Modeling the impact of awareness on the mitigation of algal bloom in a lake

A. K. Misra¹ · P. K. Tiwari¹ · Ezio Venturino²

Received: 27 February 2015 / Accepted: 22 July 2015 / Published online: 28 September 2015
© Springer Science+Business Media Dordrecht 2015

Abstract The proliferation of algal bloom in water bodies due to the enhanced concentration of nutrient inflow is becoming a global issue. A prime reason behind this aquatic catastrophe is agricultural runoff, which carries a large amount of nutrients that make the lakes more fertile and cause algal blooms. The only solution to this problem is curtailing the nutrient loading through agricultural runoff. This could be achieved by raising awareness among farmers to minimize the use of fertilizers in their farms. In view of this, in this paper, we propose a mathematical model to study the effect of awareness among the farmers of the mitigation of algal bloom in a lake. The growth rate of awareness among the farmers is assumed to be proportional to the density of algae in the lake. It is further assumed that the presence of awareness among the farmers reduces the inflow rate of nutrients through agricultural runoff and helps to remove the detritus by cleaning the bottom of the lake. The results evoke that raising awareness among farmers may be a plausible factor for the mitigation of algal bloom in the lake. Numerical simulations identify the most critical parameters that influence the blooms and provide indications to possibly mitigate it.

Keywords Mathematical model · Algal bloom · Awareness · Stability · Sensitivity analysis

1 Introduction

Fertilizer runoff from farm lands has become a leading cause of eutrophication around the world. To fulfill the increasing demands for food, various fertilizers containing nitrogen and phosphorus are being used in farming to increase the crop yield. The usage of these fertilizers in the farm increases the crop yield but their excessive use does not increase the crop yield in

✉ A. K. Misra
akmisra_knp@yahoo.com

¹ Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi 221005, India

² Dipartimento di Matematica “Giuseppe Peano”, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy

the same proportion [1]. Only 50 % of the nutrients used as fertilizers are consumed by plants, and the remainder is left as residue in soil. These surplus fertilizers erode with the soil to nearby water bodies such as lakes and ponds and enrich the water bodies with nutrients [2–4]. The high nutrient loading makes the water body more fertile and hence intensifies the growth of algae. Due to the short life span of algae, a high concentration of algae, called detritus, accumulates at the bottom of the water bodies. Detritus is then acted upon by bacteria for decomposition and the process consumes a lot of dissolved oxygen present in the water, giving rise to hypoxia. With a lack of sufficient dissolved oxygen in the water, a large proportion of aquatic organisms die.

The occurrence of algal blooms has become severe and widespread around the world [5–8]. Although many countries have passed legislation to control nutrient loading from point sources, the impairment of water bodies due to non-point input of nutrients causes huge changes in the hydrological regime. The non-point source comprises mainly agricultural runoff, which comes from an extensive area of land and thus it is difficult to monitor. The control of nutrient influx from non-point sources requires agriculture management practices and optimum use of fertilizers in farms so that the crop yield can be maximized and the runoff of nutrients through agriculture farms can be minimized.

Soil itself contains various kinds of nutrients required for the growth of a crop yield but due to lack of awareness, farmers employ a large amount of fertilizers to increase the crop yield [9]. Thus, farmers must be aware about the optimum concentration of fertilizer needed for good crop production. Also, they must be motivated to adopt soil testing techniques to determine the nutrient requirement of their farming fields. The proper estimate of fertilizers required for efficient production in fields will help farmers to optimize the amount of fertilizer applied to crops. This will not only provide economic benefit to the farmers, by decreasing fertilizer application costs, but also decrease the runoff of nutrients to surrounding water bodies and hence offer environmental benefits. Therefore, raising awareness among farmers about optimal fertilizer use and motivating them for soil testing can be an effective avenue for lessening the non-point influx of nutrients in water bodies. By inducing behavioral changes among the farmers, the problem of nutrient enrichment of a lake through agricultural runoff can be controlled. To keep the water sources alive, the aware farmers remove the mud and thus detritus from the lakes. This removal of detritus from lakes due to the awareness among farmers reduces the amount of nutrients in the lakes, which are formed due to the decomposition of detritus by bacteria. Recently, some researchers have shown that behavioral changes induced by media can help in controlling the prevalence of various diseases [10, 11 and references therein]. Misra and Verma [12] studied the impact of educational programs on the abatement of carbon dioxide emissions. They showed that educational programs are helpful for the reduction of anthropogenic carbon dioxide emissions. These works motivated us to investigate the effect of creating awareness among farmers on the mitigation of algal bloom in lakes.

Two nutrients in human-derived sources, phosphorus and nitrogen, are of most concern in algal bloom. In freshwaters, phosphorus is the least abundant among the nutrients needed in large quantity by photosynthetic organisms, so it is the primary nutrient that limits their growth. Phosphorus can also limit or co-limit algal growth in estuarine and marine environments that are sustaining high nitrogen inputs. Prevention of algal bloom has drawn wide attention by the public and researchers [13]. Although many experts have judged that the excessive nutrient concentration caused by eutrophication of a water body is the main reason of algal bloom, few of them provide an explanation from the aspect of evolutionary

mechanism. Gao et al. [14], in laboratory experiments, examined the influence of phosphorus and temperature on the algal growth rate and environmental biomass, showing that both the growth rate and environmental biomass change with temperature and nutrient concentration. Havens et al. [15] showed that dual nitrogen and phosphorus input reductions are usually required for effective long-term control and management of algal blooms. Schindler [16] suggested that protection and restoration of features of a lake's community and its catchment require management of eutrophication through the control of nutrient sources. Chen et al. [17] used a mathematical model to provide different insights to use nitrogen and phosphorus more effectively in the control and elimination of blue-green algae blooms. In the past few decades, several studies have been conducted to study algal bloom [18–21], eutrophication [22–24], and their effects on aquatic ecosystems [25–29]. Voinov and Tonkikh [24] presented a non-linear mathematical model to study eutrophication in lakes by considering the densities of algae and macrophytes, which compete for nutrients. In this study, the qualitative behavior of the growth of algae and macrophytes is discussed. Hallam [30] studied a non-linear mathematical model by considering nutrients, phytoplankton, and zooplankton as dynamical variables. He assumed that nutrients are supplied only by the dead part of phytoplankton and zooplankton. Chakraborty et al. [31, 32] studied the dynamics of one and two phytoplankton populations by considering that nutrients are supplied via external sources. Amemiya et al. [33] presented a mathematical model to study the effect of nutrient loading in a lake on the dynamics of phytoplankton, zooplankton, and fish population. They showed that nutrient loading may give rise to regime shifts. Shukla et al. [34] proposed a non-linear mathematical model for the growth of algae and its effect on the depletion of dissolved oxygen by considering the Michaelis–Menten-type interaction between nutrients and density of algae. In this study, it is shown that an increase in the inflow rate of nutrients not only enhances the growth of algae but also depletes the level of dissolved oxygen in a lake. It may be noted that in all the above studies either the constant influx rate of nutrients or supply of nutrients through the dead part of algae and macrophytes to the lakes is considered. In some of these studies, it is suggested that by controlling the influx of nutrients, the density of algae can be controlled and thus the level of dissolved oxygen can be maintained but the mechanisms to control the inflow rate is not discussed. However, the problem of algal bloom control by creating awareness among farmers has not yet been addressed via mathematical models. In this study, a mathematical model to assess the impact of awareness among farmers on the mitigation of algal bloom in lakes is proposed and analyzed. It is assumed that the inflow rate of nutrients coming into a lake through agricultural runoff decreases as the awareness among farmers increases. Further, it is assumed that farmers, aware of the adverse effects of algal bloom, remove detritus from the lake.

The main goal here is the assessment of the impact of awareness among farmers about the optimum use of fertilizers in their farming land. The aim is twofold: to maximize the crop yield and to minimize the leaching of nutrients through agricultural runoff. In this way, the inflow rate of nutrients into a lake due to agricultural runoff can be controlled. By reducing the concentration of nutrients in a lake, we can control the density of algae in it. As aware farmers also remove mud from the bottom of the lake to clean it, they also reduce the density of detritus in the lake. Thus, as detritus is converted into nutrients via decomposition by bacteria, the concentration of nutrients is also in check. Therefore, by raising awareness among the farmers, the density of algae can be controlled in two ways: first by controlling the inflow rate of nutrients, hindering their growth; secondly by removing the detritus from bottom of the lake, again diminishing their conversion into nutrients.

2 Mathematical model

In the region under consideration, let N , A and S , respectively, be the cumulative concentration, in a lake, of nutrients, the density of algae and the density of detritus, at any time $t > 0$. Let E be the measure of awareness among the farmers in the same region at time t and E_0 be the baseline awareness that is present among the farmers. Due to awareness among the farmers, the inflow rate of nutrients into the lake is modeled by a decreasing function of measure of awareness among the farmers, i.e., $q\left(1 - \frac{pE}{k+E}\right)$. Nutrients are taken up by algae for their growth following a Holling type II interaction, i.e., $\frac{\beta_1 NA}{\beta_{12} + \beta_{11}N}$. It is assumed that the density of algae wholly depends on the concentration of nutrients in the lake so their growth rate is proportional to the same interaction term. It is further assumed that the density of algae decreases due to mortality, either natural or due to intraspecific competition. Therefore, the decay rate of algae is assumed to be proportional to A as well as A^2 . When algae die out, they sink to the bottom of the lake and detritus is formed, so the growth rate of detritus must be proportional to A as well as A^2 . There is a natural depletion of detritus due to the process of decay caused by bacterial or fungal action in the lake. Detritus is decomposed by micro-organisms and nutrients are formed, thus the growth rate of nutrients also depends on the density of detritus in the lake. To clean the lake, the aware farmers remove the mud from the bottom of the lake. Therefore, the decay rate in detritus is considered proportional to the density of detritus and the awareness among the farmers. The measure of awareness among the farmers is assumed to grow with the impact of algal bloom, i.e., ultimately to be proportional to the density of algae in the lake. There is a decrease in the measure of awareness among the farmers due to memory fading or some other psychological factors. Combining all the above facts together, the dynamics of the system is governed by the following differential equations:

$$\begin{aligned} \frac{dN}{dt} &= q\left(1 - \frac{pE}{k+E}\right) - \alpha_1 N - \frac{\beta_1 NA}{\beta_{12} + \beta_{11}N} + \pi \delta S, \\ \frac{dA}{dt} &= \frac{\theta_1 \beta_1 NA}{\beta_{12} + \beta_{11}N} - \alpha_1 A - \beta_{10} A^2, \\ \frac{dS}{dt} &= \pi_1 \alpha_1 A + \pi_2 \beta_{10} A^2 - \delta S - \gamma S E, \\ \frac{dE}{dt} &= \phi A - \phi_0 (E - E_0). \end{aligned} \quad (1)$$

The initial conditions are taken to be positive values and in particular, the awareness starts at level E_0 .

In system (1), the constant q is the inflow rate of nutrients through agricultural runoff and the constant p is the efficacy of awareness to control the inflow rate of nutrients in the lake. The parameter k is the half saturation constant, which denotes the measure of awareness among the farmers at which the reduction in the inflow rate of nutrients is half of the maximum possible reduction that can ever be achieved via raising awareness among the farmers. The constant α_0 is the loss rate of nutrients due to sinking from the epilimnion down to the hypolimnion and therefore making these nutrients unavailable for algae uptake. The constant β_1 denotes the maximum uptake rate of nutrients by algae whereas the constant β_{12} limits this uptake rate and β_{11} is a constant of proportionality. The proportionality constant π represents the recycling of detritus into nutrients. The proportionality constant θ_1 represents the growth in the density of algae due to uptake of nutrients. The parameters α_1 and β_{10} are the rates at which algae

respectively die, naturally or by intraspecific competition. In model (1), π_1 and π_2 are proportionality constants, representing the conversion of dead algae into detritus while the constant δ is the natural detritus depletion rate. The constant γ is the removal rate of detritus from the lake by aware farmers. The constants ϕ and ϕ_0 represent the growth rate and decrease rate in the awareness, respectively among the farmers. All the above constants are assumed to be positive and $p, \pi, \pi_1, \pi_2 \in (0, 1)$. Also, $\theta_1 \leq 1$. From the second equation of model (1), the following condition must be satisfied:

$$\theta_1 \beta_1 - \beta_{11} \alpha_1 > 0 \quad (2)$$

otherwise the system settles to $A=0, S=0, E=E_0$ and $N=N$, the latter to be defined below.

3 Mathematical analysis

3.1 Equilibrium analysis

System (1) has two non-negative equilibria. They are as follows:

- (i) Trivial equilibrium $E_1 = (\hat{N}, 0, 0, E_0)$ with $\hat{N} = \frac{q}{\alpha_0} \left(1 - \frac{p E_0}{k + E_0}\right)$.
- (ii) Interior equilibrium $E_2 = (N^*, A^*, S^*, E^*)$.

Equilibrium E_1 is always feasible. Equilibrium E_2 is feasible if the following condition is satisfied:

$$q \left(1 - \frac{p E_0}{k + E_0}\right) (\theta_1 \beta_1 - \beta_{11} \alpha_1) - \beta_{12} \alpha_0 \alpha_1 > 0. \quad (3)$$

Inequality (3) gives a condition for mitigation of algal bloom in lakes. From this condition, we can determine an expression for the critical value of E_0 (say E_c) as follows:

$$E_c = \frac{k \{q(\theta_1 \beta_1 - \beta_{11} \alpha_1) - \beta_{12} \alpha_0 \alpha_1\}}{\beta_{12} \alpha_0 \alpha_1 - (1-p)q(\theta_1 \beta_1 - \beta_{11} \alpha_1)}. \quad (4)$$

Expression (4) defines a critical value of measure of baseline awareness among the farmers, which depends on the efficacy of awareness among the farmers and the inflow rate of nutrients in the lake. For $E_0 < E_c$, algal bloom occurs in the lake while instead it will be mitigated for any value of $E_0 > E_c$.

The feasibility of equilibrium E_1 is trivial, since $p < 1$. In the following, we show the feasibility of equilibrium E_2 . The components of the coexistence equilibrium are found as follows. From the last equilibrium equation of (1), we have:

$$E = E_0 + \frac{\phi}{\phi_0} A. \quad (5)$$

From the second equilibrium equation, it follows:

$$N = \frac{\beta_{12} (\alpha_1 + \beta_{10} A)}{(\theta_1 \beta_1 - \beta_{11} \alpha_1) - \beta_{11} \beta_{10} A}. \quad (6)$$

Using Eq. (5) into the third equilibrium equation, we find:

$$S = \frac{(\pi_1 \alpha_1 + \pi_2 \beta_{10} A) A}{\delta + \gamma \left(E_0 + \frac{\phi}{\phi_0} A \right)}. \quad (7)$$

Now, using Eqs. (5–7) in the first equilibrium equation, we obtain the following equation in A :

$$F(A) = \theta_1 q \left(1 - \frac{p \left(E_0 + \frac{\phi}{\phi_0} A \right)}{k + \left(E_0 + \frac{\phi}{\phi_0} A \right)} \right) + \frac{\pi \delta \theta_1 A (\pi_1 \alpha_1 + \pi_2 \beta_{10} A)}{\delta + \gamma \left(E_0 + \frac{\phi}{\phi_0} A \right)} - (\alpha_1 + \beta_{10} A) A - \frac{\theta_1 \beta_{12} \alpha_0 (\alpha_1 + \beta_{10} A)}{(\theta_1 \beta_1 - \beta_{11} \alpha_1) - \beta_{11} \beta_{10} A} = 0. \quad (8)$$

From Eq. (8), the following facts may be noted:

- (i) $F(0) > 0$, provided condition (3) is satisfied,
- (ii) $F(A) \rightarrow -\infty$ as $A \rightarrow \tilde{A}$, where $\tilde{A} = \frac{\theta_1 \beta_1 - \beta_{11} \alpha_1}{\beta_{11} \beta_{10}}$, and
- (iii) $F(A) < 0$ in the open interval $(0, \tilde{A})$.

The above facts imply that there exists a unique positive root $A = A^*$ of Eq. (8) in the open interval $(0, \tilde{A})$. Now, using this value of $A = A^*$ in Eqs. (5, 6) and (7) we get the positive values of E^* , N^* , and S^* , respectively.

Remark 1 To see the effects of the parameters q , p , and E_0 on the equilibrium density of algae in the lake, we determine the rate of change of the equilibrium density of algae, A^* with respect to these parameters. The numerical simulations show that $\frac{dA^*}{dq} > 0$, $\frac{dA^*}{dp} < 0$ and $\frac{dA^*}{dE_0} < 0$. Thus, the density of algae in the lake increases with an increase in the inflow rate of nutrients but it decreases with an increase in efficacy and baseline level of awareness among the farmers.

3.2 Stability analysis

The local stability behavior of the equilibria of model system (1) is given in the following theorem, showing that there is a transcritical bifurcation for which equilibrium E_2 emanates from equilibrium E_1 whenever the latter becomes unstable.

Theorem 1 *Equilibrium E_1 is unstable (or stable) whenever equilibrium E_2 is feasible (or is not feasible). Equilibrium E_2 , whenever feasible, is locally asymptotically stable if the following conditions are satisfied:*

$$A_4 > 0 \quad \text{and} \quad A_3(A_1 A_2 - A_3) - A_1^2 A_4 > 0, \quad (9)$$

where A_i 's, $i=1-4$ are given in the proof.

Proof of this theorem is given in [Appendix A](#).

In the following, we prove that equilibrium E_2 is globally asymptotically stable. For this we need the following lemma [35].

Lemma. The region of attraction for all solutions initiating in the positive orthant is given by the set [35];

$$\Omega = \left\{ (N, A, S, E) : 0 < N + A + S \leq \frac{q}{\delta_m}, E_0 \leq E \leq \left(E_0 + \frac{\phi}{\phi_0} \frac{q}{\delta_m} \right) \right\}, \quad (10)$$

where $\delta_m = \min\{\alpha_0, (1-\pi_1)\alpha_1, (1-\pi)\delta\}$.

Theorem 2 *Equilibrium E_2 , if feasible, is globally asymptotically stable inside the region of attraction Ω , provided the following conditions are satisfied:*

$$\left[\frac{\beta_1 \beta_{11} q / \delta_m}{(\beta_{12} + \beta_{11} q / \delta_m)(\beta_{12} + \beta_{11} N^*)} \right]^2 N^* < \frac{4\alpha_0 \beta_{10}}{9\theta_1}, \quad (11)$$

$$\left[\frac{pqk}{(k + E_0)(k + E^*)} \right]^2 < \frac{16\alpha_0 \beta_{10} \phi_0^2 N^*}{81\theta_1 \phi^2}, \quad (12)$$

$$\left[\frac{\pi\delta}{\delta + \gamma E_0} \right]^2 < \frac{16\alpha_0}{81} \min \left\{ \frac{m_3 \phi_0}{\gamma^2 S^{*2}}, \frac{\beta_{10} N^*}{\theta_1 [\pi_1 \alpha_1 + \pi_2 \beta_{10} (q/\delta_m + A^*)]^2} \right\}, \quad (13)$$

where m_3 is given in the proof.

Proof of this theorem is given in [Appendix B](#).

4 Numerical simulations

For the validation of our analytical findings regarding the feasibility and stability of equilibrium E_2 , numerical computation is carried out by choosing parameter values in (1) as given in Table 1.

It is found that for this set of parameter values, the condition for feasibility of equilibrium E_2 is satisfied and its components are given by:

$$N^* = 0.2003 \text{ mg l}^{-1}, \quad A^* = 0.0040 \text{ mg l}^{-1}, \quad S^* = 0.0030 \text{ mg l}^{-1}, \quad E^* = 57.95\%.$$

For the set of parameter values given in Table 1, the value for E_c is 92.30%. Eigenvalues of the matrix J_{E_2} are -0.1947 , $-0.0091 + 0.0113 i$, $-0.0091 - 0.0113 i$ and -0.5316 .

As the eigenvalues are either negative or with negative real parts, equilibrium E_2 is locally asymptotically stable. Further, to see the effect of the model parameters on the dynamics of the model variables, system (1) is solved numerically by using MatLab 7.5.0 for the above set of parameter values.

For the above parameter values, a three-dimensional graph for the concentration of nutrients, density of algae and measure of awareness among the farmers is plotted in

Table 1 Parameter values in the model system (1)

Parameter	Value	Parameter	Value
q	0.05 mg l^{-1}	α_1	0.45 day^{-1}
p	$0.33 \text{ mg l}^{-1} \text{ percent}^{-1}$	β_{10}	$0.1 \text{ l mg}^{-1} \text{ day}^{-1}$
k	0.6%	π_1	0.9
α_0	0.2 day^{-1}	π_2	0.5
β_1	0.9 day^{-1}	δ	0.01 day^{-1}
β_{12}	0.2 mg l^{-1}	γ	$0.9 (\text{percent day})^{-1}$
β_{11}	1	ϕ	0.75%
π	0.1	ϕ_0	0.008 day^{-1}
θ_1	1	E_0	$0.2 (20\%)$

Fig. 1, by taking different initial values. From this figure, it is clear that all the solution trajectories that initiate inside the region of attraction approach the point (N^*, A^*, E^*) . This shows that equilibrium E_2 is globally asymptotically stable in the N - A - E phase space. Similarly, the global asymptotic stability of equilibrium E_2 in other spaces can be shown.

Figure 2 is drawn to demonstrate the impact of the inflow rate of nutrients, efficacy, and baseline level of awareness among the farmers on the equilibrium density of algae in the lake. From this figure, it is evident that the equilibrium density of algae in the lake increases with an increase in the inflow rate of nutrients but it decreases with an increase in the efficacy and baseline level of awareness among the farmers.

Fig. 1 An illustration of the global stability of equilibrium E_2 in the N - A - E phase space

Fig. 2 Variation of algae with respect to time for different values of q , p , and E_0 . The algae equilibrium value is an increasing function of the inflow rate of nutrients but a decreasing function of the efficacy and baseline level of awareness

Further, from expression (4), it is evident that the efficacy of awareness among the farmers and the inflow rate of nutrients in the lake have a crucial impact on the critical value of baseline awareness among the farmers and hence on the mitigation of algal bloom in the lake. To make it apparent, the variation of the critical value of baseline awareness among the farmers with respect to these parameters is drawn in Fig. 3. From this figure, it is clear that for a fixed value of p , upon increasing the value of q , the value of E_c increases. On the other hand, for a fixed value of q , upon increasing the value of p , the value of E_c decreases. Therefore, from this figure, the requisite values of these parameters for the mitigation of algal bloom in the lake may be obtained.

5 Sensitivity analysis

Sensitivity analysis reveals how the model variables respond to change in parameter values. In order to see the behavior of the model variables with a change in model parameters, a basic sensitivity analysis of model system (1) was carried out,

following [36, 37]. Let us denote X_w as the sensitivity function of state variable X w.r.t. w , i.e., $X_w(t) = \frac{\partial}{\partial w} X(t, w)$. To perform the sensitivity analysis, p , γ and ϕ are chosen as sensitive parameters. The sensitivity systems with respect to these parameters are as follows:

$$\begin{aligned}\dot{N}_p(t, p) &= -\frac{qE(t, p)}{k + E(t, p)} - \frac{pqk}{(k + E(t, p))^2} E_p(t, p) - \alpha_0 N_p(t, p) \\ &\quad - \frac{\beta_1 N(t, p)}{\beta_{12} + \beta_{11} N(t, p)} A_p(t, p) - \frac{\beta_1 \beta_{12} A(t, p)}{(\beta_{12} + \beta_{11} N(t, p))^2} N_p(t, p) \\ &\quad + \pi \delta S_p(t, p), \\ \dot{A}_p(t, p) &= \frac{\theta_1 \beta_1 N(t, p)}{\beta_{12} + \beta_{11} N(t, p)} A_p(t, p) + \frac{\theta_1 \beta_1 \beta_{12} A(t, p)}{(\beta_{12} + \beta_{11} N(t, p))^2} N_p(t, p) \\ &\quad - \alpha_1 A_p(t, p) - 2 \beta_{10} A(t, p) A_p(t, p), \\ \dot{S}_p(t, p) &= \pi_1 \alpha_1 A_p(t, p) + 2 \pi_2 \beta_{10} A(t, p) A_p(t, p) - \delta S_p(t, p) \\ &\quad - \gamma S_p(t, p) E(t, p) - \gamma S(t, p) E_p(t, p), \\ \dot{E}_p(t, p) &= \phi A_p(t, p) - \phi_0 E_p(t, p); \end{aligned} \quad (14)$$

$$\begin{aligned}\dot{N}_\gamma(t, \gamma) &= -\frac{pqk}{(k + E(t, \gamma))^2} E_\gamma(t, \gamma) - \alpha_0 N_\gamma(t, \gamma) - \frac{\beta_1 N(t, \gamma)}{\beta_{12} + \beta_{11} N(t, \gamma)} A_\gamma(t, \gamma) \\ &\quad - \frac{\beta_1 \beta_{12} A(t, \gamma)}{(\beta_{12} + \beta_{11} N(t, \gamma))^2} N_\gamma(t, \gamma) + \pi \delta S_\gamma(t, \gamma), \\ \dot{A}_\gamma(t, \gamma) &= \frac{\theta_1 \beta_1 N(t, \gamma)}{\beta_{12} + \beta_{11} N(t, \gamma)} A_\gamma(t, \gamma) + \frac{\theta_1 \beta_1 \beta_{12} A(t, \gamma)}{(\beta_{12} + \beta_{11} N(t, \gamma))^2} N_\gamma(t, \gamma) \\ &\quad - \alpha_1 A_\gamma(t, \gamma) - 2 \beta_{10} A(t, \gamma) A_\gamma(t, \gamma), \\ \dot{S}_\gamma(t, \gamma) &= \pi_1 \alpha_1 A_\gamma(t, \gamma) + 2 \pi_2 \beta_{10} A(t, \gamma) A_\gamma(t, \gamma) - \delta S_\gamma(t, \gamma) \\ &\quad - S(t, \gamma) E(t, \gamma) - \gamma S_\gamma(t, \gamma) E(t, \gamma) - \gamma S(t, \gamma) E_\gamma(t, \gamma), \\ \dot{E}_\gamma(t, \gamma) &= \phi A_\gamma(t, \gamma) - \phi_0 E_\gamma(t, \gamma); \end{aligned} \quad (15)$$

and

$$\begin{aligned}\dot{N}_\phi(t, \phi) &= -\frac{pqk}{(k + E(t, \phi))^2} E_\phi(t, \phi) - \alpha_0 N_\phi(t, \phi) - \frac{\beta_1 N(t, \phi)}{\beta_{12} + \beta_{11} N(t, \phi)} A_\phi(t, \phi) \\ &\quad - \frac{\beta_1 \beta_{12} A(t, \phi)}{(\beta_{12} + \beta_{11} N(t, \phi))^2} N_\phi(t, \phi) + \pi \delta S_\phi(t, \phi), \\ \dot{A}_\phi(t, \phi) &= \frac{\theta_1 \beta_1 N(t, \phi)}{\beta_{12} + \beta_{11} N(t, \phi)} A_\phi(t, \phi) + \frac{\theta_1 \beta_1 \beta_{12} A(t, \phi)}{(\beta_{12} + \beta_{11} N(t, \phi))^2} N_\phi(t, \phi) \\ &\quad - \alpha_1 A_\phi(t, \phi) - 2 \beta_{10} A(t, \phi) A_\phi(t, \phi), \\ \dot{S}_\phi(t, \phi) &= \pi_1 \alpha_1 A_\phi(t, \phi) + 2 \pi_2 \beta_{10} A(t, \phi) A_\phi(t, \phi) - \delta S_\phi(t, \phi) \\ &\quad - \gamma S_\phi(t, \phi) E(t, \phi) - \gamma S(t, \phi) E_\phi(t, \phi), \\ \dot{E}_\phi(t, \phi) &= A(t, \phi) + \phi A_\phi(t, \phi) - \phi_0 E_\phi(t, \phi). \end{aligned} \quad (16)$$

For the sensitivity systems (14)–(16), the semi-relative sensitivity as well as the logarithmic sensitivity solutions for the set of parameter values given in Table 1 are

Fig. 3 Variation of the critical value of baseline awareness E_c with respect to p and q . Fixing the value of p and increasing q , the value of E_c increases; but for a fixed value of q , by increasing p we obtain lower values of E_c

plotted in Figs. 4 and 5, respectively. The semi-relative sensitivity solutions (i.e., $wX_w(t, w)$) determine the change that doubling a parameter yields in the value of a state variable. It is apparent from Fig. 4 that doubling the efficacy of awareness among the farmers, p , reduces the concentration of nutrients by 0.022 mg l^{-1} and density of algae by 0.0028 mg l^{-1} in 5 days. The density of detritus and measure of awareness among the farmers show a slight decrease on doubling this parameter. Further, doubling the value of the removal rate of detritus due to presence of awareness among the farmers, γ , yields a negligible decrease in the concentration of nutrients, density of algae and measure of awareness but the density of detritus decreases by 0.027 mg l^{-1} in 5 days. For the growth rate of awareness among the farmers ϕ , a doubling in its value increases the awareness by 214.4% in 5 days. The concentration of nutrients, density of algae and density of detritus decrease by 0.023 mg l^{-1} , 0.0019 mg l^{-1} and 0.064 mg l^{-1} , respectively, in 5 days by doubling the value of ϕ . It is clear from this figure that out of these three parameters, the parameter p has maximum effect on the concentration of nutrients and density of algae in the lake.

Moreover, the logarithmic sensitivity solutions (i.e., $\frac{w}{X(t,w)} X_w(t, w)$), quantify the percentage change in the value of a state variable induced by doubling a parameter. It is apparent from Fig. 5 that doubling the value of p decreases the

concentration of nutrients by 13.25% in 5 days. The density of algae decreases by 8.51% in 5 days. The density of detritus and measure of awareness among the farmers decrease by a negligible percentage on doubling this parameter. The concentration of nutrients, density of algae, and measure of awareness show a slight decrease on doubling the parameter γ . However, the density of detritus shows a significant decrease of 98.12% in 5 days. The concentration of nutrients, density of algae, and density of detritus decrease by 14.85%, 5.97%, and 236.1%, respectively, in 5 days on doubling the parameter ϕ ; the measure of awareness instead increases by 331.5%. Note here that the parameter p has a larger effect in comparison to the other two parameters on the density of algae in the lake. Therefore, it plays a crucial role for the control of the density of algae in the lake. However, the impacts of γ and ϕ are also important.

To see the effect of the parameters p , q , k , α_0 , β_1 , β_{11} , β_{12} , θ_1 and α_1 on the value of E_c , the normalized forward sensitivity indices of E_c to these parameters are calculated and the values are evaluated for the set of parameter values given in

Fig. 4 Semi-relative sensitivity analysis. The parameter p leads to the highest reduction in algal density, but overall, the system is most sensitive to the parameter ϕ than to all the other parameters, as also a sizeable reduction in detritus is observed

Table 1. The normalized forward sensitivity index of a variable to a parameter is a ratio of the relative change in the variable to the relative change in the parameter, i.e., the normalized forward sensitivity index of a variable α that depends differentiably on a parameter w is defined as: $X_w^\alpha = \frac{\partial \alpha}{\partial w} \times \frac{w}{\alpha}$. The sensitivity indices are plotted in Fig. 6. This figure shows that when the parameters q , k , β_1 and θ_1 increase, keeping the other parameters constant, the value of E_c increases as they have positive indices. When the parameters p , α_0 , β_{11} , β_{12} and α_1 increase while keeping other parameters constant, the value of E_c decreases as they have negative indices. A low value of E_c enhances the possibility of the awareness to exceed it, thereby avoiding algal blooms. Therefore, it is imperative to prevent an increase in the parameters β_1 and θ_1 , while increasing α_1 should instead be fostered. Clearly these parameters are related to nutrients uptake and algal mortality. It is therefore highly improbable that they change in the direction that one would like. Any external measure aiming at reducing the former parameter and enhancing the latter should therefore be taken into serious consideration.

Fig. 5 Logarithmic sensitivity analysis. As in Fig. 4, the largest reduction in the algae population is provided by the parameter p , but the system is most sensitive to the parameter ϕ , for which a reduction in detritus and an increase in awareness are obtained

Fig. 6 Sensitivity indices of the critical threshold E_c with respect to all the model parameters. A lower value of E_c is preferable since it enhances the possibility of awareness to be above it and therefore to avoid algal blooms. Therefore, above all an increase in the parameters β_1 and θ_1 must be prevented by all means, while an increase in α_1 should instead be favored

6 Conclusions

Nutrient enrichment through agricultural runoff has become a great threat to water bodies. The problem of nutrient loading in lakes due to use of chemical fertilizers and pesticides in agricultural farms is increasing day by day. In this paper, a non-linear mathematical model is proposed and analyzed to study the control of algal bloom in a lake by imparting awareness among the farmers. It is assumed that the growth rate of measure of awareness among the farmers is proportional to the density of algae in the lake. Aware farmers adopt soil testing techniques and use the required amount of fertilizers in their farms and thus the inflow rate of nutrients reduces. The farmers also make efforts to remove detritus from the nearby water bodies to keep their source of water alive. The model analysis shows that as the inflow rate of nutrients in the lake increases, the equilibrium density of algae increases but it decreases upon increments in the values of efficacy and baseline awareness among the farmers.

The eradication of algal bloom from the lake depends on the critical value of baseline measure of awareness among the farmers. If it is smaller than the critical

value E_c , then algal bloom occurs but it can be avoided when the awareness exceeds the critical value. Numerical simulations show that the most critical parameters involved in this phenomenon are intrinsic to the algae dynamics, so that any external measure tending to alter them in the direction for which the critical value E_c can be more easily attained should be fostered. Further, sensitivity analysis shows that the efficacy of awareness has a significant impact on the mitigation of algal bloom from the lake. It is well known that the eradication of algal bloom is not possible without control of the inflow rate of nutrients in the lake. By imparting awareness among the farmers about algal bloom and increasing their knowledge about the benefits of optimal use of fertilizers in farms, the amount of nutrients runoff can be reduced, and consequently algal bloom can be controlled.

Acknowledgments The authors are thankful to the referees for their valuable suggestions, which have improved the quality of the paper. A.K. Misra thankfully acknowledges the help and support received from DST-Centre for Interdisciplinary Mathematical Sciences, Banaras Hindu University, Varanasi, India; P.K. Tiwari acknowledges the University Grants Commission, New Delhi, India, for their financial assistance in the form of Senior Research Fellowship (19-12/2010(i)EU-IV).

Appendix A

The Jacobian matrix J of system (1) is given by:

$$J = \begin{pmatrix} -a_{11} & a_{12} & \pi\delta & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & -\gamma S \\ 0 & \phi & 0 & -\phi_0 \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= \alpha_0 + \frac{\beta_1 \beta_{12} A}{(\beta_{12} + \beta_{11} N)^2}, \quad a_{12} = -\frac{\beta_1 N}{\beta_{12} + \beta_{11} N}, \quad a_{14} = -\frac{qpk}{(k + E)^2}, \\ a_{21} &= \frac{\theta_1 \beta_1 \beta_{12} A}{(\beta_{12} + \beta_{11} N)^2}, \quad a_{22} = \frac{\theta_1 \beta_1 N}{\beta_{12} + \beta_{11} N} - \alpha_1 - 2\beta_{10} A, \\ a_{32} &= \pi_1 \alpha_1 + 2\pi_2 \beta_{10} A, \quad a_{33} = -(\delta + \gamma E). \end{aligned}$$

The eigenvalues of matrix J_{E_1} (Jacobian matrix J evaluated at equilibrium E_1) are:

$$-\alpha_0, \frac{\theta_1 \beta_1 q}{\alpha_0} \left(1 - \frac{pE_0}{k + E_0} \right), -\alpha_1, -(\delta + \gamma E_0) \text{ and } -\phi_0.$$

Equilibrium E_1 is unstable (or stable) if condition (3) holds, i.e., whenever equilibrium E_2 is feasible (or is not feasible).

Let the Jacobian evaluated at equilibrium E_2 be denoted by J_{E_2} ; the corresponding characteristic equation is $\sum_{i=0}^4 A_{4-i} \lambda^i = 0$, where:

$$\begin{aligned}
 A_1 &= \phi_0 + \beta_{10} A^* + \delta + \gamma E^* + \alpha_0 + \frac{\beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2}, \\
 A_2 &= \phi_0 \beta_{10} A^* + (\phi_0 + \beta_{10} A^*) (\delta + \gamma E^*) + \frac{\theta_1 \beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} \frac{\beta_1 N^*}{\beta_{12} + \beta_{11} N^*} \\
 &\quad + (\phi_0 + \beta_{10} A^* + \delta + \gamma E^*) \left(\alpha_0 + \frac{\beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} \right), \\
 A_3 &= \phi_0 \beta_{10} A^* (\delta + \gamma E^*) + (\phi_0 + \delta + \gamma E^*) \frac{\theta_1 \beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} \frac{\beta_1 N^*}{\beta_{12} + \beta_{11} N^*} \\
 &\quad + \left(\alpha_0 + \frac{\beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} \right) \{ \phi_0 \beta_{10} A^* + (\delta + \gamma E^*) (\phi_0 + \beta_{10} A^*) \} \\
 &\quad - \pi \delta (\pi_1 \alpha_1 + 2\pi_2 \beta_{10} A^*) \frac{\theta_1 \beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} + \frac{qpk\phi}{(k + E^*)^2} \frac{\theta_1 \beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2}, \\
 A_4 &= \phi_0 (\delta + \gamma E^*) \left\{ \beta_{10} A^* \left(\alpha_0 + \frac{\beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} \right) + \frac{\theta_1 \beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} \frac{\beta_1 N^*}{\beta_{12} + \beta_{11} N^*} \right\} \\
 &\quad - \pi \delta \phi_0 (\pi_1 \alpha_1 + 2\pi_2 \beta_{10} A^*) \frac{\theta_1 \beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} + \pi \delta \phi \gamma S^* \frac{\theta_1 \beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2} \\
 &\quad + \frac{qpk\phi (\delta + \gamma E^*)}{(k + E^*)^2} \frac{\theta_1 \beta_1 \beta_{12} A^*}{(\beta_{12} + \beta_{11} N^*)^2}.
 \end{aligned}$$

Clearly, $A_1 > 0$ and it is found that $A_1 A_2 - A_3 > 0$. Thus, using Routh-Hurwitz criterion, all the roots of the characteristic equation are either negative or with negative real parts if A_4 and $A_3(A_1 A_2 - A_3) - A_1^2 A_4$ are positive. Thus, equilibrium E_2 is locally asymptotically stable if the conditions stated in (9) are satisfied. Hence the proof.

Appendix B

Let us consider the following positive definite function:

$$V = \frac{1}{2} (N - N^*)^2 + m_1 \left(A - A^* - \ln \frac{A}{A^*} \right) + \frac{m_2}{2} (S - S^*)^2 + \frac{m_3}{2} (E - E^*)^2,$$

where m_1 , m_2 and m_3 are positive constants to be chosen appropriately.

Now, differentiating the above equation with respect to time t along the solutions of model system (1) and rearranging the terms, we have:

$$\begin{aligned}
\frac{dV}{dt} = & - \left[\alpha_0 + \frac{\beta_1 \beta_{12} A}{(\beta_{12} + \beta_{11} N)(\beta_{12} + \beta_{11} N^*)} \right] (N - N^*)^2 \\
& - m_1 \beta_{10} (A - A^*)^2 - m_2 (\delta + \gamma E) (S - S^*)^2 - m_3 \phi_0 (E - E^*)^2 \\
& - \frac{kpq}{(k + E)(k + E^*)} (N - N^*) (E - E^*) + \pi \delta (N - N^*) (S - S^*) \\
& + \left[\frac{m_1 \theta_1 \beta_1 \beta_{12}}{(\beta_{12} + \beta_{11} N)(\beta_{12} + \beta_{11} N^*)} - \frac{\beta_1 N^*}{\beta_{12} + \beta_{11} N^*} \right] (N - N^*) (A - A^*) \\
& - m_2 \gamma S^* (S - S^*) (E - E^*) \\
& + m_2 [\pi_1 \alpha_1 + \pi_2 \beta_{10} (A + A^*)] (A - A^*) (S - S^*) \\
& + m_3 \phi (A - A^*) (E - E^*).
\end{aligned}$$

Choosing $m_1 = \frac{N^*}{\theta_1}$, we have:

$$\begin{aligned}
\frac{dV}{dt} = & - \left[\alpha_0 + \frac{\beta_1 \beta_{12} A}{(\beta_{12} + \beta_{11} N)(\beta_{12} + \beta_{11} N^*)} \right] (N - N^*)^2 \\
& - \frac{\beta_{10} N^*}{\theta_1} (A - A^*)^2 - m_2 (\delta + \gamma E) (S - S^*)^2 - m_3 \phi_0 (E - E^*)^2 \\
& - \frac{kpq}{(k + E)(k + E^*)} (N - N^*) (E - E^*) + \pi \delta (N - N^*) (S - S^*) \\
& - \frac{\beta_1 \beta_{11} N N^*}{(\beta_{12} + \beta_{11} N)(\beta_{12} + \beta_{11} N^*)} (N - N^*) (A - A^*) \\
& \quad - m_2 \gamma S^* (S - S^*) (E - E^*) \\
& + m_2 [\pi_1 \alpha_1 + \pi_2 \beta_{10} (A + A^*)] (A - A^*) (S - S^*) \\
& + m_3 \phi (A - A^*) (E - E^*).
\end{aligned}$$

$\frac{dV}{dt}$ can be made negative definite inside the region of attraction Ω , if the following conditions are satisfied:

$$\left[\frac{\beta_1 \beta_{11} q / \delta_m N^*}{(\beta_{12} + \beta_{11} q / \delta_m)(\beta_{12} + \beta_{11} N^*)} \right]^2 < \frac{4\alpha_0 \beta_{10} N^*}{9\theta_1}, \quad (17)$$

$$\left[\frac{pqk}{(k + E_0)(k + E^*)} \right]^2 < \frac{4m_3 \alpha_0 \phi_0}{9}, \quad (18)$$

$$\pi^2 \delta^2 < \frac{4m_2 \alpha_0 (\delta + \gamma E_0)}{9}, \quad (19)$$

$$m_2 [\pi_1 \alpha_1 + \pi_2 \beta_{10} (q/\delta_m + A^*)]^2 < \frac{4\beta_{10} N^* (\delta + \gamma E_0)}{9\theta_1}, \quad (20)$$

$$m_2 [\gamma S^*]^2 < \frac{4m_3 \phi_0 (\delta + \gamma E_0)}{9}, \quad (21)$$

$$m_3 \phi^2 < \frac{4\beta_{10} \phi_0 N^*}{9\theta_1}. \quad (22)$$

From inequalities (18) and (22), we can choose a positive value of m_3 provided

$$\left[\frac{pqk}{(k + E_0)(k + E^*)} \right]^2 < m_3 < \frac{16\alpha_0\beta_{10} \phi_0^2 N^*}{81\theta_1 \phi^2}.$$

Now, from inequalities (19)–(21), we can choose a positive value of m_2 provided

$$\left[\frac{\pi\delta}{\delta + \gamma E_0} \right]^2 < m_2 < \frac{16\alpha_0}{81} \min \left\{ \frac{m_3 \phi_0}{\gamma^2 S^{*2}}, \frac{\beta_{10} N^*}{\theta_1 [\pi_1 \alpha_1 + \pi_2 \beta_{10} (q/\delta_m + A^*)]^2} \right\}.$$

Thus, $\frac{dV}{dt}$ is negative definite inside the region of attraction, provided conditions (11)–(13) are satisfied. Hence the proof.

References

- Huang, Y., Sass, R.L., Sun, W., Zhang, W., Sass, Y.Y.: Reducing nitrogen fertilizer use to mitigate negative environmental impact in China. James A. Baker III Institute for Public Rice Production (2010)
- Carpenter, S.R., Kraft, C.E., Wright, R., He, X., Soranno, P.A., Hodgson, J.R.: Resilience and resistance of a lake phosphorus cycle before and after food web manipulation. *Am. Nat.* **140**(5), 781–798 (1992)
- Eutrophication – Oregon State University, <http://people.oregonstate.edu/muirp/eutrophi.htm>, retrieved on 22.06.2013.
- Nitrogen in Agriculture – European Commission – Europa, <http://ec.europa.eu/agriculture/envir/report/en/nitrogen/report.htm>, retrieved on 15.06.2013.
- Carpenter, S.R., Caraco, N.F., Correll, D.L., Howarth, R.W., Sharpley, A.N., Smith, V.H.: Nonpoint pollution of surface waters with phosphorus and nitrogen. *Ecol. Appl.* **8**, 559–568 (1998)
- DeAngelis, D.L., Bartell, S.M., Brenkert, A.L.: Effects of nutrient recycling and food-chain length on resilience. *Am. Nat.* **134**(5), 778–805 (1989)
- Frankie, U., Hutter, K., Johnk, K.: A physical-biological coupled model for algal dynamics in lakes. *Bull. Math. Biol.* **61**, 239–272 (1999)
- Jones, R.A., Lee, G.F.: Recent advances in assessing impact of phosphorus loads on eutrophication related water quality. *Water Res.* **16**, 503–515 (1982)
- Farouque, M.G., Takeya, H.: Farmers' perception of integrated soil fertility and nutrient management for sustainable crop production: a study of rural areas in Bangladesh. *J. Agri. Edu.* **48**(3), 111–122 (2007)
- Liu, Y., Cui, J.: The impact of media coverage on the dynamics of infectious diseases. *Int. J. Biomath.* **1**, 65–74 (2008)
- Misra, A.K., Sharma, A., Li, J.: A mathematical model for control of vector borne diseases through media campaigns. *Discrete Cont. Dyn. Sys. Ser. B.* **18**(7), 1909–1927 (2013)
- Misra, A.K., Verna, M.: Impact of environmental education on mitigation of carbon dioxide emissions: a modeling study. *Int. J. Global Warming*, **7**(4), 466–486 (2015)
- Hallegraeve, G.M.: A review of harmful algae blooms and the apparent global increase. *Phycologia* **32**, 79–99 (1993)

14. Gao, X., Ren, J.C., Zong, Z.X.: Studies on the nutrient energetics of *Microcystis aeruginosa*. Acta Sci. Nat. Univ. Pekin. **30**(4), 461–469 (1994)
15. Havens, K.E., James, R.T., East, T.L., Smith, V.H.: N:P ratios, light limitation, and cyanobacterial dominance in a subtropical lake impacted by non-point source nutrient pollution. Environ. Pollut. **122**, 379–390 (2003)
16. Schindler, D.W.: Recent advances in the understanding and management of eutrophication. Limnol. Oceanogr. **51**, 356–363 (2006)
17. Chen, S., Chen, X., Peng, Y., Peng, K.: A mathematical model of the effect of nitrogen and phosphorus on the growth of blue-green algae population. Appl. Math. Model. **33**, 1097–1106 (2009)
18. Baek, S.H., Shimode, S., Han, M.-S., Kikuchi, T.: Growth of dinoflagellates, *Ceratium furca* and *Ceratium fusus* in Sagami Bay, Japan: the role of nutrients. Harmful Algae **7**(6), 729–739 (2008)
19. Bartell, S.M., Lefebvre, G., Kaminski, G., Carreau, M., Campbell, K.R.: An ecosystem model for assessing ecological risks in Quebec rivers, lakes, and reservoirs. Ecol. Model. **124**, 43–67 (1999)
20. Misra, A.K.: Modeling the depletion of dissolved oxygen due to algal bloom in a lake by taking Holling type III interaction. Appl. Math. Comp. **217**, 8367–8376 (2011)
21. Shukla, J.B., Misra, A.K., Chandra, P.: Mathematical modeling and analysis of the depletion of dissolved oxygen in eutrophied water bodies affected by organic pollutants. Nonlinear Analysis: RWA **9**, 1851–1865 (2008)
22. Jayaweera, M., Asaeda, T.: Modeling of biomanipulation in shallow, eutrophic lakes: an application to Lake Bleiswijkse Zoom, The Netherlands. Ecol. Model. **85**, 113–127 (1996)
23. Jørgensen, S.E.: Fundamentals of Ecological Modeling. Elsevier Science Publishers B.V., Amsterdam (1988)
24. Voinov, A.A., Tonkikh, A.P.: Qualitative model of eutrophication in macrophyte lakes. Ecol. Model. **35**, 211–226 (1987)
25. Misra, A.K.: Mathematical modeling of the survival of biological species in eutrophied water bodies. Proc. Nat. Acad. Sci. India Sect A **78**(IV), 331–340 (2008)
26. Pauer, J.J., Auer, M.T.: Nitrification in the water column and sediments of a hypereutrophic lake and adjoining river system. Water Res. **34**(4), 1247–1254 (2000)
27. Peeters, J.H.C., Eilers, P.H.C.: The relationship between light intensity and photosynthesis. A simple mathematical model. Hydrobiol. Bull. **12**, 134–136 (1978)
28. Rinaldi, S., Soncini-sessa, R., Stehfest, H., Tamura, H.: Modeling and Control of River Quality, p. 380. McGraw-Hill Inc, U.K (1979)
29. Smith, I.R.: A simple theory of algal deposition. Freshwater Biol. **445**–449 (1982)
30. Hallam, T.G.: Structural sensitivity of grazing formulations in nutrient controlled plankton models. J. Math. Biol. **5**, 269–280 (1978)
31. Chakraborty, S., Chatterjee, S., Venturino, E., Chattopadhyay, J.: Recurring plankton bloom dynamics modeled via toxin-producing phytoplankton. J. Biol. Phys. **33**, 271–290 (2007)
32. Chakraborty, S., Roy, S., Chattopadhyay, J.: Nutrient-limited toxin production and the dynamics of two phytoplankton in culture media: a mathematical model. Ecol. Model. **213**, 191–201 (2008)
33. Amemiya, T., Enomoto, T., Rossberg, A.G., Yamamoto, T., Inamori, Y., Itoh, K.: Stability and dynamical behavior in a lake-model and implications for regime shifts in real lakes. Ecol. Model. **206**, 54–62 (2007)
34. Shukla, J.B., Misra, A.K., Chandra, P.: Modeling and analysis of the algal bloom in a lake caused by discharge of nutrients. Appl. Math. Comp. **196**(2), 782–790 (2008)
35. Misra, A.K.: Modeling the depletion of dissolved oxygen in a lake due to submerged macrophytes. Nonlinear Anal.: Model. Control **15**(2), 185–198 (2010)
36. Bortz, D.M., Nelson, P.W.: Sensitivity analysis of a nonlinear lumped parameter model of HIV infection dynamics. Bull. Math. Biol. **66**, 1009–1026 (2004)
37. Misra, A.K., Verma, M.: Modeling the impact of mitigation options on methane abatement from rice fields. Mitig. Adapt. Strateg. Glob. Change **19**, 927–945 (2014)