{n}^* \cdot \nabla \hat{f}, \quad (\text{A.3})$$

we used the relation $\partial \hat{h}(\omega_t) / \partial \omega_t = -\sigma^2 \omega_t \hat{h}(\omega_t)$ for the Gaussian profile;

$$\frac{\partial \hat{p}}{\partial \omega_s} = \frac{\hat{h}(\omega_t)}{\cos \theta} \hat{u}^* \cdot \nabla \hat{f}. \quad (\text{A.4})$$

We can verify (11) by putting (A.1), (A.3) and (A.4) into it, and (12) by putting (A.2) and (A.3) into it. We conclude that (11) and (12) are *necessary* for the 3D X-ray transform with TOF measurement.

B. The solution to the first consistency equation

Applying the method of characteristics (John 1982, Courant and Hilbert 1989), we can obtain from (11) the following characteristic equations:

$$d\phi = \frac{d\omega_s}{\omega_t \sec \theta - \omega_z \tan \theta} = \frac{d\omega_t}{-\omega_s \cos \theta} = \frac{d\hat{p}}{\sigma^2 \omega_s \omega_t \cos \theta \hat{p}}, \quad d\omega_z = 0, \quad \text{and} \quad d\theta = 0. \quad (\text{B.1})$$

Since ω_z and θ are invariant along the characteristic curves, we can determine the characteristic curves by solving the three ordinary differential equations (ODEs)

$$\frac{d\omega_s}{\omega_t \sec \theta - \omega_z \tan \theta} = \frac{d\omega_t}{-\cos \theta \omega_s}, \quad d\phi = \frac{d\omega_t}{-\cos \theta \omega_s}, \quad \frac{d\hat{p}}{\hat{p}} = -\sigma^2 \omega_t d\omega_t. \quad (\text{B.2})$$

The solutions to the above three ODEs are

$$\begin{aligned}\sqrt{\omega_s^2 + (\omega_t \sec \theta - \omega_z \tan \theta)^2} &= C_1, \\ \phi + \tan^{-1} \frac{\omega_t \sec \theta - \omega_z \tan \theta}{\omega_s} &= C_2, \\ \hat{p} &= C_3 e^{-\sigma^2 \omega_t^2 / 2}.\end{aligned}\quad (\text{B.3})$$

The three constants C_1 , C_2 and C_3 can be determined from initial conditions. We applied the solution of the first ODE in solving the second ODE in (B.3). The first two solutions (with constant-valued ω_z and θ) determine the parameterized character curves with parameter, e.g., ϕ , and the third one determines the sinogram data along the character curves. The solution to PDE (11) (also called the integral surface) consists of the union of the characteristic curves with different values of C_1 and C_2 (John 1982, Courant and Hilbert 1989). By choosing the initial argument $(\omega_{s,0}, \omega_{t,0}, \omega_z; \phi_0, \theta)$ for the sinogram data $\hat{p}$, we obtain the three constants:

$$\begin{aligned}C_1 &= \sqrt{\omega_{s,0}^2 + (\omega_{t,0} \sec \theta - \omega_z \tan \theta)^2}, \\ C_2 &= \phi_0 + \tan^{-1} \frac{\omega_{t,0} \sec \theta - \omega_z \tan \theta}{\omega_{s,0}}, \\ C_3 &= \frac{1}{e^{-\sigma^2 \omega_{t,0}^2 / 2}} \hat{p}(\omega_{s,0}, \omega_{t,0}, \omega_z; \phi_0, \theta).\end{aligned}\quad (\text{B.4})$$

Putting (B.4) into (B.2), we obtain

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{e^{-\sigma^2 \omega_t^2 / 2}}{e^{-\sigma^2 \omega_{t,0}^2 / 2}} \hat{p}(\omega_{s,0}, \omega_{t,0}, \omega_z; \phi_0, \theta),$$

$$\begin{aligned}\sqrt{\omega_s^2 + (\omega_t \sec \theta - \omega_z \tan \theta)^2} &= \sqrt{\omega_{s,0}^2 + (\omega_{t,0} \sec \theta - \omega_z \tan \theta)^2}, \\ \phi + \tan^{-1} \frac{\omega_t \sec \theta - \omega_z \tan \theta}{\omega_s} &= \phi_0 + \tan^{-1} \frac{\omega_{t,0} \sec \theta - \omega_z \tan \theta}{\omega_{s,0}}.\end{aligned}\quad (\text{B.5})$$

We can then obtain (19) from (B.5) by representing ω_s and ω_t using $\omega_{s,0}$, $\omega_{t,0}$ and ϕ while keeping the same sign for ω_s and $\omega_{s,0}$.

C. The solution to the second consistency equation

From (12), we can obtain the following characteristic equations:

$$d\theta = \frac{d\omega_t}{-(\omega_t \tan \theta - \omega_z \sec \theta)} = \frac{d\hat{p}}{\tan \theta \hat{p} + \sigma^2 (\omega_t \tan \theta - \omega_z \sec \theta) \omega_t \hat{p}}, \quad d\omega_s = 0, \quad d\omega_z = 0, \quad \text{and} \quad d\theta = 0. \quad (\text{C.1})$$

The variables ω_s , ω_z and θ are invariant along the characteristic curves, and we can rewrite (C.1) as the following two ODEs:

$$\frac{d\omega_t}{d\theta} + \tan \theta \omega_t = \sec \theta \omega_z, \quad \frac{d\hat{p}}{\hat{p}} = \tan \theta d\theta - \sigma^2 \omega_t d\omega_t. \quad (\text{C.2})$$

The solutions to the two first-order ODEs are

$$\omega_t \sec \theta - \omega_z \tan \theta = C_1, \quad \hat{p} = C_2 \frac{e^{-\sigma^2 \omega_t^2 / 2}}{\cos \theta}. \quad (\text{C.3})$$

By choosing the initial argument $(\omega_s, \omega_{t,0}, \omega_z; \phi, \theta_0)$, we obtain the two constants:

$$C_1 = \omega_{t,0} \sec \theta_0 - \omega_z \tan \theta_0, \quad C_2 = \frac{\cos \theta_0}{e^{-\sigma^2 \omega_{t,0}^2 / 2}} \hat{p}(\omega_s, \omega_{t,0}, \omega_z; \phi, \theta_0). \quad (\text{C.4})$$

Putting (C.4) into (C.3), we obtain

$$\hat{p}(\omega_s, \omega_t, \omega_z; \phi, \theta) = \frac{e^{-\sigma^2 \omega_t^2 / 2}}{e^{-\sigma^2 \omega_{t,0}^2 / 2}} \frac{\cos \theta_0}{\cos \theta} \hat{p}(\omega_s, \omega_{t,0}, \omega_z; \phi, \theta_0), \quad (\text{C.5})$$

$$\omega_t \sec \theta - \omega_z \tan \theta = \omega_{t,0} \sec \theta_0 - \omega_z \tan \theta_0.$$

After representing ω_t as a function of $\omega_{t,0}$ and θ , we obtain the solution (23).

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Figure 1.

Data parameterization for a multi-ring TOF-PET scanner. The LOR between detectors A and B is parameterized in sinogram format by the variables s , t , z , ϕ and θ . the TOF profile is centered at the most likely position $\vec{x} = \vec{x}_0 + t\hat{n}$ along $\hat{n}$, and $\vec{x}_0$ is the center of the LOR.

Figure 2.

A unified picture for converting among different types of PET data using the twelve FORCES: 3D TOF, 2D TOF, 3D non-TOF and 2D non-TOF data.

Figure 3.

Geometric interpretation for the two Fourier consistency equations: FCE1 and FCE2. For a fixed frequency point $\vec{\xi} = [\xi_x, \xi_y, \xi_z]^T$, we have the sinogram frequencies ω_s , ω_t and ω_z with $\vec{P} = \omega_z \hat{u}^*$, $\vec{Q} = \omega_t \hat{n}^*$, $\vec{R} = \omega_z \hat{e}_z^*$ and $\vec{\xi} = \vec{P} + \vec{Q} + \vec{R}$. By respectively changing the azimuthal angle ϕ and co-polar angle θ , we obtain the $\vec{P}$, $\vec{Q}$ and $\vec{R}$ traces that have the same locus $\vec{\xi}$ for CE1 and CE2. In (b), we have $\chi = \omega_t \sec \theta - \omega_z \tan \theta$, and

$$\chi = \vec{\xi} \cdot \hat{n}^* / \|\hat{n}^{ast}\| = -\xi_x \sin \phi + \xi_y \cos \phi \text{ which states that } \chi \text{ is independent of } \theta.$$

Figure 4.
The transverse slice (left) and the coronal slice (right) through the center of the NEMA ICE phantom

(a) Transverse TOF-PET data

(b) Axial TOF-PET data

Figure 5.

Comparison of inverse rebinned transverse (a) and axial (b) TOF sinograms with the corresponding directly computed TOF sinograms. The horizontal and vertical axes in (a) respectively represent radial variable s and azimuthal angle φ ; the axes in (b) respectively represent axial variable z and the tangent of angle θ . The columns show the TOF bin indices 0, 2, 4 and 6; the rows show the inverse rebinned TOF sinogram (FORCE), the directly computed sinogram (Reference) and their difference (difference).

(a) Rebinned sinogram and the non-TOF sinogram reference

(b) Horizontal profile of mean and variance through $\phi = 45^\circ$

Figure 6.

Comparison the rebinned sinograms and the corresponding reference non-TOF sinograms. The columns show the sample, mean and variance sinograms; the rows show the non-TOF and rebinned sinograms. The horizontal and vertical axes in (a) represent radial variable s and azimuthal angle ϕ , respectively. The horizontal profiles of the mean and variance sinograms through $\phi = 45^\circ$ are shown in (b).

Figure 7. The reconstructed images of non-TOF, FORCE and TOF (top). The mean and variance reconstructions are shown in the middle and bottom rows. The 3 types of reconstructed images have the same CRC in the mean images by selecting different iteration numbers.

Figure 8.
Central horizontal profiles of mean and variance reconstructions.

Figure 9.

Calculated CRC vs standard deviation for the non-TOF, FORCE and TOF reconstructions of the four hot rods. Each marker represents one iteration.