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Joint confidence region estimation of L-moment ratios with an extension to right censored data

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Abstract

L-moments, defined as specific linear combinations of expectations of order statistics, have been advocated by Hosking [7] and others in the literature as meaningful replacements to that of classic moments in a wide variety of applications. One particular use of L-moments is to classify distributions based on the so-called L-skewness and L-kurtosis measures and given by an L-moment ratio diagram. This method parallels the classic moment-based plot of skewness and kurtosis corresponding to the Pearson system of distributions. In general, these methods have been more descriptive in nature and failed to consider the corresponding variation and covariance of the point estimators. In this note, we propose two procedures to estimate the $100(1 - \alpha)\%$ joint confidence region of L-skewness and L-kurtosis, given both complete and censored data. The procedures are derived based on asymptotic normality of L-moment estimators or through a novel empirical characteristic function (c.f.) approach. Simulation results are provided for comparing the performance of these procedures in terms of their respective coverage probabilities. The new and novel c.f.-based confidence region provided superior coverage probability as compared to the standard bootstrap procedure across all parameter settings. The proposed methods are illustrated via an application to a complete Buffalo snow fall data set and to a censored breast cancer data set, respectively.

Keywords

L-moment; inversion theorem; characteristic function; bootstrap; asymptotic normality

1. Introduction

For a random variable X with continuous cumulative distribution function F , L-moments are defined as expectations of certain linear combinations of order statistics given as

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$$\lambda_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E X_{r-k:r}, \quad (1)$$

$$\tau_2 = \frac{\lambda_2}{\lambda_1} \text{ and } \tau_r = \frac{\lambda_r}{\lambda_2}, \quad r=3, \dots, \quad (2)$$

where $X_{1:r}, X_{2:r}, \dots, X_{r:r}$ denote the order statistics from an independent and identically distributed random sample of size r from c.d.f. F . Hosking [7] provided for a complete theoretical examination. Specifically named L-moment ratios of interest given by Equation (2) include coefficient of L-variation (L-CV, $\tau_2 = \lambda_2/\lambda_1$), L-skewness, τ_3 and L-kurtosis, τ_4 , respectively. These L-moment ratios and their estimators are analogs to the classic product moment-based measures. Collectively, L-moments summarize the properties of a probability distribution and observed data, and provide alternatives for parameter estimation and hypothesis testing.

Similar to the traditional product moment ratio diagram which compares the sample skewness and kurtosis with their theoretical counterparts for a range of assumed distributions, Hosking [7] introduced the L-moment ratio diagram (LMRD), which plots L-skewness against L-kurtosis for a range of distributions. Several examples of LMRD have been developed for choosing an appropriate distribution, given observed data; e.g. see Vogel *et al.* [20], Chow and Watt [2], Gingras and Adamowski [5] and Vogel and Wilson [21], to name a few. Thorough comparisons between product moments and L-moments have been preformed by Royston [16] and Vogel and Fennessey [19]. Simulation results showed that LMRDs are preferred over product moment diagrams in many cases because of the fact that the L-moment ratios τ_2 , τ_3 , and τ_4 estimators are nearly unbiased for all underlying distributions. To date, the issue as to how to measure the precision of the sample L-moment ratios has not been well addressed from a perspective of formal statistical inference such as providing a reliable joint confidence region. Hosking [7] demonstrated the asymptotic properties of sample L-moment estimators, including consistency and asymptotic normality. The large sample theory of L-moments makes the theory straightforward to apply to generate approximate large sample confidence intervals for any single L-moment and confidence regions for any combination of L-moments, where the variance-covariance matrix can be readily estimated by conventional bootstrap methodology. Liou *et al.* [13] and Wu *et al.* [26] provided a method to create a joint acceptance region for sample estimates of τ_3 and τ_4 given by stochastic simulation. Unfortunately, the investigators only limited themselves to the normal, Gumbel and Pearson Type III distributions.

The aim of this note is to establish the $100(1 - \alpha)\%$ joint confidence region of any combination of L-moments given a single data set, which is either derived based on asymptotic normality of L-moment estimators or through a novel empirical characteristic function (c.f.) approach. We are particularly focused on τ_3 and τ_4 because of their excellent selective capability in the context of the LMRD among a set of widely used candidate distributions. Furthermore, the methods in this note can be readily extended to right censored data. The remainder of this note is organized as follows. In Section 2, we briefly

describe several sample L-moment estimators. Our proposed joint confidence region estimators are introduced in Section 3. The first estimator is described using exact bootstrap methodology based upon the asymptotic normality of L-moment estimators. The second nonparametric estimator is developed through a characteristic approach in conjunction with the classic inversion theorem. In Section 4, we examine the coverage probability of the joint confidence region estimators, using a simulation study, and finally we illustrate our procedure with a complete Buffalo snow fall data set and a censored breast cancer data set.

2. L-moment estimators

Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ denote the order statistics from an independent and identically distributed sample $X_1, X_2, \dots, X_n$ of size n from a continuous distribution F spanning the entire real line. Hosking [7] introduced the U-statistic based L-moment estimator ($\hat{\lambda}_r, r = 1, 2, \dots$) and corresponding sample L-moment ratio estimator ($\hat{\tau}_r$), which can be re-expressed as the linear combinations

$$\hat{\lambda}_r = \sum_{j=1}^n X_{j:n} u_{j(r)} \quad (3)$$

$$\hat{\tau}_r = \frac{\hat{\lambda}_r}{\hat{\lambda}_2}, \quad (4)$$

where

$$u_{j(r)} = \frac{\sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{j-1}{r-1-k} \binom{n-j}{k}}{r \binom{n}{r}}. \quad (5)$$

The sample L-moments $\hat{\lambda}_r$ are unbiased estimators of λ_r , but the sample L-moment ratios $\hat{\tau}_r$ are not unbiased. However, for most distributions the biases are negligible for sample sizes of 20 or more. Furthermore, the sample L-moment estimators have an asymptotic multivariate normal distribution; e.g. for more details about asymptotic properties of sample L-moments see Hosking [7] and Hosking and Wallis [9].

Hosking [7] also described another class of estimators, which are termed plotting-position estimators. These are not unbiased but asymptotically equivalent to the sample L-moments in Equation (3). Generally the plotting-position estimators are not preferred to the unbiased sample L-moments other than for specific situations; See more discussions about the comparison between two estimators in Hosking and Wallis [8]. Thus in this note, we focus on the sample L-moment estimator at (3), given a complete data set.

Wang *et al.* [23,24] extended the estimation of L-moments to right-censored data. Let $0 < T_{(1)} < T_{(2)} < \dots < T_{(n)}$ be the order statistics corresponding to the independent and identically distributed sample of n failure or censoring times $T_1, T_2, \dots, T_n$, and $\delta_{(1)}, \delta_{(2)}, \dots, \delta_{(n)}$ are censoring indicators corresponding to the ordered T_i 's, respectively. A value of $\delta_{(i)} = 1$

indicates that $T_{(i)}$ is uncensored, whereas a value of $\delta_{(i)} = 0$ indicates that $T_{(i)}$ is censored. The estimators of L-moments λ_r and L-moment ratios τ_r for right-censored data are given by

$$\hat{\lambda}_r = \sum_{j=1}^n T_{(j)} u_{j(r)}, \quad (6)$$

$$\hat{\tau}_r = \frac{\hat{\lambda}_r}{\hat{\lambda}_2}, \quad (7)$$

where

$$u_{j(r)} = \sum_{k=0}^{r-1} r^{-1} (-1)^k \binom{r-1}{k} [B_{p,q}(1 - \hat{S}_n(T_{(j)})) - B_{p,q}(1 - \hat{S}_n(T_{(j-1)}))] \quad (8)$$

and $T_{(0)} = 0$, $p = r - k$, $q = k + 1$ and $B_{p,q}(x)$ is the cumulative distribution function for a beta function with shape parameters p and q as parametrized in Casella and Berger [1]. The derivation of Equations (6) and (7) follows after plugging the classic Kaplan–Meier survival function estimator $\hat{S}$ into the definitions (1) and (2), respectively; see [11] for the original derivation of the survival function estimator. Wang *et al.* [23,24] also proved the consistency and asymptotic normality of $\hat{\lambda}_r$ and τ_r and demonstrated the advantages of L-moments in distribution characterization and parameter estimation given right-censored data.

3. Joint confidence region

The goal of this note is to construct a $100(1 - \alpha)\%$ confidence region of L-skewness and L-kurtosis on an LMRD. This regions serves a particularly important role in aiding an appropriate distribution for a given data set. The methodology, however, can be readily generalized to any set of L-moments or L-moment ratios as necessary for a particular application. For example, diagrams of τ_4 and τ_6 are particularly effective in analysis of symmetrical distributions ($\tau_3 = 0$). In this section, we propose two different methods, one is derived using the exact bootstrap methodology developed by Hutson and Ernst [10] and the other is derived via a characteristic function approach developed by Wang *et al.* [25] in conjunction with the conventional inversion theorem.

3.1 Exact bootstrap-based joint confidence region

Probably the most straightforward way to construct the confidence region is to utilize the asymptotic normality properties of L-moment estimators. Instead of the standard bootstrap, we utilize the exact bootstrap method to avoid the Monte Carlo error arising from random resampling with respect to the parameter estimation corresponding to the asymptotic bivariate distribution. The mean vector and covariance matrix of sample L-skewness and L-kurtosis can be derived via the standard delta method. Towards this end, let us denote the mean vector for the order statistics $\mathbf{X} = (X_{1:n}, X_{2:n}, \dots, X_{n:n})'$ as

$$\boldsymbol{\mu} = (\mu_{1:n}, \mu_{2:n}, \dots, \mu_{n:n})', \quad (9)$$

where $\mu_{r:n} = EX_{r:n}$ and the variance-covariance matrix of the set of order statistics as

$$\Sigma = \begin{pmatrix} \sigma_{1:n} & \sigma_{12:n} & \cdots & \sigma_{1n:n} \\ \sigma_{21:n} & \sigma_{2:n} & \cdots & \sigma_{2n:n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1:n} & \sigma_{n2:n} & \cdots & \sigma_{n:n} \end{pmatrix}, \quad (10)$$

where $\sigma_{r:n}^2 = \text{var}(X_{r:n})$ and $\sigma_{rs:n} = \sigma_{sr:n} = \text{cov}(X_{r:n}, X_{s:n})$. The main results from Hutson and Ernst [10] corresponding to the estimators for Equations (9) and (10) are as follows:

Lemma 3.1—The exact bootstrap estimator of $\mu_{r:n}$, $\sigma_{r:n}^2$ and $\sigma_{rs:n}$ $1 \leq r < s \leq n$, are

$$\hat{\mu}_r = \hat{\mu}_{r:n} = \sum_{j=1}^n w_{j(r)} X_{j:n}, \quad (11)$$

$$\hat{\Sigma}_{rr} = \hat{\sigma}_{r:n}^2 = \sum_{j=1}^n w_{j(r)} (X_{j:n} - \hat{\mu}_{r:n})^2, \quad (12)$$

$$\hat{\sigma}_{rs:n} = \hat{\sigma}_{sr:n} = \sum_{j=2}^n \sum_{i=1}^{j-1} w_{ij(rs)} (X_{i:n} - \hat{\mu}_{r:n})(X_{j:n} - \hat{\mu}_{s:n}) + \sum_{j=1}^n v_{j(rs)} (X_{j:n} - \hat{\mu}_{r:n})(X_{j:n} - \hat{\mu}_{s:n}), \quad (13)$$

where

$$w_{j(r)} = r \binom{n}{r} \left\{ B_{r,n-r+1} \left(\frac{j}{n} \right) - B_{r,n-r+1} \left(\frac{j-1}{n} \right) \right\}, \quad (14)$$

$$\begin{aligned} w_{ij(rs)} = & {}_nC_{rs} \sum_{k=0}^{s-r-1} \binom{s-r-1}{k} \frac{(-1)^{s-r-1-k}}{s-k-1} \left\{ \left(\frac{i}{n} \right)^{s-k-1} - \left(\frac{i-1}{n} \right)^{s-k-1} \right\} \\ & \times \left\{ B_{k+1,n-s+1} \left(\frac{j}{n} \right) - B_{k+1,n-s+1} \left(\frac{j-1}{n} \right) \right\}, \end{aligned} \quad (15)$$

$$\begin{aligned} v_{j(rs)} = & {}_nC_{rs} \sum_{k=0}^{s-r-1} \binom{s-r-1}{k} \frac{(-1)^{s-r-1-k}}{s-k-1} \left[B_{s,n-s+1} \left(\frac{j}{n} \right) - B_{s,n-s+1} \left(\frac{j-1}{n} \right) \right. \\ & \left. - \left(\frac{j-1}{n} \right)^{s-k-1} \left\{ B_{k+1,n-s+1} \left(\frac{j}{n} \right) - B_{k+1,n-s+1} \left(\frac{j-1}{n} \right) \right\} \right], \end{aligned} \quad (16)$$

with

$${}_nC_{rs} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

and $B_{a,b}(x) = \int_0^x t^{a-1} (1-t)^{b-1} dt$ is the incomplete beta function.

The mean and variance–covariance matrix of any set of m ($m < \infty$) arbitrarily chosen L-moment estimates $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_m)$ can be readily estimated as follows:

Definition 3.2—Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ denote the order statistics from an i.i.d. sample of size n , either complete or censored, from a continuous distribution F with L-moments λ_r and finite variance. Let $\hat{\lambda}_r, r = 1, 2, \dots, m$ be L-moments estimators calculated from Equations (3) or (6). Let $\tau_r = \lambda_r/\lambda_2$ and $\hat{\tau}_r = \hat{\lambda}_r/\hat{\lambda}_2, r = 3, 4, \dots, m$. Then

- i. The exact bootstrap estimates of $\kappa = E(\hat{\lambda})$ and $\Lambda = \text{Var}(\hat{\lambda})$ for the vector of L-moment estimates $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_m)' = \mathbf{u}'\mathbf{X}$ are

$$\hat{\kappa} = \hat{E}(\hat{\lambda}) = \mathbf{u}'\hat{\mu}, \quad (17)$$

$$\hat{\Lambda} = \widehat{\text{Var}}(\hat{\lambda}) = \mathbf{u}'\hat{\Sigma}\mathbf{u}, \quad (18)$$

where $\mathbf{u} = (\mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots, \mathbf{u}_{(m)})$ with the weights $\mathbf{u}_{(r)}$ being calculated by Equations (5) or (8) for complete and right-censored data, respectively.

- ii. The exact bootstrap estimates of the mean $\gamma = E(\hat{\tau})$ and variance–covariance matrix $\mathbf{T} = \text{Var}(\hat{\tau})$ for the vector $\hat{\tau} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\tau}_3, \hat{\tau}_4, \dots, \hat{\tau}_m)'$ are

$$\hat{\gamma} = \hat{E}(\hat{\tau}) = \begin{cases} \hat{\kappa}_r & \text{if } r \leq 2; \\ \frac{\hat{\kappa}_r}{\hat{\kappa}_2} & \text{if } r \geq 3; \end{cases} \quad (19)$$

$$\hat{\mathbf{T}} = \widehat{\text{Var}}(\hat{\tau}) = \begin{cases} \hat{\Lambda}_{rs} & \text{if } r \leq 2, s \leq 2; \\ \frac{\hat{\Lambda}_{rs} - \hat{\tau}_r \hat{\Lambda}_{2s}}{\hat{\lambda}_2} & \text{if } r \geq 3, s \leq 2; \\ \frac{(\hat{\Lambda}_{rs} - \hat{\tau}_r \hat{\Lambda}_{2s} - \hat{\tau}_s \hat{\Lambda}_{2r} + \hat{\tau}_r \hat{\tau}_s \hat{\Lambda}_{22})}{\hat{\lambda}_2^2} & \text{if } r \geq 3, s \geq 3. \end{cases} \quad (20)$$

The derivation of Part (i) of Definition 3.2 can be readily derived through standard linear algebra on the basis that $\hat{\lambda} = \mathbf{u}\mathbf{X}$, which is a set of linear combinations of order statistics. Part (ii) of Definition 3.2 is obtained through standard multivariate delta method noting that $\hat{\tau}$ is a function of $\hat{\lambda}$, which follows an asymptotic multivariate normal distribution. The readers are referred to Theorem 3 of Hosking [7] for more details about asymptotic properties of L-moment and L-moment ratio estimators. For right-censored data, the L-moment estimator described in Wang *et al.* [23,24] follows a similar form of asymptotic normal distribution. The exact bootstrap mean and variance of the L-moment estimators are also provided in Wang *et al.* [23,24], which can be readily substituted in Equations (17) and (18). The consistency of the estimators in Definition 3.2 should be guaranteed since it is in essence a standard bootstrap procedure.

Note that Definition 3.2 can be used for the inference of any single L-moment ratio or any subset of the first m ($m < \infty$) L-moment ratio. In this note, we focus on the estimation of mean and variance matrix of τ_3 and τ_4 , which can be expressed as a special case of Definition 3.2 with $r = 3$ and $s = 4$. Thus, we are now enabled to estimate the $100(1 - \alpha)\%$

confidence region of $\boldsymbol{\eta} = (\tau_3, \tau_4)$ based on the asymptotic normality of L-moments. It is worth noting that for this case, only the mean and covariance matrix of $\hat{\tau}_3$ and $\hat{\tau}_4$ need to be calculated, even though the mean vectors and covariance matrices of the first m L-moments and L-moment ratios are fully provided in Definition 3.2.

Definition 3.3—The $100(1 - \alpha)\%$ asymptotic confidence region of $\boldsymbol{\eta} = (\tau_3, \tau_4)$ based on the sample L-skewness and L-kurtosis $\hat{\boldsymbol{\eta}} = (\hat{\tau}_3, \hat{\tau}_4)$ is expressed as

$$(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}})' \hat{\mathbf{P}}^{-1} (\boldsymbol{\eta} - \hat{\boldsymbol{\eta}}) \leq \chi_{2,\alpha}^2, \quad (21)$$

where $\mathbf{P}$ is the variance estimator of $\hat{\boldsymbol{\eta}}$ calculated by Equation (20) and $\chi_{2,\alpha}^2$ is the upper quantile of the χ_2^2 distribution at a significance level α .

Definition 3.3 represents a standard equiprobable ellipse from a bivariate normal distribution.

3.2 C.f.-based joint confidence region

A nonparametric confidence region using a c.f. approach is now developed. A general framework of deriving confidence intervals for L-estimates via c.f.s has been provided by Wang *et al.* [25]. Towards this end, consider L-moment estimator

$\hat{\lambda}_r = \mathbf{u}_{(r)}' \mathbf{X} = \sum_{j=1}^n X_{j:n} u_{j(r)}$ as a linear form $\mathbf{u}' \mathbf{X}$ for a random $n \times 1$ vector $\mathbf{X} = (X_{1:n}, X_{2:n}, \dots, X_{n:n})'$, with a mean vector $\boldsymbol{\mu} = (\mu_{1:n}, \mu_{2:n}, \dots, \mu_{n:n})'$ and variance–covariance matrix $\boldsymbol{\Sigma}$. Direct utilization of the results from Wang *et al.* [25] yields the c.g.f. of $\hat{\lambda}_r$, as

$$\begin{aligned} \phi_{\hat{\lambda}_r}(t) = & \exp \left(\frac{1}{2} (2it \mathbf{u}_{(r)}' \boldsymbol{\mu} - t^2 \mathbf{u}_{(r)}' \boldsymbol{\Sigma} \mathbf{u}_{(r)}) \right) \\ & \times \left(1 + \frac{1}{6} \sum_{j=1}^n (it)^3 u_{j(r)}^3 \kappa_{3j} + \frac{1}{24} \sum_{j=1}^n (it)^4 u_{j(r)}^4 \kappa_{4j} + \frac{1}{72} \sum_{i=1}^n \sum_{j=1}^n (it)^6 u_{i(r)}^3 u_{j(r)}^3 \kappa_{3i} \kappa_{3j} \right), \end{aligned} \quad (22)$$

where κ_{3j} and κ_{4j} are the third and fourth cumulant of $X_{j:n}$, respectively. We refer the reader to Knight [12] and Wang *et al.* [25] for detailed derivation. Equation (22) allows us to derive the c.g.f. of the L-moment estimator for any continuous distribution since the moments of order statistics have been well developed; see details in David and Nagaraja [3].

We can further develop the sampling distribution of L-moment estimators via classic inversion theorem. Towards this end, consider a c.d.f. F continuous everywhere, and $dF = f(x) dx$, the inversion theorem states:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) \phi_x(t) dt, \quad (23)$$

$$F(x) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(itx) \phi_x(-t) - \exp(-itx) \phi_x(t)}{it} dt, \quad (24)$$

where $i = \sqrt{-1}$. More details about the utilization of the inversion theorem can be found in Gil-Pelaez [4], Shephard [18], and Waller *et al.* [22].

Now consider sample data $\mathbf{x} = (x_{1:n}, x_{2:n}, \dots, x_{n:n})$ from a continuous distribution F . Direct substitution of exact bootstrap estimators for the mean vector and covariance matrix of order statistics provided at Lemma 3.1 into Equation (22) and inversion theorem yields the following results:

Definition 3.4—An empirical “plug-in” estimator for the c.g.f. of the r th L -moment

estimator $\hat{\lambda}_r = \mathbf{u}_{(r)}' \mathbf{X} = \sum_{j=1}^n X_{j:n} u_{j(r)}$ is given as

$$\begin{aligned} \hat{\phi}_{\hat{\lambda}_r}(t) = & \exp \left(\frac{1}{2} (2it \mathbf{u}_{(r)}' \hat{\boldsymbol{\mu}} - t^2 \mathbf{u}_{(r)}' \hat{\boldsymbol{\Sigma}} \mathbf{u}_{(r)}) \right) \\ & \times \left(1 + \frac{1}{6} \sum_{j=1}^n (it)^3 u_{j(r)}^3 \hat{\kappa}_{3j} + \frac{1}{24} \sum_{j=1}^n (it)^4 u_{j(r)}^4 \hat{\kappa}_{4j} + \frac{1}{72} \sum_{i=1}^n \sum_{j=1}^n (it)^6 u_{i(r)}^3 u_{j(r)}^3 \hat{\kappa}_{3i} \hat{\kappa}_{3j} \right), \end{aligned} \quad (25)$$

where

$$\hat{\kappa}_{3j} = 2(\hat{\mu}_{j:n})^3 - 3\hat{\mu}_{j:n}\hat{\mu}_{j:n}^2 + \hat{\mu}_{j:n}^3, \quad (26)$$

$$\hat{\kappa}_{4j} = -6(\hat{\mu}_{j:n})^4 + 12(\hat{\mu}_{j:n})^2 \hat{\mu}_{j:n}^2 - 3(\hat{\mu}_{j:n}^2)^2 - 4\hat{\mu}_{j:n}\hat{\mu}_{j:n}^3 + \hat{\mu}_{j:n}^4. \quad (27)$$

Note that $\hat{\mu}_{j:n}^k = E(\widehat{X_{j:n}^k}) = \sum_{i=1}^n w_{i(j)} x_{i:n}^k$, $k = 2, 3, 4$ is the k th moment estimator of $X_{j:n}$ with $w_{i(j)}$ defined at (14)

Definition 3.5—The inversion theorem-based kernel estimators for the p.d.f. and c.d.f. for

the r th L -moment estimator $\hat{\lambda}_r = \mathbf{u}_{(r)}' \mathbf{X} = \sum_{j=1}^n X_{j:n} u_{j(r)}$ given the observed data $\mathbf{x} = (x_{1:n}, x_{2:n}, \dots, x_{n:n})$ are defined as

$$\hat{f}_{\hat{\lambda}_r}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) \hat{\phi}_{\hat{\lambda}_r}(t) dt, \quad (28)$$

$$\hat{F}_{\hat{\lambda}_r}(x) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{\exp(itx) \hat{\phi}_{\hat{\lambda}_r}(-t) - \exp(-itx) \hat{\phi}_{\hat{\lambda}_r}(t)}{it} dt, \quad (29)$$

According to Slutsky's theorem, the asymptotic properties of $\hat{\phi}_{\hat{\lambda}_r}(t)$ at (25), $\hat{f}_{\hat{\lambda}_r}(x)$ at (28) and $\hat{F}_{\hat{\lambda}_r}(x)$ at (29), are essentially determined by the moment estimators of order statistics at (11)–(13). Under the well-known regularity conditions for the asymptotic consistency of sample quantile function as stated in Serfling [17], the asymptotic consistency of the estimators at (25), (28) and (29) can be readily proved; see Theorem 5.1 and 5.2 in Wang *et al.* [25] for more details.

Given an observed data set, the integration can be readily programmed by commonly used softwares. In this note, we utilized the *NIntegrate* function in *Mathematica*. In *R*, we recommend the *myintegrate* function from library *elliptic*. According to the well-known inversion theorem, only the real part of the kernel estimator needs to be considered as usually the imaginary part is extremely small that can be safely ignored. In reality, the real part can be extracted by the *Real* function from *Mathematica* or by the *as.real* function from *R*.

Definition 3.6—The Bonferroni-corrected asymptotic $100(1 - \alpha)\%$ simultaneous confidence intervals for τ_3 and τ_4 based on our nonparametric approach are defined as

$$\frac{\hat{F}_{l_3}^{-1}(\alpha/4)}{\hat{\lambda}_2} \leq \tau_3 \leq \frac{\hat{F}_{l_3}^{-1}(1 - \alpha/4)}{\hat{\lambda}_2} \quad (30)$$

$$\frac{\hat{F}_{l_4}^{-1}(\alpha/4)}{\hat{\lambda}_2} \leq \tau_4 \leq \frac{\hat{F}_{l_4}^{-1}(1 - \alpha/4)}{\hat{\lambda}_2}, \quad (31)$$

where $\hat{F}_{\hat{\lambda}_r}^{-1}(\cdot)$ is the inverse of $F_{\hat{\lambda}_r}(\cdot)$ defined at (29) above.

Even though the calculation of the percentile interval requires numerical inversion of the c.d.f., it is straightforward using common statistical software packages. Examples are provided in a later section.

4. Simulation and examples

We performed a simulation study to evaluate the proposed methods in terms of empirical coverage probabilities. Empirical coverage probability is a good approximation of true coverage probability, which is calculated as the proportion of simulations where the confidence region simultaneously covers the set of true parameters τ_3 and τ_4 . We choose a wide range of underlying distributions including exponential, Rayleigh, uniform, normal, Gumbel, gamma distributions. Among this distributions, exponential, Rayleigh, uniform, normal, Gumbel distributions have unique L-skewness and L-kurtosis. Gamma and Weibull distributions share the same scale parameter but different shape parameters, so that the coefficients of L-skewness differ. For each underlying distribution, we generated 1000 simulations for sample sizes $n = 35, 50$, and 100 and calculate the 95% joint confidence region utilizing the exact bootstrap-based method described in Section 3.1 and the characteristic function-based method described in Section 3.2. The simulation results are displayed in Table 1. Note that according to the large sample theory, an empirical probability will be within 0.936 and 0.964 at the 95% nominal confidence region, given 1000 simulations. The exact bootstrap-based joint confidence regions tend to be anti-conservative and not very satisfactory for many distributions, such as normal, gamma, particularly for small to median sample sizes ($n = 50$). It is generally true that as sample size goes up, coverage probabilities of both exact bootstrap-based and c.f.-based joint confidence regions tend to reach the nominal level of 95%. It is also worth noting that the performance of both methods gets less satisfied with higher coefficient of L-skewness for both gamma

and Weibull distributions. This is presumably because of the well-known observation that asymptotic normality of τ_3 and τ_4 estimators gets worse as τ_3 increases. According to Table 1, sample size of $n = 35$ is fairly good enough for the c.f.-based method, given a underlying distribution with L-skewness less than 0.2. More than 50 observations may be required if the coefficient of L-skewness is greater than 0.2. For exact bootstrap-based method, at least 100 observations is needed except for a underlying normal distribution, since this method heavily depends on asymptotic normal approximation. Overall, no matter the sample size and the type of underlying distribution, the c.f.-based joint confidence region generally provides superior coverage probability as compared to the exact bootstrap-based method.

We utilized the classic Buffalo snow fall data to illustrate our procedure of constructing the joint confidence region for τ_3 and τ_4 for complete data. This data set consists of 63

observations of the annual snowfall amounts in Buffalo, New York as observed from $\frac{1910}{11}$ to $\frac{1970}{73}$, which has been extensively analysed by Parzen [15] and Miecznikowski *et al.* [14] and the references therein. The sample τ_3 and τ_4 (plotted as the “+” sign) and the joint confidence regions derived either using asymptotic properties (the ellipse) or using the c.f. (the rectangle) are shown in Figure 1. Consistent with the conclusion by Parzen [15] who used a density-quantile estimation procedure, the position of sample L-skewness and L-kurtosis suggests the snow fall data set is from a normal distribution. Both exact bootstrap-based and c.f.-based joint confidence regions exclude the possibility of uniform distribution and the c.f.-based confidence region also suggests that the set of data is not likely from a logistic distribution.

To illustrate our method, given right-censored data, we used a small survival data set measured in months for $n = 18$ breast cancer patients; See Gottlieb *et al.* [6] for additional information pertaining to the data. As shown in Figure 2, the τ_3 and τ_4 estimates suggest a underlying uniform distribution. This agrees with the straight kernel quantile function curve presented in Wang *et al.* [23,24]. The associated 95% joint confidence regions suggest that the set of data are not likely from a normal or logistic function, but may be from a generalized Pareto distribution.

5. Conclusion and discussion

In this note, we develop two procedures to estimate the $100(1 - \alpha)\%$ joint confidence region of τ_3 and τ_4 , given a complete or censored data set. We highly recommend the c.f. approach, since in our simulation study, the c.f.-derived joint confidence regions display superior performance than those derived using exact bootstrap in terms of coverage probabilities. Given two small examples utilized in this study, the conclusions using the LMRD are consistent with those in prior publications with regard to assessment of the underlying distributional form. Our methodology can be readily generalized to any bivariate combination of L-moments or L-moment ratios.

It is not an uncommon issue that the joint confidence region of L-skewness and L-kurtosis cover several candidate distributions, thus selecting the most suitable distribution under such a situation may not be straightforward. As a very general model/distribution selection

procedure, the utilization of LMRD may be less powerful than the well designed goodness-of-fit test for a single null distribution. However, an efficient set of the estimator of L-skewness and L-kurtosis together with a satisfactory joint confidence region still shed much light on what to choose for the alternative distribution, given a specific goodness-of-fit test.

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Figure 1.
Buffalo snow fall data utilized in Parzen [15]. Ellipse: exact bootstrap-based confidence region; Rectangle: characteristic function-based confidence region.

Figure 2.

Breast cancer survival data from Gottlieb *et al.* [6]. Ellipse: exact bootstrap-based confidence region; rectangle: characteristic function-based confidence region.

Table 1

Empirical joint coverage probabilities of τ_3 and τ_4 with $\alpha = 0.05$.

	τ_3	τ_4		$n = 35$	50	100
exponential	0.333	0.167	EB	0.892	0.895	0.912
			CF	0.920	0.907	0.932
Rayleigh	0.114	0.105	EB	0.910	0.920	0.938
			CF	0.952	0.951	0.948
uniform	0.000	0.000	EB	0.929	0.920	0.931
			CF	0.972	0.962	0.959
normal	0.000	0.123	EB	0.894	0.900	0.894
			CF	0.948	0.946	0.917
Gumbel	0.170	0.150	EB	0.878	0.900	0.920
			CF	0.906	0.924	0.912
gamma(0.5, 2)	0.464	0.226	EB	0.871	0.869	0.901
			CF	0.883	0.892	0.935
gamma(1, 2)	0.333	0.167	EB	0.884	0.909	0.922
			CF	0.917	0.921	0.934
gamma(2, 2)	0.235	0.142	EB	0.887	0.890	0.907
			CF	0.906	0.913	0.927
gamma(5, 2)	0.147	0.129	EB	0.904	0.906	0.917
			CF	0.940	0.938	0.949
Weibull(0.4, 1)	0.726	0.516	EB	0.807	0.801	0.813
			CF	0.767	0.822	0.915
Weibull(0.8, 1)	0.423	0.218	EB	0.878	0.889	0.891
			CF	0.875	0.894	0.918
Weibull(2, 1)	0.114	0.105	EB	0.913	0.932	0.910
			CF	0.957	0.953	0.930

Notes: EB: exact bootstrap; CF: characteristic function; $\text{gamma}(\alpha, \beta): F_{\alpha, \beta}(x) = ((\beta^{-\alpha} / \Gamma(\alpha)) \int_0^x t^{\alpha-1} e^{-t/\beta} dt; \text{Weibull}(k, \lambda): F_{k, \lambda}(x) = 1 - e^{-(x/\lambda)^k}$.