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Some new analysis results for a class of interface problems

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Abstract

Interface problems modeled by differential equations have many applications in mathematical biology, fluid mechanics, material sciences, and many other areas. Typically, interface problems are characterized by discontinuities in the coefficients and/or the Dirac delta function singularities in the source term. Due to these irregularities, solutions to the differential equations are not smooth or discontinuous. In this paper, some new results on the jump conditions of the solution across the interface are derived using the distribution theory and the theory of weak solutions. Some theoretical results on the boundary singularity in which the singular delta function is at the boundary are obtained. Finally, the proof of the convergency of the Immersed Boundary method is presented. The IB method is shown to be first order convergent in L^∞ norm.

Keywords

Immersed Boundary (IB) method; Immersed Interface Method (IIM); jump conditions; discontinuous coefficient; Dirac delta function; weak solution; boundary singularity; convergence of IB method; equivalent boundary conditions

1 Introduction

Interface problems arise in many areas of science and mathematics. Whenever there are different materials such as oil and water, gas and water, or bubbles in a fluid, we have interface problems. Another kind of interface problems is the Peskin's Immersed Boundary (IB) method, see for example, [11] for a review and references therein. The IB method is not only a mathematical modeling tool in which a complicated boundary condition can be

treated as a source distribution but also a numerical method in which a discrete delta function is used. The IB method is robust, simple, and has been applied to many problems in mathematical biology, fluid mechanics, material science, and other areas. However, the IB method is only first order accurate in general. The Immersed Interface Method (IIM) [7, 6, 8] is a second order accurate method particularly designed for interface problems. In the IIM, jump conditions are enforced near the interface to get high order accuracy. Knowing the jump conditions is crucial in the Immersed Interface Method.

Mathematically, there are two characteristics in an interface problem. One is that a discontinuity exists in the coefficient of the differential equation along a co-dimensional space. The other one is singular source terms such as the Dirac delta function or its derivatives along a co-dimensional space. Below is a model interface problem in one dimension (1D),

$$(\beta(x)u')' = f(x) + c\delta(x-\alpha) + \bar{c}\delta'(x-\alpha), \quad 0 < x < 1, \quad 0 < \alpha < 1, \quad (1)$$

where

$$\beta(x) = \begin{cases} \beta^-(x) & \text{if } x < \alpha, \\ \beta^+(x) & \text{if } x > \alpha. \end{cases} \quad (2)$$

along with two boundary conditions at $x = 0$ and $x = 1$. In the Immersed Boundary (IB) method developed by Peskin [10, 11], the Dirac delta function is intensively used. Often a delta function singularity corresponds to a source distribution, while the derivative of the delta function singularity corresponds to a dipole distribution. For Stokes and Navier-Stokes equations, a delta function distribution along a surface indicates that the derivative of the velocity has jump discontinuities across the interface; while the derivative of the delta function indicates that the pressure has a jump discontinuity.

There are three goals in this paper. The first one is to derive the jump conditions for (1); the second one is to consider the limiting case as α approaches one of boundary points. The third one is to provide a convergence proof for the IB method. Since there are no other regularities except at the interface α , we know that the solution is piecewise smooth excluding the interface α , that is, $u(x) \in C^2(0, 1) \setminus \alpha$. The discontinuity is at the interface α . We can derive the jump condition of the solution and its derivative across the interface. We define

$$\beta^- = \beta(\alpha-) = \lim_{x \rightarrow \alpha, x < \alpha} \beta(x), \quad \beta^+ = \beta(\alpha+) = \lim_{x \rightarrow \alpha, x > \alpha} \beta(x), \quad \text{and} \quad [\beta] = \beta^+ - \beta^-. \quad (3)$$

Jump conditions are crucial in the immersed interface method [7, 6, 8]. If $c = 0$ or β is continuous at α , that is $\beta^- = \beta^+$, then the jump conditions have been derived, see for example, [8]. We will derive the more challenging case when $c \neq 0$ and $\beta^- \neq \beta^+$. To our best knowledge, we have not seen the jump conditions and the derivation in the literature for the difficult case. We also report theoretical results and the derivation for the boundary

singularity problem for which α is at one of the boundary points. We will see that the effect of the delta function singularity can be absorbed by suitable boundary conditions.

In this paper, we will use the notations $u'(x) = u_x(x) = \frac{du}{dx}(x)$ whenever it is convenient. The rest of the paper is organized as follows. In the next section, we define the weak solution and derive the jump conditions. In Section 3, we discuss the case when the delta function singularity appears at one of the boundaries. We show that it can be translated as a contribution from the boundary conditions. In Section 4, we show the convergence of the IB method. Our result confirms that the IB method is first order accurate in L^∞ in general. We conclude in the last section.

2 Deriving the jump conditions

To use the IIM, we must know or derive jump conditions. The jump conditions describe the discontinuity of the solution and its derivative. The information is necessary to obtain second order accuracy of the finite difference scheme as seen in [6]. We first state one of the main results of this paper in the following theorem.

Theorem 2.1—Assume $\beta(x) \in C^1[0, a) \cup C^1(a, 1]$, $f(x) \in C(0, 1)$. Then the following jump conditions hold:

$$[u] = \frac{2\bar{c}}{\beta^- + \beta^+}, \quad [\beta u_x] = c. \quad (4)$$

We prove the theorem by considering the following cases although the last case can cover all the situations. The purpose is to show that for simpler cases, the proofs are also simpler.

2.1 The singularity from $\delta(x - a)$ only

In this simplest case, we assume that $[\beta] = 0$, $c \neq 0$. Thus the ODE is simply $(\beta u_x)_x = f(x) + c\delta(x - a)$. From the differential equation theory, see for example, [4], we know that the solution is continuous, i.e., $[u] = 0$. Integrating the differential equation from a^- to a^+ , we get

$$\int_{a^-}^{a^+} (\beta u_x)_x dx = \int_{a^-}^{a^+} f(x) dx + \int_{a^-}^{a^+} c\delta(x - a) dx. \quad (5)$$

Notice that $f(x)$ is continuous in the neighborhood of a and consequently the first integration in the right hand side vanishes. Also, from the definition of the Dirac delta function, we have the following property of the delta function,

$$\int_{-\varepsilon}^{+\varepsilon} g(x)\delta(x - a) dx = g(a) \quad (6)$$

for any continuous function $g(x)$. Thus from (5), we get

$$\beta^+ u_x^+(\alpha) - \beta^- u_x^-(\alpha) = c, \quad \text{or} \quad [\beta u_x] = c.$$

Thus, the theorem is true for this case.

2.2 Singularities from $\delta(x - \alpha)$ and $\delta'(x - \alpha)$

In this case, we assume that $[\beta] = 0$, and both c and $\bar{c}$ can be non-zero. Thus the ODE is $(\beta u_x)_x = f(x) + c\delta(x - \alpha) + \bar{c}\delta'(x - \alpha)$. The presence of the derivative of the delta function corresponds to a double layer in the potential theory. Thus, the solution typically has a finite jump across the interface α . We can not simply integrate the differential equation from α^- to α^+ since $\int c\delta(x - \alpha)dx$ does not exist. Similar to the process in deriving the weak form in the finite element method, we multiply a testing function $\phi(x) \in C_0^\infty(0, 1)$ to both sides of the differential equation and then integrate to get

$$\int_{\alpha^-}^{\alpha^+} (\beta u_x)_x \phi(x) dx = \int_{\alpha^-}^{\alpha^+} f(x) \phi(x) dx + \int_{\alpha^-}^{\alpha^+} c\delta(x - \alpha) \phi(x) dx + \int_{\alpha^-}^{\alpha^+} \bar{c}\delta'(x - \alpha) \phi(x) dx \quad (7)$$

$$= c\phi(\alpha) - \bar{c}\phi'(\alpha).$$

The right hand side is obtained from the definition of the weak derivatives of the delta function. We manipulate the left hand side of (7) as follows,

$$\begin{aligned} LHS &= (\beta u_x) \phi \Big|_{\alpha^-}^{\alpha^+} - \int_{\alpha^-}^{\alpha^+} \beta u_x \phi'(x) dx \\ &= [\beta u_x] \phi(\alpha) - \beta(\alpha) \left((u\phi') \Big|_{\alpha^-}^{\alpha^+} - \int_{\alpha^-}^{\alpha^+} u \phi''(x) dx \right) \quad (8) \\ &= [\beta u_x] \phi(\alpha) - \beta(\alpha) [u] \phi'(\alpha). \end{aligned}$$

Since $\phi(x)$ is arbitrary and the equation (7) is the same as (8), we have to have

$$[\beta u_x] = c, \quad \beta(\alpha) [u] = \bar{c}, \quad \text{i.e.} \quad [u] = \frac{\bar{c}}{\beta(\alpha)} = \frac{2\bar{c}}{\beta^- + \beta^+}.$$

Thus, the theorem is also true for this case.

2.3 Singularities from $\delta(x - \alpha)$, $\delta'(x - \alpha)$, and a discontinuity in β

Now the coefficient $\beta(x)$ has a discontinuity at α . For simplicity of proof, we assume that β is piecewise constant

$$\beta(x) = \begin{cases} \beta^- & \text{if } x < \alpha, \\ \beta^+ & \text{if } x > \alpha. \end{cases} \quad (9)$$

Before we derive the jump conditions, we define a weak solution using the standard regularization technique. We will see that the solution defined from the jump conditions is the same as the weak solution. First we define a regularized Heaviside function $H_\epsilon(x)$ as follows

$$H_\varepsilon(x) = \begin{cases} 0 & \text{if } x < -\varepsilon, \\ \frac{1}{2} \left(1 + \frac{x}{\varepsilon} + \frac{1}{\pi} \sin \frac{\pi x}{\varepsilon} \right) & \text{if } |x| \leq \varepsilon, \\ 1 & \text{if } x > \varepsilon. \end{cases} \quad (10)$$

Note that $H_\varepsilon(0) = \frac{1}{2}$. We define the regularized coefficient as

$$\beta_\varepsilon(x) = \beta^- + [\beta] H_\varepsilon(x - \alpha). \quad (11)$$

Note that $\beta_\varepsilon(\alpha) = \frac{1}{2}(\beta^- + \beta^+)$. We define the regularized solution $u_\varepsilon(x)$ as the solution of the following boundary value problem,

$$(\beta_\varepsilon(x) u_\varepsilon')' = f(x) + c\delta(x - \alpha) + \bar{c}\delta'(x - \alpha), \quad 0 < x < 1, \quad 0 < \alpha < 1. \quad (12)$$

Since $\beta_\varepsilon(x)$ is continuous now, we have

$$[u_\varepsilon] = \frac{2\bar{c}}{\beta_\varepsilon(\alpha+) + \beta_\varepsilon(\alpha-)}, \quad [\beta_\varepsilon u_\varepsilon'] = c. \quad (13)$$

Note that, we have

$$\lim_{\varepsilon \rightarrow 0} [u_\varepsilon] = \lim_{\varepsilon \rightarrow 0} \frac{2\bar{c}}{\beta_\varepsilon(\alpha+) + \beta_\varepsilon(\alpha-)} = \frac{2\bar{c}}{\beta^+ + \beta^-}. \quad (14)$$

Definition 2.2—The weak solution to (1) is defined as

$$u_w(x) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x). \quad (15)$$

It is obvious that the weak solution satisfies the jump conditions (4). We now prove that from the original differential equations, we can derive the same jump conditions as that of the weak solution.

2.3.1 The derivation of the general jump conditions—As before, we multiply a testing function $\phi(x) \in C_0^\infty(0, 1)$ to both side of the differential equation and then integrate to get,

$$\begin{aligned} \int_{\alpha^-}^{\alpha^+} (\beta u_x)_x \phi(x) dx &= \int_{\alpha^-}^{\alpha^+} f(x) \phi(x) dx + \int_{\alpha^-}^{\alpha^+} c\delta(x - \alpha) \phi(x) dx + \int_{\alpha^-}^{\alpha^+} \bar{c}\delta'(x - \alpha) \phi(x) dx \\ &= c\phi(\alpha) - \bar{c}\phi'(\alpha). \end{aligned} \quad (16)$$

The right hand side is obtained from the definition of the weak derivatives. We manipulate the left hand side of (16) as follows

$$\begin{aligned} LHS &= (\beta u_x) \phi \Big|_{\alpha^-}^{\alpha^+} - \int_{\alpha^-}^{\alpha^+} \beta(x) u_x \phi'(x) dx \\ &= [\beta u_x] \phi(\alpha) - \int_{\alpha^-}^{\alpha^+} \beta(x) u_x \phi'(x) dx. \end{aligned}$$

We can not take $\beta(x)$ out of the integration anymore since it is discontinuous. We represent $\beta(x)$ and $u(x)$ as follows, see also [2],

$$\begin{aligned} \beta(x) &= \beta^- + [\beta] H(x-\alpha), \\ u(x) &= u^- + [u] H(x-\alpha) + [u_x] H(x-\alpha)(x-\alpha) + \frac{1}{2} [u_{xx}] H(x-\alpha)(x-\alpha)^2 + \dots \end{aligned}$$

Differentiating the above representation, we get

$$u_x(x) = [u] \delta(x-\alpha) + [u_x] \delta(x-\alpha)(x-\alpha) + [u_x] H(x-\alpha) + \dots$$

Plugging the expressions above into the integral of LHS , we have

$$\begin{aligned} \int_{\alpha^-}^{\alpha^+} \beta(x) u_x \phi'(x) dx &= \int_{\alpha^-}^{\alpha^+} (\beta^- + [\beta] H(x-\alpha)) ([u] \delta(x-\alpha) + [u_x] \delta(x-\alpha)(x-\alpha) + [u_x] H(x-\alpha) + \dots) \phi'(x) dx \\ &= \int_{\alpha^-}^{\alpha^+} \beta^- [u] \delta(x-\alpha) \phi'(x) dx + \int_{\alpha^-}^{\alpha^+} [\beta] [u] H(x-\alpha) \delta(x-\alpha) \phi'(x) dx + \int_{\alpha^-}^{\alpha^+} \beta^- [u_x] H(x-\alpha) \phi'(x) dx + \int_{\alpha^-}^{\alpha^+} [\beta] [u_x] H^2(x-\alpha) \phi'(x) dx + \dots \\ &= \beta^- [u] \phi'(\alpha) + \frac{1}{2} \int_{\alpha^-}^{\alpha^+} \frac{d}{dx} ([\beta] [u] H^2(x-\alpha)) \phi'(x) dx + 0 + 0 + \dots \\ &= \beta^- [u] \phi'(\alpha) + \frac{1}{2} [\beta] [u] H^2(x-\alpha) \phi'(\alpha) - \int_{\alpha^-}^{\alpha^+} [\beta] [u] H^2(x-\alpha) \phi''(x) dx \\ &= \beta^- [u] \phi'(\alpha) + \frac{1}{2} [\beta] [u] \phi'(\alpha) + 0 \\ &= \frac{\beta^- + \beta^+}{2} [u] \phi'(\alpha). \end{aligned}$$

Since $\phi(x) \in C_0^\infty$ is arbitrary, by matching the coefficients in front of $\phi(\alpha)$ and $\phi'(\alpha)$ in (16) and using the relation above, we have the jump conditions (4).

3 The boundary singularity analysis

In this section, we discuss what happens when a delta function appears at one of the boundary points. We call such an irregularity a *boundary singularity* for short. This problem is worth discussing for several reasons. First of all, in some real applications, the interface is quite close to the boundary. Secondly, for many applications, sources may indeed appear on parts of boundaries. Thirdly, it is theoretically significant to change the irregular problem with a d function on the right hand side to a regular partial differential equation in order to make it more convenient to solve. The discussions here have some similarities of the limiting value theory, see for example [5].

Our discussions in this section indicate that we can translate the delta function singularity on a boundary into a corresponding boundary condition. Without loss of generality, we consider that the boundary singularity is at the boundary $x = 0$. In this case, the discontinuity in the coefficient is irrelevant.

Note that it is rarely seen a dipole (derivative of the delta function) singularities at boundaries. From another point of view, the two pieces of the solutions are almost arbitrarily independent due to arbitrary parameters c and c for the dipole case. Therefore, we only consider the case when a delta function singularity appears at one of boundary points. Thus, the problem can be written as

$$(\beta(x)u')' = f(x) + c\delta(x), \quad 0 < x < 1. \quad (17)$$

For simplicity, we assume a Dirichlet boundary condition at $x = 1$ with $u(1) = \eta$ and a linear boundary condition at $x = 0$. That is, a Dirichlet, or Neumann, or a mixed boundary condition. Note that the solution can be decomposed as the sum of two problems:

$$(\beta(x)u_1')' = f(x) \quad 0 < x < 1, \quad (18)$$

with homogenous but the same type of boundary conditions; and

$$(\beta(x)u_2')' = c\delta(x), \quad 0 < x < 1, \quad (19)$$

with the original boundary conditions. Since $u_1(x)$ is the solution to a standard problem, we can treat it easily. Thus, we will simply discuss the second equation. To further simplify the discussion, we assume that $\beta = 1$. The problem is simplified to

$$u'' = c\delta(x), \quad 0 < x < 1. \quad (20)$$

We first define the weak solution of (20).

Definition 3.1—We consider

$$u''_\alpha = c\delta(x - \alpha), \quad 0 < x < 1, \quad 0 < \alpha < 1, \quad (21)$$

with a boundary condition at $x = 0$ and $x = 1$. The weak solution of (20) is defined as $u(x) = \lim_{\alpha \rightarrow 0} u_\alpha(x)$.

To consider the effect of the delta function singularity at one boundary, we use the following procedure.

- Using the jump condition and the boundary condition to obtain the analytical solution for $u_\alpha(x)$.
- Take the limit of the solution on the right interval $(\alpha, 1]$ as α approaches zero.
- Using the limiting solution to derive the equivalent equations and equivalent boundary condition.

3.1 Dirichlet boundary conditions

In this case, the interface problem has the following form:

$$\begin{aligned} u_{xx}(x) &= c\delta(x-\alpha), \\ u(0) &= \gamma \quad 0 < \alpha < 1, \quad u(1) = \eta \quad 0 < \alpha < 1. \end{aligned} \quad (22)$$

Theorem 3.2—The limiting solution above $\lim_{\alpha \rightarrow 0} u_{\alpha}(x)$ is the solution of the following problem

$$u_{xx} = 0, \quad u(0) = \gamma, \quad u(1) = \eta. \quad (23)$$

That is, for a Dirichlet boundary condition, the boundary singularity has no effect on the solution.

Proof—We know that the solution has the following form

$$u_{\alpha}(x) = \begin{cases} v_1 + v_2 x, & 0 < x < \alpha, \\ v_3 + v_4 x, & \alpha < x < 1. \end{cases} \quad (24)$$

Using the known jump conditions $[u] = 0$ and $[u_x] = c$ and the boundary condition $u(0) = \gamma$, we have:

$$\begin{aligned} v_1 &= \gamma \\ v_1 + v_2 \alpha &= v_3 + v_4 \alpha \\ v_2 + c &= v_4 \\ v_3 + v_4 &= \eta. \end{aligned}$$

Solving the linear system we get

$$\begin{aligned} v_1 &= \gamma \\ v_2 &= c\alpha + \eta - \gamma - c \\ v_3 &= \gamma - c\alpha \\ v_4 &= c\alpha + \eta - \gamma. \end{aligned}$$

Consequently, the analytical solution is:

$$u_{\alpha}(x) = \begin{cases} \gamma + (c\alpha + \eta - \gamma - c)x, & 0 < x < \alpha, \\ \gamma - c\alpha + (c\alpha + \eta - \gamma)x, & \alpha < x < 1. \end{cases} \quad (25)$$

When α approaches zero, considering the analytical result on the right interval $(\alpha, 1]$, the solution is as follows:

$$\bar{u}(x) = \gamma + (\eta - \gamma)x. \quad (26)$$

It is the solution of the equivalent regular problem:

$$\begin{aligned} u_{xx}(x) &= 0 \\ u(0) &= \gamma \\ u(1) &= \eta. \end{aligned} \quad (27)$$

This completes the proof.

3.2 Neumann boundary conditions

We consider the problem with a Neumann boundary condition at $x = 0$.

$$\begin{aligned} u_{xx}(x) &= c\delta(x-\alpha), \\ u_x(0) &= \gamma, \quad 0 < \alpha < 1, \\ u(1) &= \eta \quad 0 < \alpha < 1. \end{aligned} \quad (28)$$

Theorem 3.3—The limiting solution above $\lim_{\alpha \rightarrow 0} u_\alpha(x)$ is the solution of the following problem

$$u_{xx} = 0, \quad u_x(0) = \gamma + c, \quad u(1) = \eta. \quad (29)$$

In other words, the boundary source strength was added to the derivative boundary condition.

Proof—We know that the solution has the following form:

$$u_\alpha(x) = \begin{cases} v_1 + v_2x, & 0 < x < \alpha, \\ v_3 + v_4x, & \alpha < x < 1. \end{cases}$$

Using the known jump conditions $[u] = 0$ and $[u_x] = c$ and the boundary condition $u_x(0) = \gamma$, we have:

$$\begin{aligned} v_2 &= \gamma \\ v_1 + v_2\alpha &= v_3 + v_4\alpha \\ v_2 + c &= v_4 \\ v_3 + v_4 &= \eta. \end{aligned}$$

Solving the linear system we get:

$$\begin{aligned} v_1 &= \eta + c\alpha - c - \gamma \\ v_2 &= \gamma \\ v_3 &= \eta - c - \gamma \\ v_4 &= \gamma + c. \end{aligned}$$

Consequently, the analytical solution is:

$$u_{\alpha}(x)=\begin{cases} \eta+c\alpha-c-\gamma+\gamma x, & 0<x<\alpha, \\ \eta-c-\gamma+(\gamma+c)x, & \alpha<x<1. \end{cases}$$

When α approaches zero, considering the right interval $(\alpha, 1]$, the approximate solution is:

$$\bar{u}(x)=\eta-c-\gamma+(\gamma+c)x.$$

This is the weak solution of the equivalent equation for the Neumann boundary condition without the d function:

$$\begin{aligned} u_{xx}(x) &= 0, \\ u_x(0) &= \gamma + c, \\ u(1) &= \eta. \end{aligned} \quad (30)$$

That completes the proof.

3.3 Mixed boundary conditions

We now consider a mixed (Robin) boundary condition at $x = 0$ of the following problem:

$$\begin{aligned} u_{xx}(x) &= c\delta(x-\alpha), \\ \sigma u(0) + \beta u_x(0) &= \gamma, \\ u(1) &= \eta \quad 0 < \alpha < 1. \end{aligned} \quad (31)$$

Theorem 3.4—The limiting solution above $\lim_{\alpha \rightarrow 0} u_{\alpha}(x)$ is the solution of the following problem

$$u_{xx}=0, \quad \sigma u(0) + \beta u_x(0) = c\beta + \gamma, \quad u(1) = \eta. \quad (32)$$

Again, we see that the source of the boundary singularity was added to the derivative part of the mixed boundary condition.

Proof—We know that the solution has the following form:

$$u_{\alpha}(x)=\begin{cases} v_1+v_2x, & 0<x<\alpha, \\ v_3+v_4x, & \alpha<x<1. \end{cases}$$

Using the known jump conditions $[u] = 0$ and $[u_x] = c$ and the boundary condition $\sigma u(0) + \beta u_x(0) = \gamma$, we have:

$$\begin{aligned}
 \sigma v_1 + \beta v_2 &= \eta \\
 v_1 + v_2 \alpha &= v_3 + v_4 \alpha \\
 v_2 + c &= v_4 \\
 v_3 + v_4 &= \eta.
 \end{aligned}$$

Solving the linear system we get

$$\begin{aligned}
 v_1 &= \frac{\beta(\eta + c\alpha - c) - \gamma}{\beta - \sigma} \\
 v_2 &= \frac{\gamma + \sigma(c - \eta - c\alpha)}{\beta - \sigma} \\
 v_3 &= \frac{\beta(\eta - c) + c\sigma\alpha - \gamma}{\beta - \sigma} \\
 v_4 &= \frac{\gamma + c\beta - \sigma(\eta + c\alpha)}{\beta - \sigma}.
 \end{aligned}$$

Therefore, the analytical solution is:

$$u_\alpha(x) = \begin{cases} \frac{\beta(\eta + c\alpha - c) - \gamma}{\beta - \sigma} + \frac{\gamma + \sigma(c - \eta - c\alpha)}{\beta - \sigma} x & 0 < x < \alpha, \\ \frac{\beta(\eta - c) + c\sigma\alpha - \gamma}{\beta - \sigma} + \frac{\gamma + c\beta - \sigma(\eta + c\alpha)}{\beta - \sigma} x & \alpha < x < 1. \end{cases}$$

When α approaches zero, using the same technique, the limiting solution is:

$$\bar{u}(x) = \frac{\beta(\eta - c) - \gamma}{\beta - \sigma} + \frac{\gamma + c\beta - \sigma\eta}{\beta - \sigma} x. \quad (33)$$

Since $\sigma u(0) + \beta u_x(0) = c\beta + \gamma$, the equivalent regular differential equation for the mixed boundary condition without the d function is:

$$\begin{aligned}
 u_{xx}(x) &= 0 \\
 \sigma u(0) + \beta u_x(0) &= c\beta + \gamma, \quad u(1) = \eta.
 \end{aligned}$$

This completes the proof.

4 A proof of the convergence of the IB method

It is well known that Peskin's IB method is only first order accurate in L^∞ norm. In [9], the author has proved the first order accuracy of the IB method for the Stokes equations, see also [3]. However, there are few theoretical proofs on the IB method for elliptic interface problems. This is the main motivation of this section. We will give a proof for the 1D model,

$$u'' = c\delta(x - \alpha) \quad 0 < x < 1, \quad 0 < \alpha < 1, \quad u(0) = u(1) = 0, \quad (34)$$

in this section. The proof for the 2D problem is more challenging and is under investigation. Note that the analytic solution to the equation above is

$$u(x) = \begin{cases} -cx(1-\alpha) & \text{if } x \leq x_k, \\ -c\alpha(1-x) & \text{otherwise.} \end{cases} \quad (35)$$

Given a uniform Cartesian grid $x_i = ih$, $i = 0, 1, \dots, n$, $h = 1/n$, the IB method leads to the following system of linear equations,

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = c\delta_h(x_i - \alpha), \quad i = 1, 2, \dots, n-1, \quad (36)$$

where U_i is the finite difference approximation of the solution $u(x_i)$, and $\delta_h(x_i - \alpha)$ is a discrete delta function applied to the grid point x_i . In the matrix-vector form, the above finite difference equations can be written as $\mathbf{A}\mathbf{U} = \mathbf{F}$, where $\mathbf{A}$ is the tri-diagonal matrix formed by the discrete Laplacian. It is well known that $-\mathbf{A}$ is a symmetric positive definite matrix and diagonally dominant. Note that, a discrete delta function has a compact support in the neighborhood of the interface, that is, $\delta_h(x) = 0$ only if $|x| > Wh$, where W is a constant. Commonly used discrete delta functions include the hat discrete delta function ($\delta^{hat}(x)$ with $W = 1$) and the cosine discrete delta function ($\delta^{cosine}(x)$ with $W = 2$), see for example, [8]. Note that, when we use the hat delta function, the result is the same as that of the IIM for the simple model. The solution to the finite difference equations is the same as a true solution if there are no round-off errors, that is

$$\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1}))}{h^2} = c\delta_h^{hat}(x_i - \alpha), \quad i = 1, 2, \dots, n-1, \quad (37)$$

see for example, [1, 8]. But this is not the case for other discrete delta functions.

We define the error vector as $\mathbf{E} = \{E_i\}$, where $E_i = u(x_i) - U_i$. The local truncation error is defined as $\mathbf{T} = \{T_i\}$,

$$T_i = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1}))}{h^2} - c\delta_h(x_i - \alpha). \quad (38)$$

With the definition, we have $\mathbf{A}\mathbf{U} = \mathbf{F}$, $\mathbf{A}\mathbf{u} = \mathbf{F} + \mathbf{T}$, and therefore $\mathbf{A}\mathbf{E} = \mathbf{T}$. For the hat discrete delta function, we have $|T_i| = 0$ for all i 's. For other discrete delta functions, we have $|T_i| = O(1/h)$ for a few grid points neighboring the interface a . So the interesting question is: Why is the IB method still first order accurate, that is, $\|\mathbf{E}\|_\infty = O(h)$? To answer this question, we first introduce the following lemma.

Lemma 4.1

Let $\mathbf{A}\mathbf{y} = \mathbf{e}_k$, where $\mathbf{e}_k$ is the k -th unit base vector, then

$$y_i = \begin{cases} -hx_i(1-x_k) & \text{if } i \leq k, \\ -hx_k(1-x_i) & \text{otherwise.} \end{cases} \quad (39)$$

The significance of this lemma is that the solution is order h smaller than the concentrated source.

Proof

We note the following identity

$$A\mathbf{y} = h \frac{\mathbf{e}_k}{h} = h \delta_h^{\text{hat}} \mathbf{e}_k. \quad (40)$$

For this simple case, the IB and IIM are identical. Thus, from the Immersed Interface Method see [1, 8], we know that $\mathbf{y}$ is the exact discrete solution at the grid points of the following boundary value problem

$$u'' = h \delta(x - x_k), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \quad (41)$$

whose solution is

$$y(x) = \begin{cases} -hx(1-x_k) & \text{if } x \leq x_k, \\ -hx_k(1-x) & \text{otherwise.} \end{cases} \quad (42)$$

This completes the proof. Note that $|y(x)| \leq h$. From this lemma, we have the following corollary.

Corollary 4.2

Let $A\mathbf{y} = \mathbf{r}$, then $|y(x)| \leq h W$, where W is the number of non-zero components of $\mathbf{r}$.

Notice that for a discrete delta function, it should satisfy at least the zeroth moment equation, see [2], that is

$$\sum_i \delta_h(x_i - \alpha) = 1, \quad (43)$$

corresponding to the continuous case $\int \delta_h(x - \alpha) dx = 1$. Now we are ready to prove the main theorem.

Theorem 4.3

The Immersed Boundary method for (34) is first order accurate, that is

$$\|\mathbf{E}\|_{\infty} \leq \bar{C}h, \quad (44)$$

where $\bar{C}$ is a constant.

Proof

We can decompose the local truncation error into two groups

$$\mathbf{T} = \mathbf{T}^{reg} + \mathbf{T}^{irreg}, \quad (45)$$

where $\|\mathbf{T}^{reg}\|_{\infty} = 0$ corresponds to the local truncation errors at regular grid points where $\delta_h(x_i - \alpha) = 0$ and the true solution is piecewise linear. Note that, we have

$$\sum_{i=1}^{n-1} T_i = \sum \mathbf{T}^{reg} + \sum \mathbf{T}^{irreg} = O + \sum \mathbf{T}^{irreg}. \quad (46)$$

On the other hand, we also have

$$\sum_{i=1}^{n-1} T_i = \sum_{i=1}^{n-1} c \delta_h^{hat}(x_i - \alpha) - \sum_{i=1}^{n-1} c \delta_h(x_i - \alpha) = 0, \quad (47)$$

since the finite difference method using the discrete delta function gives the exact solution at the grid points. Thus, we have $\sum_i T_i^{irreg} = 0$. We divide T_i^{irreg} into two groups, one with all positive T_i 's denoted as $T_i^{irreg,+}$ the other one is all the negatives denoted as $T_i^{irreg,-}$. Since $\sum T_i^{irreg,+} + \sum T_i^{irreg,-} = 0$, $T_i^{irreg,+}$ and $T_i^{irreg,-}$ must have the same order of the magnitude ($O(1/h)$) although those indexed i are different except that $|x_i - \alpha|$ is true for all irregular grid points. Because the solution is linear with c , we have

$$\mathbf{E} = A^{-1} \mathbf{T} = A^{-1} (\mathbf{T}^{irreg,+} + \mathbf{T}^{irreg,-}). \quad (48)$$

From the solution expression, we know that, assuming that $x_k < \alpha$,

$$\begin{aligned} E_k &= -x_k \left(\sum_i T_i^{irreg,+} (1-x_i) + \sum_j T_j^{irreg,-} (1-x_j) \right) \\ &= -x_k (1-\alpha) \left(\sum_i T_i^{irreg,+} + \sum_j T_j^{irreg,-} \right) + O(Wh) \\ &= O(Wh), \end{aligned}$$

after we expand all x_i 's and x_k 's at α and since all related x_i and x_j are within Wh distance from the interface α . The proof when $x_k > \alpha$ is similar except that we need to use the solution for $x > \alpha$. This completes the proof.

5 Conclusions and acknowledgements

In this paper, we have derived the new jump condition when both single and double layer singularities are present along with a discontinuity in the coefficient of a general 1D interface problem. We discussed the effect of the boundary delta function singularities on the solution and derived the equivalent regular differential equations and equivalent boundary conditions. We also gave a convergence proof of the immersed boundary (IB)

method in the L^∞ norm. We showed that the IB method is first order accurate if a general discrete delta function is used.

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