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A comment on sample size calculations for binomial confidence intervals

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Abstract

In this article we examine sample size calculations for a binomial proportion based on the confidence interval width of the Agresti–Coull, Wald and Wilson Score intervals. We pointed out that the commonly used methods based on known and fixed standard errors cannot guarantee the desired confidence interval width given a hypothesized proportion. Therefore, a new adjusted sample size calculation method was introduced, which is based on the conditional expectation of the width of the confidence interval given the hypothesized proportion. With the reduced sample size, the coverage probability can still maintain at the nominal level and is very competitive to the converge probability for the original sample size.

Keywords

binomial proportion; sample size calculation; expected width; Agresti–Coull interval; Wald interval; Wilson Score interval

1. Introduction

A common problem in biostatistics is the estimation of a binomial parameter, π , based on a binomial observation, $Y \sim B(n, \pi)$, e.g. π may represent the fraction surviving beyond one year, the sensitivity or specificity of a diagnostic test, or the proportion of values outside the normal range, to name just a few common applications. In early stage biomedical investigations of pilot studies, researchers are oftentimes more inclined to base their inferences about $\pi = \pi_0$ through the use of the $(1 - \alpha) \times 100\%$ confidence interval for π versus a formal hypothesis test of $H_0 : \pi = \pi_0$. The reasons for this preference are varied, but generally involve particulars in early phase investigations where very little is known about π or historically confidence intervals are the preferred approach within a given subspecialty, e.g. diagnostic testing.

It is well known that the maximum-likelihood estimator for π is given by $\hat{\pi} = Y/n$ and that asymptotically $\sqrt{n}(\hat{\pi} - \pi) / \sqrt{\pi(1 - \pi)} \sim N(0, 1)$. Historically, this result was the basis for the large sample $(1 - \alpha) \times 100\%$ approximate confidence interval for π given by

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$$\hat{\pi} \pm z_{\alpha/2} \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}, \quad (1)$$

where $z_{\alpha/2}$ is the upper $(1 - \alpha/2)$ th quantile from a standard normal distribution, oftentimes given as $\Phi^{-1}(1 - \alpha/2)$. The interval given by Equation (1) is referred to as a Wald interval for π [1].

In recent years, there have been several improvements to the approximate interval given by Equation (1), e.g. see Piegorsch [5] for an excellent survey of these results. Most notable improvement is a straightforward correction developed by Brown *et al.* [3,4], which extends the approach of Agresti and Coull [1], and is given by

$$\tilde{\pi} \pm z_{\alpha/2} \sqrt{\frac{\tilde{\pi}(1-\tilde{\pi})}{\tilde{n}}}, \quad (2)$$

where the modified estimator of π is given by $\tilde{\pi} = (Y + z_{\alpha/2}^2/2) / (n + z_{\alpha/2}^2)$ and $\tilde{n} = n + z_{\alpha/2}^2$, for details see Brown *et al.* [3,4]. This approach improves the coverage probabilities of the confidence interval for π substantially as compared with the Wald-type interval for a general level α .

Wilson [6] gave an interval estimation approach which is the inversion of the score test for π , and is given by

$$\frac{\hat{\pi} + (1/2n)z_{1-\alpha/2}^2 \pm z_{\alpha/2} \sqrt{\hat{\pi}(1-\hat{\pi})/n + (z_{\alpha/2}^2/4n^2)}}{1 + (1/n)z_{\alpha/2}^2}. \quad (3)$$

By simple derivation, it can be expressed as

$$\tilde{\pi} \pm \frac{z_{\alpha/2}}{\tilde{n}} \sqrt{n\hat{\pi}(1-\hat{\pi}) + \frac{z_{\alpha/2}^2}{4}}, \quad (4)$$

where $\tilde{\pi}$, $\tilde{n}$ and $\hat{\pi}$ are as above.

This interval is also called the Wilson Score interval. The coverage probability is closer to the nominal value compared with the Wald interval and it has good properties even for a small number of trials and/or an extreme probability.

In terms of deriving an estimate for sample size n required to carry out a given study, the basic recommendation based on Equation (1) is to solve the equation

$$\delta_0 = z_{\alpha/2} \sqrt{\frac{\pi_0(1-\pi_0)}{n}} \quad (5)$$

for n at some desired interval width $w_0 = 2\delta_0$ and given some hypothesized and ‘known’ π_0 , e.g. see the oft-cited article by Arkin and Wachtel [2]. This method is widely used even

though it is not particularly accurate, justifying Piegorsch's [5] comment that it is 'methodus non-gratus'. Improvements in sample size estimation based on more accurate intervals such as the one developed by Brown *et al.* [3,4] at Equation (2) are provided in Piegorsch [5]. We argue, however, that this is the incorrect parameterization of the problem at hand even if the more accurate interval estimators are utilized, i.e. Is this the correct approach to 'guarantee' the appropriate confidence interval width given π_0 is known? Given that the calculated interval width is a random variable, it is impossible to correctly ensure *a priori*, the fixed width w_0 . We can, however, calculate the distribution of the random variable corresponding to the interval width conditional on π_0 and its associated quantities. Furthermore, we illustrate in later sections that the desired width as defined as w_0 may not even be achievable, i.e. the range of support for the random variable $\tilde{W}=2\tilde{\delta}$ falls below w_0 , where $\tilde{W}_W$, $\tilde{W}_{AC}$ and $\tilde{W}_S$ denote the estimated width for Wald, Agresti–Coull and Wilson Score methods, respectively, and the corresponding half interval lengths are given by

$$\tilde{\delta}_W = z_{\alpha/2} \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}, \quad (6)$$

$$\tilde{\delta}_{AC} = z_{\alpha/2} \sqrt{\frac{\tilde{\pi}(1-\tilde{\pi})}{\tilde{n}}}, \quad (7)$$

and

$$\tilde{\delta}_S = \frac{z_{\alpha/2}}{\tilde{n}} \sqrt{n\hat{\pi}(1-\hat{\pi}) + \frac{z_{\alpha/2}^2}{4}}. \quad (8)$$

Note also that even if the range of $\tilde{W}=2\tilde{\delta}$ does encompass w_0 , the $P(\tilde{W} \leq w_0)$ is quite high or even equal to 1. To summarize, the expected width of $\tilde{W}$ is less than w_0 . This will be illustrated in more detail in the next few sections.

In the following sections, we will introduce a new sample size calculation method, from which the improved interval is based upon, i.e. solve

$$w_E = E(\tilde{W}|\pi_0), \quad (9)$$

for n corresponding to a desired width of w_E given a fixed sample size, versus a cutpoint w_0 that may not even fall within the support of $\tilde{W}$.

2. Specific case $\pi_0 = \frac{1}{2}$

The specific case that $\pi_0 = \frac{1}{2}$ is very important theoretically given that it is oftentimes recommended as the default conservative approach given that nothing is known about π since it provides the maximum w_0 for a given n . It has been noted by Brown *et al.* [3,4] that

Agresti–Coull confidence interval outperforms the Wald intervals. Hence, we will use the Agresti–Coull interval for derivation of the theorems in this section and then generalize the properties in Section 3 for the other two intervals. For the Agresti–Coull interval, the probability distribution of $\tilde{W}_{AC}$ is given by

$$P\left(\tilde{W}_{AC}=w(y)|\pi_0\right)=\binom{n}{y}\pi_0^y(1-\pi_0)^{n-y}, \quad (10)$$

where

$$w(y)=2z_{\alpha/2}\sqrt{\frac{\tilde{\pi}(y)(1-\tilde{\pi}(y))}{n+z_{\alpha/2}^2}}, \quad (11)$$

and

$$\tilde{\pi}(y)=\frac{y+z_{\alpha/2}^2/2}{n+z_{\alpha/2}^2}, \quad y=0, 1, 2, \dots, n, \quad (12)$$

The sample space for $\tilde{W}_{AC}$ is given by the set $\mathcal{W}=\{w(0), w(1), \dots, w(n)\}$ with associated binomial probabilities given in Equation (10).

In order to develop our argument for the appropriate sample size metric, we first present two very straightforward theorems: **THEOREM 1a** ($n = \text{odd}$) *Given $\pi_0 = \frac{1}{2}$ and a sample size of $n = 2m + 1$, $m = 0, 1, 2, \dots$, the maximum of $\tilde{W}_{AC}$ is strictly less than w_0 .*

Proof The proof is straightforward in that the maximum for $\tilde{W}_{AC}$ occurs at the value of $\tilde{\pi}$ where $\sqrt{\tilde{\pi}(1-\tilde{\pi})}$ is maximized, which occurs at the values for which $Y = (n-1)/2$ and $Y = (n+1)/2$ corresponding to values for $\tilde{\pi}$ of $\left((n-1)/2 + z_{\alpha/2}^2/2\right) / (n + z_{\alpha/2}^2)$ and $\left((n+1)/2 + z_{\alpha/2}^2/2\right) / (n + z_{\alpha/2}^2)$, respectively. The width w_0 is maximized for the value of π_0 for which $\sqrt{\pi_0(1-\pi_0)}$ is maximized, which occurs at $\pi_0 = \frac{1}{2}$. Noting that $\tilde{\pi}$ at $Y = (n-1)/2$ and $Y = (n+1)/2$ are strictly smaller than $\pi_0 = \frac{1}{2}$ completes the proof.

THEOREM 1b ($n = \text{even}$) *Given $\pi_0 = \frac{1}{2}$ and a sample size of $n = 2m$, $m = 0, 1, 2, \dots$, the maximum value that $\tilde{W}_{AC}$ may attain is equal to w_0 .*

Proof The proof follows a similar logic as Theorem 1a in that the maximum for $\tilde{W}_{AC}$ occurs at $Y = n/2$.

COROLLARY 1b For sample sizes of $n = 2m$, $m = 0, 1, 2, \dots$, the

$$P\left(\tilde{W}_{AC} < w_0 | \pi_0 = \frac{1}{2}\right) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Proof The proof follows from Theorem 1b in that

$$P\left(\tilde{W}_{AC} < w_0 | \pi_0 = \frac{1}{2}\right) = 1 - P\left(\tilde{W}_{AC} = w_0 | \pi_0 = \frac{1}{2}\right), \text{ which occurs at } Y = n/2.$$

THEOREM 2 The $E\left(\tilde{W}_{AC} | \pi_0 = \frac{1}{2}\right) < w_0$.

Proof Expand $\tilde{W}_{AC}$ in a Taylor series about $\pi_0 = \frac{1}{2}$. Such that

$$2z_{\alpha/2} \sqrt{\frac{\tilde{\pi}(1-\tilde{\pi})}{\tilde{n}}} = \frac{2z_{\alpha/2}}{\sqrt{\tilde{n}}} \left[\frac{1}{2} - \left(\tilde{\pi} - \frac{1}{2}\right)^2 - \left(\tilde{\pi} - \frac{1}{2}\right)^4 - 2\left(\tilde{\pi} - \frac{1}{2}\right)^6 - O\left(\left(\tilde{\pi} - \frac{1}{2}\right)^8\right) \right]. \quad (13)$$

Taking expectations on both sides of Equation (13) and noting that the values for

$$E\left(\tilde{\pi} - \frac{1}{2}\right)^{2k} > 0 \text{ for all } k \text{ in the summand completes the proof.}$$

The implications of Theorems 1a, b and 2 may be summarized as follows:

1. From Theorems 1a and b: If we anticipate that $\pi_0 = \frac{1}{2}$, then solving for n from Equation (7) will guarantee a high probability that the calculated width $\tilde{W}_{AC}$ will be less than the desired width w_0 . If the sample size turns out to be odd then it is *impossible* for the calculated interval to be as wide or wider than w_0 , i.e. $\tilde{W}_{AC} < w_0$ for odd sample sizes.
2. From Theorem 2: If $\pi_0 = \frac{1}{2}$, then the value for the width of the confidence interval that we would expect from repeated experiments is less than the desired w_0 .

3. General case and applicability to other intervals

In this section, we will investigate the properties of $\tilde{W}_{AC}$ for the case $\pi_0 \neq \frac{1}{2}$.

THEOREM 3 $P\left(\tilde{W}_{AC} < w_0 | \pi_0 \neq \frac{1}{2}\right) \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty.$

Proof Given $\pi_0 \neq \frac{1}{2}$, the probability distribution of W_{AC} is given in Equation (10). By

Equation (11), we can simplify $P\left(\tilde{W}_{AC} < w_0 | \pi_0 \neq \frac{1}{2}\right)$ as

$$P\left(\tilde{W}_{AC} < w_0 | \pi_0 \neq \frac{1}{2}\right) = P\left(\tilde{\pi}(1-\tilde{\pi}) < \pi_0(1-\pi_0) | \pi_0 \neq \frac{1}{2}\right). \quad (14)$$

Based on the property of function $f(x) = x(1-x)$, we can conclude that Equation (14) can be further simplified for $\pi_0 < \frac{1}{2}$ and $\pi_0 > \frac{1}{2}$, respectively.

$$\begin{aligned} & P\left(\tilde{\pi}(1-\tilde{\pi}) < \pi_0(1-\pi_0) \mid \pi_0 < \frac{1}{2}\right) \\ &= P\left(\tilde{\pi} < \pi_0 \text{ or } \tilde{\pi} > 1-\pi_0 \mid \pi_0 < \frac{1}{2}\right) \\ &= P\left(Y < \pi_0 \left(n + z_{\alpha/2}^2\right) - \frac{z_{\alpha/2}^2}{2} \text{ or } Y > (1-\pi_0) \left(n + z_{\alpha/2}^2\right) - \frac{z_{\alpha/2}^2}{2} \mid \pi_0 < \frac{1}{2}\right), \end{aligned} \quad (15)$$

where $P(Y=y|\pi_0) = \binom{n}{y} \pi_0^y (1-\pi_0)^{n-y}$. Through this derivation, the probability function in Equation (14) can be calculated through the binomial density function and is closer to 0.5 as $n \rightarrow \infty$.

THEOREM 4 The $E\left(\tilde{W}_{AC} \mid \pi_0\right) < w_0$ where $\pi_0 \in (0, 1)$

Proof Following the proof of Theorem 2, by taking the expectation on both sides of the equation for Taylor expansion at π_0 , we can reach the conclusion since $E(\tilde{\pi} - \pi_0)^{2k} > 0$ for $\pi_0 \neq \frac{1}{2}$ case. Theorem 2 gives us the truth that $E\left(\tilde{W}_{AC} \mid \pi_0 = \frac{1}{2}\right) < w_0$, which allows us to complete the proof.

From Theorems 3 and 4, we learn that although the $\tilde{W}_{AC}$ is not strictly less than the desired width w_0 for $\pi_0 \neq \frac{1}{2}$, the value for the width of the confidence interval that we would expect from repeated experiments is still less than the desired w_0 .

Using the same rationale as the Agresti–Coull interval, we will arrive at the same

conclusions for the Wald and Wilson Score intervals as Theorems 1a and b that given $\pi_0 = \frac{1}{2}$, the maximum of $\tilde{W}_W$ and $\tilde{W}_S$ are strictly less than w_0 for $n = \text{odd}$, while the maximum value $\tilde{W}_W$ and $\tilde{W}_S$ may attain is equal to w_0 for $n = \text{even}$. For the conditional expectation, $E\left(\tilde{W}_S \mid \pi_0\right) < w_0$ where $\pi_0 \in (0, 1)$, which can be developed from the Taylor expansion following the proof of Theorem 2.

4. Comparison of fixed versus random interval approaches

In this section, we examined the fixed confidence interval approaches, including the Agresti–Coull, Wald and Wilson Score confidence intervals and compared them with our approach for calculating sample sizes based on the expectation of $\tilde{W}$ given π_0 . Different w_0 's and the confidence interval widths given π_0 were calculated for sample sizes 10, 11, 100, 101, 200, 201, 1000, 1001, 2000 and 2001 for the cases $\pi_0 = 0.5, 0.95$ and 0.975 in Tables 1–9. The probability that $\tilde{W}$ is less than w_0 given π_0 was obtained by summing up the probabilities for which $\tilde{W}$ is less than w_0 . The conditional expectations of $\tilde{W}$ given π_0 were also calculated. The n^* 's, which are the suggested sample sizes by us based on

$E(\tilde{W}_{AC}|\pi_0) > w_0$, were given for the corresponding sample sizes. Additionally, we investigated the coverage probabilities for the given sample size and the suggested sample size n^* through 100,000 simulations.

In Table 1, the $\tilde{W}_{AC}$'s are less than w_0 for all odd sample sizes, thus w_0 is not attainable. For even sample sizes, the $P(\tilde{W}_{AC} < w_0 | \pi_0 = \frac{1}{2})$ increases as a function of n as per Corollary 1b. In terms of the practical implications, we would likely require a sample size of $n^* = n - 1$ to carry out the study based on the metric $E(\tilde{W}_{AC} | \pi_0 = \frac{1}{2})$. Although the preferred sample sizes are smaller than the original sample sizes, we find that the coverage probabilities are closer to nominal level from an examination of the tables. The quantities of interest for $\pi_0 = 0.95$ and $\pi_0 = 0.975$ cases are obtained in Tables 2 and 3. In general, the $P(\tilde{W}_{AC} < w_0 | \pi_0)$ becomes closer to 0.5 as the sample size increases. We notice that n^* would be $n - k$, where k is a positive number and will increase as n gets larger and π_0 gets further from $\frac{1}{2}$. For example, for Table 2 we find empirically $n^* = \lfloor n - \log(n) + 1 \rfloor + 1$. It is interesting to point out that the coverage probabilities for n and n^* are very competitive for all kinds of sample sizes and parameter settings.

As we can see from the results in Tables 4–9, the majority of the preferred sample sizes for the Wald and Wilson Score confidence intervals are less than the ones we have from Agresti–Coull interval in the corresponding parameter settings. n^* 's are not attainable in Tables 5 and 6, which means we cannot find sample sizes such that the expected value of confidence interval width is greater than the desired confidence interval width. For $\pi_0 = \frac{1}{2}$, we find empirically the suggested sample size to carry out the study

$$n_W^* = n_S^* = n_{AC}^* - 1 = n - 2. \quad (16)$$

For $\pi_0 \neq \frac{1}{2}$, the differences between n and n^* are larger than the ones we have from the Agresti–Coull interval. Therefore, if researchers are inclined to choose a sample size calculation method from the Agresti–Coull, Wald or Wilson Score interval without adjustment, we suggest the Agresti–Coull interval since the obtained sample size would be closer to the sample size based on the conditional expectation of the random variable $\tilde{W}$ given π_0 .

The adjusted sample size calculation method can guarantee that the expected confidence interval width given a hypothesized proportion will not be less than the desired confidence interval width. With the reduced sample size, the coverage probability can still maintain at the nominal level. It would be appealing to extend this idea to other discrete or continuous distributions.

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Table 1

Behavior of the Agresti–Coull confidence interval width $\tilde{W}_{AC}$ as compared to the fixed value w_0 at $\pi_0 = \frac{1}{2}$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_{AC} < w_0 \mid \pi_0 = \frac{1}{2}\right)$	$E\left(\tilde{W}_{AC} \mid \pi_0 = \frac{1}{2}\right)$	n^*	CP_n	CP_{n^*}
10	0.5268	0.7539	0.5125	9	0.9783	0.9621
11	0.5088	1.0000	0.4955	10	0.9342	0.9794
100	0.1923	0.9204	0.1914	99	0.9439	0.9564
101	0.1914	1.0000	0.1905	100	0.9545	0.9438
200	0.1373	0.9437	0.1369	199	0.9454	0.9536
201	0.1369	1.0000	0.1366	200	0.9518	0.9449
1000	0.0619	0.9748	0.0618	999	0.9458	0.9494
1001	0.0618	1.0000	0.0618	1000	0.9510	0.9466
2000	0.0438	0.9822	0.0438	1999	0.9478	0.9514
2001	0.0438	1.0000	0.0438	2000	0.9510	0.9488

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_{AC} \mid \pi_0 = \frac{1}{2}\right) > w_0$; CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* .

Table 2

Behavior of the Agresti–Coul confidence interval width $\tilde{W}_{AC}$ as compared to the fixed value w_0 at $\pi_0 = 0.95$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_{AC} < w_0 \mid \pi_0 = 0.95\right)$	$E\left(\tilde{W}_{AC} \mid \pi_0 = 0.95\right)$	n^*	CP_n	CP_{n^*}
10	0.4002	0.5987	0.3948	9	0.9882	0.9289
11	0.3790	0.5688	0.3737	10	0.9842	0.9886
100	0.0959	0.4360	0.0945	97	0.9663	0.9699
101	0.0954	0.6070	0.0940	98	0.9641	0.9691
200	0.0644	0.4547	0.0638	196	0.9659	0.9516
201	0.0642	0.5766	0.0636	197	0.9662	0.9703
1000	0.0274	0.4797	0.0273	995	0.9522	0.9519
1001	0.0274	0.5346	0.0273	996	0.9510	0.9518
2000	0.0192	0.4857	0.0192	1994	0.9537	0.9490
2001	0.0192	0.5245	0.0192	1995	0.9537	0.9488

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_{AC} \mid \pi_0 = 0.95\right) > w_0$; CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* .

Table 3

Behavior of the Agresti–Coull confidence interval width $\tilde{W}_{AC}$ as compared to the fixed value w_0 at $\pi_0 = 0.975$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_{AC} < w_0 \mid \pi_0 = 0.975\right)$	$E\left(\tilde{W}_{AC} \mid \pi_0 = 0.975\right)$	n^*	CP_n	CP_{n^*}
10	0.3831	0.7763	0.3801	9	0.9750	0.9804
11	0.3613	0.7569	0.3582	10	0.9693	0.9750
100	0.0777	0.5422	0.0763	97	0.9608	0.9650
101	0.0771	0.5357	0.0758	98	0.9864	0.9634
200	0.0497	0.4383	0.0490	195	0.9626	0.9668
201	0.0496	0.6115	0.0489	196	0.9626	0.9664
1000	0.0200	0.4724	0.0199	991	0.9466	0.9588
1001	0.0200	0.5509	0.0199	992	0.9474	0.9581
2000	0.0139	0.4805	0.0139	1990	0.9480	0.9564
2001	0.0139	0.5361	0.0139	1991	0.9469	0.9563

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_{AC} \mid \pi_0 = 0.975\right) > w_0$, CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* .

Table 4

Behavior of the Wald confidence interval width $\tilde{W}_w$ as compared to the fixed value $\pi_0 = \frac{1}{2}$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_w < w_0 \mid \pi_0 = \frac{1}{2}\right)$	$E\left(\tilde{W}_w \mid \pi_0 = \frac{1}{2}\right)$	n^*	CP_n	CP_{n^*}
10	0.6198	0.7539	0.5856	8	0.8907	0.9307
11	0.5910	1.0000	0.5617	9	0.9342	0.8198
100	0.1960	0.9204	0.1950	98	0.9439	0.9466
101	0.1950	1.0000	0.1941	99	0.9545	0.9567
200	0.1386	0.9437	0.1382	198	0.9454	0.9456
201	0.1382	1.0000	0.1379	199	0.9518	0.9538
1000	0.0620	0.9748	0.0619	998	0.9458	0.9457
1001	0.0619	1.0000	0.0619	999	0.9510	0.9502
2000	0.0438	0.9822	0.0438	1998	0.9478	0.9487
2001	0.0438	1.0000	0.0438	1999	0.9510	0.9519

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_w \mid \pi_0 = \frac{1}{2}\right) > u_0$; CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* .

Table 5

Behavior of the Wald confidence interval width $\tilde{W}_w$ as compared to the fixed value w_0 at $\pi_0 = 0.95$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_w < w_0 \mid \pi_0 = 0.95\right)$	$E\left(\tilde{W}_w \mid \pi_0 = 0.95\right)$	n^*	CP_n	CP_{n^*}
10	0.2702	0.5987	0.1608	NA ^b	0.4022	NA ^b
11	0.2576	0.5688	0.1595	NA ^b	0.4311	NA ^b
100	0.0854	0.4360	0.0829	93	0.8770	0.9439
101	0.0850	0.6070	0.0825	94	0.8819	0.9447
200	0.0604	0.4547	0.0596	194	0.9241	0.9175
201	0.0603	0.5766	0.0594	195	0.9256	0.9188
1000	0.0270	0.4797	0.0269	994	0.9436	0.9474
1001	0.0270	0.5346	0.0269	995	0.9431	0.9542
2000	0.0191	0.4857	0.0191	1994	0.9458	0.9447
2001	0.0191	0.5245	0.0191	1995	0.9458	0.9463

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_w \mid \pi_0 = 0.95\right) > w_0$; CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* ; ^b n^* and CP_{n^*} are not attainable for the corresponding sample size.

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Table 6

Behavior of the Wald confidence interval width $\tilde{W}_w$ as compared to the fixed value w_0 at $\pi_0 = 0.975$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_w < w_0 \pi_0 = 0.975\right)$	$E\left(\tilde{W}_w \pi_0 = 0.975\right)$	n^*	CP_n	CP_{n^*}
10	0.1935	0.7763	0.0864	NA ^b	0.2256	NA ^b
11	0.1845	0.7569	0.0862	NA ^b	0.2428	NA ^b
100	0.0612	0.5422	0.0565	80	0.9163	0.8635
101	0.0609	0.5357	0.0563	81	0.9180	0.8664
200	0.0433	0.4383	0.0420	187	0.8724	0.9410
201	0.0432	0.6115	0.0419	188	0.8740	0.9477
1000	0.0194	0.4724	0.0193	989	0.9497	0.9481
1001	0.0193	0.5509	0.0192	990	0.9287	0.9488
2000	0.0137	0.4805	0.0136	1989	0.9397	0.9500
2001	0.0137	0.5361	0.0136	1990	0.9393	0.9508

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_w | \pi_0 = 0.975\right) > w_0$, CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* ; ^b n^* and CP_{n^*} are not attainable for the corresponding sample size.

Table 7

Behavior of the Wilson (Score) confidence interval width $\tilde{W}_s$ as compared to the fixed value w_0 at $\pi_0 = \frac{1}{2}$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_s < w_0 \mid \pi_0 = \frac{1}{2}\right)$	$E\left(\tilde{W}_s \mid \pi_0 = \frac{1}{2}\right)$	n^*	CP_n	CP_{n^*}
10	0.5268	0.7539	0.5066	8	0.9783	0.9307
11	0.5088	1.0000	0.4906	9	0.9342	0.9612
100	0.1923	0.9204	0.1914	98	0.9439	0.9466
101	0.1914	1.0000	0.1905	99	0.9545	0.9567
200	0.1373	0.9437	0.1369	198	0.9454	0.9456
201	0.1369	1.0000	0.1366	199	0.9518	0.9538
1000	0.0619	0.9748	0.0618	998	0.9458	0.9457
1001	0.0618	1.0000	0.0618	999	0.9510	0.9502
2000	0.0438	0.9822	0.0438	1998	0.9478	0.9487
2001	0.0438	1.0000	0.0438	1999	0.9510	0.9519

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_s \mid \pi_0 = \frac{1}{2}\right) > w_0$; CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* .

Table 8

Behavior of the Wilson (Score) confidence interval width $\tilde{W}_s$ as compared to the fixed value w_0 at $\pi_0 = 0.95$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_s < w_0 \mid \pi_0 = 0.95\right)$	$E\left(\tilde{W}_s \mid \pi_0 = 0.95\right)$	n^*	CP_n	CP_{n^*}
10	0.3393	0.5987	0.3274	9	0.9136	0.9289
11	0.3216	0.5688	0.3102	10	0.8968	0.9138
100	0.0902	0.4360	0.0884	96	0.9663	0.9422
101	0.0897	0.6070	0.0879	97	0.9641	0.9706
200	0.0622	0.4547	0.0615	195	0.9659	0.9524
201	0.0620	0.5766	0.0613	196	0.9662	0.9529
1000	0.0272	0.4797	0.0271	994	0.9522	0.9521
1001	0.0272	0.5346	0.0271	995	0.9510	0.9518
2000	0.0192	0.4857	0.0191	1994	0.9537	0.9490
2001	0.0192	0.5245	0.0191	1995	0.9537	0.9488

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_s \mid \pi_0 = 0.95\right) > w_0$; CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* .

Table 9

Behavior of the Wilson (Score) confidence interval width $\tilde{W}_s$ as compared to the fixed value w_0 at $\pi_0 = 0.975$ and $\alpha = 0.05$.

Sample size	w_0	$P\left(\tilde{W}_s < w_0 \mid \pi_0 = 0.975\right)$	$E\left(\tilde{W}_s \mid \pi_0 = 0.975\right)$	n^*	CP_n	CP_{n^*}
10	0.3108	0.7763	0.3036	9	0.9750	0.9804
11	0.2927	0.7569	0.2857	10	0.9693	0.975
100	0.0696	0.5422	0.0676	95	0.9608	0.9682
101	0.0692	0.5357	0.0672	96	0.9581	0.9664
200	0.0465	0.4383	0.0456	193	0.9626	0.9682
201	0.0463	0.6115	0.0454	194	0.9626	0.9677
1000	0.0197	0.4724	0.0196	990	0.9466	0.9485
1001	0.0196	0.5509	0.0196	991	0.9474	0.9477
2000	0.0138	0.4805	0.0138	1990	0.9480	0.9488
2001	0.0138	0.5361	0.0138	1991	0.9469	0.9488

Notes: n^* , suggested sample size based on $E\left(\tilde{W}_s \mid \pi_0 = 0.975\right) > w_0$, CP_n , coverage probability for sample size n ; CP_{n^*} , coverage probability for sample size n^* .