Gravitational vacuum condensate stars

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A new final state of gravitational collapse is proposed. By extending the concept of Bose–Einstein condensation to gravitational systems, a cold, dark, compact object with an interior de Sitter condensate $p_v = -\rho_v$ and an exterior Schwarzschild geometry of arbitrary total mass M is constructed. These regions are separated by a shell with a small but finite proper thickness ℓ of fluid with equation of state $p = +\rho$, replacing both the Schwarzschild and de Sitter classical horizons. The new solution has no singularities, no event horizons, and a global time. Its entropy is maximized under small fluctuations and is given by the standard hydrodynamic entropy of the thin shell, which is of the order $k_B \ell M c / \hbar$, instead of the Bekenstein–Hawking entropy formula, $S_{BH} = 4\pi k_B G M^2 / \hbar c$. Hence, unlike black holes, the new solution is thermodynamically stable and has no information paradox.

Cold superdense stars with a mass greater than some critical value undergo rapid gravitational collapse. Due to the impossibility of halting this collapse by any known equation of state for high-density matter, a kind of consensus has developed that a collapsing star must inevitably arrive in a finite proper time at a singular condition called a black hole.

The characteristic feature of a black hole is its event horizon, the null surface of finite area at which outwardly directed light rays hover indefinitely. For simplicity, consider an uncharged, nonrotating Schwarzschild black hole with the static, spherically symmetric line element,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad [1]$$

The functions $f(r)$ and $h(r)$ are equal in this case, and

$$f_s(r) = h_s(r) = 1 - \frac{r_s}{r}, \quad r_s \equiv \frac{2GM}{c^2}. \quad [2]$$

At the event horizon, $r = r_s$, the metric (Eq. 1) becomes singular. Since local curvature invariants remain regular at $r = r_s$, a test particle falling through the horizon experiences nothing catastrophic there (if M is large enough), and it is possible to find regular coordinates that analytically extend the exterior Schwarzschild geometry through the event horizon into the interior region.

It is important to recognize that this mathematical procedure of analytic continuation through a null hypersurface involves a physical assumption, namely that the stress-energy tensor is vanishing there. Even in the classical theory, the hyperbolic character of the Einstein equations allows generically for sources and discontinuities on the horizon that would violate this assumption. Whether such analytic continuation is mandatory, or even permissible in a more complete theory taking quantum effects into account, is still less certain.

Nonanalytic behavior is typical of quantum many-body systems at a phase transition. Quantum systems also exhibit macroscopic coherence effects that do not depend on local forces becoming large. Thus, the fact that the tidal forces on classical test bodies falling through the event horizon are arbitrarily weak (proportional to $M/r_s^3 \sim 1/M^2$) for an arbitrarily large black hole does not imply that quantum effects must be unimportant there. Electron waves restricted to the region outside an Aharonov–

Bohm solenoid, where the electromagnetic field strength vanishes and no classical forces whatsoever exist, nevertheless experience a shift in their interference fringes. Qualitatively new effects such as these arise because quantum matter has extended wave-like properties, with many-body statistical correlations that can in no way be captured by consideration of point-like test particles responding only to local forces.

A photon with asymptotic frequency ω and energy $\hbar\omega$ far from the black hole has a local energy $\hbar\omega f^{-1/2}$, which diverges at the event horizon. Unlike classical test particles, when $\hbar \neq 0$, such extremely blue-shifted photons are necessarily present in the vacuum as virtual quanta. Their effects on the geometry depend on the quantum state of the vacuum, defined by boundary conditions on the wave equation in a nonlocal way over all of space, and $\langle T_{ab} \rangle$ may be large at $r = r_s$, notwithstanding the smallness of the local classical curvature there (1–3). Because the limits $r \rightarrow r_s$ and $\hbar \rightarrow 0$ do not commute, nonanalytic behavior near the event horizon, quite different from that in the strictly classical ($\hbar \equiv 0$) situation is possible in the quantum theory.

The nonanalytic nature of the classical limit $\hbar \rightarrow 0$ may be seen also in the thermodynamic analogy for the laws of black hole mechanics. Pursuing the analogy of these classical laws with thermodynamics, Bekenstein suggested that black hole event horizons carry an intrinsic entropy proportional to their area, $4\pi r_s^2$ (4–6). To have the units of entropy, the horizon area must be multiplied by a constant with units of $c^3 k_B / \hbar G$. When Hawking found that a flux of radiation could be emitted from a black hole with a well defined temperature, $T_H = \hbar c^3 / 8\pi k_B G M$ (7, 8), the thermodynamic analogy of area with entropy received some support, since the conservation of energy in the system can be written in the form,

$$dE = T_H dS_{BH} = \frac{\hbar c^3}{8\pi G k_B M} d\left(\frac{4\pi k_B G M^2}{\hbar c}\right), \quad [3]$$

suggestive of the first law of thermodynamics.

The curious feature of Eq. 3 is that $\hbar$ cancels out and plays no dynamical role. The identification of S_{BH} with the entropy of the black hole is founded on the dynamics of purely classical relativity (i.e., $\hbar = 0$), through the area law of refs. (9–11). If this identification of rescaled classical area with entropy is to be valid in the quantum theory, then the limit $\hbar \rightarrow 0$ (with M fixed), which yields an arbitrarily low temperature, would assign to the black hole an arbitrarily large entropy, completely unlike the zero-temperature limit of any other cold quantum system.

Closely related to this paradoxical result is the fact, also pointed out by Hawking (12), that a temperature inversely proportional to $M = E/c^2$ implies a negative heat capacity: $dE/dT = -(\hbar c^5 / 8\pi G k_B) T^{-2} < 0$. However, a negative heat capacity is impossible for any system in stable equilibrium, since the heat capacity is proportional to the square of the energy fluctuations of the system. If this quantity is negative, the system cannot be in stable equilibrium at all, and the applicability of equilibrium thermodynamic relations is questionable. Attempts

Abbreviation: GBEC, gravitational Bose–Einstein condensate.

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to evaluate the entropy of an uncharged black hole directly from statistical considerations ($S = k_B \ln \Omega$) produce divergent results due to the unbounded number of wave modes infinitely close to the event horizon (13, 14). The entropy of fields in a fixed Schwarzschild background would also be expected to scale linearly with the number N of independent fields, whereas S_{BH} is independent of N .

Ignoring these myriad difficulties and nevertheless interpreting Eq. 3 literally as a thermodynamic relation implies that a black hole has an enormous entropy, $S_{\text{BH}} \approx 10^{77} k_B (M/M_\odot)^2$, far in excess of a typical stellar progenitor with a comparable mass. The associated “information paradox” and implied violation of unitarity has been characterized as so serious as to require an alteration in the principles of quantum mechanics (15, 16).

This paradoxical state of affairs arising from an area law originally derived in a strictly classical framework, together with the cancellation of $\hbar$ in Eq. 3, suggest that Eq. 3 may not be a generally valid quantum relation at all, but only the (improper) classical limit of such a relation. The cancellation of $\hbar$ here is reminiscent of a similar cancellation in the energy density of modes of the radiation field in thermodynamic equilibrium, $\hbar\omega n(\omega)\omega^2 d\omega \rightarrow k_B T \omega^2 d\omega$ in the Rayleigh–Jeans limit of very low frequencies, $\hbar\omega \ll k_B T$. Improper extension of this low-energy relation from Maxwell theory into the quantum high-frequency regime leads to a UV catastrophe, similar to that encountered in the counting of wave modes near an event horizon. Conversely, treating the atoms in a solid as classical point particles leads to equipartition of energy and an incorrect prediction (the Dulong–Petit law) of constant specific heat for crystals at low temperatures. Both of these difficulties precipitated and were resolved by the rise of a new quantum theory of matter and radiation.

In the illustrative case of the Einstein single-frequency crystal, the difference between the heat capacity $C_V(T)$ at the temperature T and its Dulong–Petit high-temperature limit, $C_V(\infty)$ is $\Delta C_V(T) = C_V(T) - C_V(\infty)$, which has the leading high-temperature behavior $\Delta C_V(T) \propto -T^{-2}$, exactly the same temperature dependence as the negative Bekenstein–Hawking heat capacity of the Schwarzschild black hole. When the term $C_V(\infty)$ arising from the discrete atomistic degrees of freedom is restored, the true heat capacity $C_V(T)$ of the crystal is positive. Application of the Einstein energy fluctuation formula to a black hole shows that the Bekenstein–Hawking heat capacity is identical with what one would obtain for a large number of weakly interacting massive bosons, each with a mass of order of the Planck mass, close to the Dulong–Petit limit, but where the term analogous to $C_V(\infty)$ has been dropped (17).

A related consideration suggesting that the Hawking emission depends on an improper UV extrapolation of classical waves into the quantum regime comes from examining the origin of these waves in the past. Although the Hawking temperature T_H is very small for a large black hole, simple kinematic considerations show that the Hawking modes observed at a time t long after the formation of the hole originate as incoming vacuum modes at a greatly blue-shifted frequency, $\omega_{\text{in}} \sim (k_B T_H / \hbar) \exp(ct/2r_s)$. In other words, the calculation of the Hawking flux at late times assumes that the local properties of the fixed space-time geometry are known with arbitrarily high precision, even at exponentially sub-Planck length and time scales, where one would normally question the semiclassical approximation and indeed the existence of any well defined metric at all. The contribution to the local stress-energy tensor, $\langle T_a^b \rangle$ of these highly blue-shifted modes diverges at the horizon, which is another form of the UV catastrophe. Only an exact balance with time-reversed highly blue-shifted ingoing modes can cancel this divergence in the precisely tuned Hartle–Hawking thermal state (18). This state is in any case unstable because of its negative heat capacity. In any other thermal state, with $T \neq T_H$, such as would be

expected to arise from fluctuations, the divergence at the horizon is not canceled, and one must expect a substantial quantum backreaction on the geometry there, which invalidates the basic assumption of an arbitrarily accurately known, fixed, classical Schwarzschild background, arbitrarily close to $r = r_s$.

Note that since it transforms as a tensor under coordinate transformations, a large local $\langle T_a^b \rangle$ near $r = r_s$ is perfectly consistent with the Equivalence Principle, except in its strongest possible form, which would admit no nonlocal effects of any kind, including those of macroscopic coherence and entanglement, known to exist in both relativistic and nonrelativistic quantum many-body systems.

In earlier models of backreaction, the black hole was immersed in a Hawking radiation atmosphere, with an effective equation of state, $p = \kappa\rho$. It was found that, due to the blue-shift effect, the backreaction of such an atmosphere on the metric near $r = r_s$ is enormous, with the interior region quite different from the vacuum Schwarzschild solution and with a large entropy of the order of S_{BH} from the fluid alone (19, 20). In fact, $S = 4(\kappa + 1)/(7\kappa + 1)S_{\text{BH}}$, becoming equal to the Bekenstein–Hawking entropy for $\kappa = 1$. Aside for accounting for the entropy S_{BH} from purely standard hydrodynamical relations, this result suggests that the maximally stiff equation of state consistent with the causal limit may play a role in the quantum theory of fully collapsed objects. Despite such very suggestive features indicating the importance of backreaction on the geometry, these models cannot be viewed as a satisfactory solution to the final state of the collapse problem, since they involve huge (Planckian) energy densities near r_s and a negative mass singularity at $r = 0$. The negative mass singularity arises because a repulsive core is necessary to counteract the self-attractive gravitation of the dense relativistic fluid with positive energy.

Recently another proposal for incorporating quantum effects near the horizon has been made (21–23), with a critical surface of a quantum phase transition replacing the classical horizon and the interior replaced by a region with equation of state, $p = -\rho < 0$. Equivalent to a positive cosmological term in Einstein’s equations, this equation of state was first proposed for the final state of gravitational collapse by Gliner (24) and considered in a cosmological context by Sakharov (25). It violates the strong energy condition $\rho + 3p \geq 0$ used in proving the classical singularity theorems. As is now well known because of the observations of distant supernovae implying an accelerating universe, the geodesic world lines of test particles in a region of space-time where $\rho + 3p < 0$ diverge from each other, mimicking the effects of a repulsive gravitational potential. Thus, such a region free of any singularities in the interior region $r < r_s$ can replace the unphysical negative mass singularity encountered in the pure $\kappa = 1$ fluid model.

Based on these considerations, in this article we show that a consistent static solution of Einstein’s equations can be constructed, with the critical surface of refs. 21 and 23 replaced by a thin shell of ultrarelativistic fluid with equation of state $p = \rho$. Because of its vacuum energy interior with $p = -\rho$ the new solution is stable and free of all singularities. Its entropy is a local maximum of the hydrodynamic entropy, whose modest value given by Eqs. 27 and 28 below is easily attainable in a physical collapse process from a stellar progenitor with a comparable mass.

The model we arrive at is that of a low-temperature condensate of weakly interacting massive bosons trapped in a self-consistently generated cavity, whose boundary layer can be described by a thin shell. The assumption required for a solution of this kind to exist is that gravity, i.e., space-time itself, must undergo a quantum vacuum rearrangement phase transition in the vicinity of $r = r_s$. In this region quantum zero-point fluctuations dominate the stress-energy tensor and become large enough to influence the geometry, regardless of the composition

of the matter undergoing the gravitational collapse. As the causal limit $p = \rho$ is reached, the interior space-time becomes unstable to the formation of a gravitational Bose–Einstein condensate (GBEC), described by a non-zero macroscopic order parameter in the effective low-energy description. Since a condensate is a single macroscopic quantum state with zero entropy, a model of a cold condensate repulsive core as the stable, nonsingular endpoint of gravitational collapse provides a resolution of the information paradox, which is completely consistent with quantum principles (17, 26). The interior and exterior regions are separated by a thin surface layer near $r = r_s$ where the vacuum condensate disorders. Any entropy in the configuration can reside only in the excitations of this boundary layer. A suggestion for the effective theory incorporating the effects of quantum anomalies that describes this fully disordered phase, where the role of the order parameter is played by the conformal part of the metric has been presented elsewhere (ref. 27 and references therein). For a recent review of investigations of other nonsingular quasi-black hole models see ref. 28.

The Vacuum Condensate Model

In an effective mean field treatment for a perfect fluid at rest in the coordinates (Eq. 1), any static, spherically symmetric collapsed object must satisfy the Einstein equations (in units where $c = 1$),

$$-G_t^t = \frac{1}{r^2} \frac{d}{dr} [r(1-h)] = -8\pi G T_t^t = 8\pi G \rho, \quad [4]$$

$$G_r^r = \frac{h}{rf} \frac{df}{dr} + \frac{1}{r^2} (h-1) = 8\pi G T_r^r = 8\pi G p, \quad [5]$$

together with the conservation equation,

$$\nabla_a T_r^a = \frac{dp}{dr} + \frac{\rho+p}{2f} \frac{df}{dr} + 2 \frac{p-p_\perp}{r} = 0, \quad [6]$$

which ensures that the other components of the Einstein equations are satisfied. In the general spherically symmetric situation the tangential pressure, $p_\perp \equiv T_\theta^\theta = T_\phi^\phi$, is not necessarily equal to the radial normal pressure $p = T_r^r$. However, for purposes of developing the simplest possibility first, we restrict ourselves in this article to the isotropic case where $p_\perp = p$. In that case, we have three first-order equations for four unknown functions of r , namely, f , h , ρ , and p . The system becomes closed when an equation of state for the fluid, relating p and ρ is specified. Because of the considerations of the introduction we allow for three different regions with the three different eqs. of state,

$$\begin{aligned} \text{I. Interior:} & \quad 0 \leq r < r_1, \quad \rho = -p, \\ \text{II. Thin shell:} & \quad r_1 < r < r_2, \quad \rho = +p, \\ \text{III. Exterior:} & \quad r_2 < r, \quad \rho = p = 0. \end{aligned} \quad [7]$$

In the interior region $\rho = -p$ is a constant from Eq. 6. Let us call this constant $\rho_v = 3H_0^2/8\pi G$. If we require that the origin is free of any mass singularity then the interior is determined to be a region of de Sitter space-time in static coordinates, i.e.,

$$\text{I. } f(r) = Ch(r) = C(1 - H_0^2 r^2), \quad 0 \leq r < r_1, \quad [8]$$

where C is an arbitrary constant, corresponding to the freedom to redefine the interior time coordinate.

The unique solution in the exterior vacuum region that approaches flat space-time as $r \rightarrow \infty$ is a region of Schwarzschild space-time (Eq. 2), namely,

$$\text{III. } f(r) = h(r) = 1 - \frac{2GM}{r}, \quad r_2 < r. \quad [9]$$

The integration constant M is the total mass of the object.

The only nonvacuum region is region II. Let us define the dimensionless variable w by $w \equiv 8\pi G r^2 p$, so that Eqs. 4–6 with $\rho = p$ may be recast in the form,

$$\frac{dr}{r} = \frac{dh}{1-w-h}, \quad [10]$$

$$\frac{dh}{h} = -\frac{1-w-h}{1+w-3h} \frac{dw}{w}, \quad [11]$$

together with $pf \propto wf/r^2$ a constant. Eq. 10 is equivalent to the definition of the (rescaled) Tolman mass function by $h = 1 - 2m(r)/r$ and $dm(r) = 4\pi G \rho r^2 dr = w dr/2$ within the shell. Eq. 11 can be solved only numerically in general. However, it is possible to obtain an analytic solution in the thin shell limit, $0 < h \ll 1$, for in this limit we can set h to zero on the right side of Eq. 11 to leading order, and integrate it immediately to obtain

$$h \equiv 1 - \frac{2m}{r} \approx \varepsilon \frac{(1+w)^2}{w} \ll 1 \quad [12]$$

in region II, where ε is an integration constant. Because of the condition $h \ll 1$, we require $\varepsilon \ll 1$, with w of order unity. Making use of Eqs. 10, 11, and 12, we have

$$\frac{dr}{r} \approx -\varepsilon dw \frac{(1+w)}{w^2}. \quad [13]$$

Because of the approximation $\varepsilon \ll 1$, the radius r hardly changes within region II, and dr is of the order εdw . The final unknown function f is given by $f = (r/r_1)^2 (w_1/w) f(r_1) \approx (w_1/w) f(r_1)$ for small ε , showing that f is also of the order ε everywhere within region II and its boundaries.

At each of the two interfaces at $r = r_1$ and $r = r_2$ the induced 3D metric must be continuous. Hence, r and $f(r)$ are continuous at the interfaces, and

$$f(r_2) \approx \frac{w_1}{w_2} f(r_1) = \frac{Cw_1}{w_2} (1 - H_0^2 r_1^2) = 1 - \frac{2GM}{r_2}. \quad [14]$$

To leading order in $\varepsilon \ll 1$ this relation implies that

$$r_1 \approx \frac{1}{H_0} \approx 2GM \approx r_2. \quad [15]$$

Thus, the interfaces describing the phase boundaries at r_1 and r_2 are very close to the classical event horizons of the de Sitter interior and the Schwarzschild exterior.

The significance of $0 < \varepsilon \ll 1$ is that both f and h are of the order ε in region II, but are nowhere vanishing. Hence, there is no event horizon, and t is a global time. A photon experiences a very large, $O(\varepsilon^{-1/2})$ but finite blue shift in falling into the shell from infinity. The proper thickness of the shell between these interface boundaries is

$$\ell = \int_{r_1}^{r_2} \frac{dr}{\sqrt{h}} \approx r_s \sqrt{\varepsilon} \int_{w_2}^{w_1} dw w^{-3/2} \sim \sqrt{\varepsilon} r_s, \quad [16]$$

and very small for $\varepsilon \rightarrow 0$. Because of Eq. 15 the constant vacuum energy density in the interior is just the total mass M divided by the volume, i.e., $\rho_v \approx 3M/4\pi r_s^3$, to leading order in ε . The energy within the shell itself,

$$E_{\text{II}} = 4\pi \int_{r_1}^{r_2} \rho r^2 dr \approx \varepsilon M \int_{w_2}^{w_1} \frac{dw}{w} (1+w) \sim \varepsilon M \quad [17]$$

is extremely small.

We can estimate the size of ε and ℓ by consideration of the expectation value of the quantum stress tensor in the static exterior Schwarzschild space-time. In the static vacuum state corresponding to no incoming or outgoing quanta at large distances from the object, i.e., the Boulware vacuum (1, 2), the stress tensor near $r = r_s$ is the negative of the stress tensor of massless radiation at the blue-shifted temperature, $T_{\text{loc}} = T_H / \sqrt{f(r)}$ and diverges as $T_{\text{loc}}^4 \sim f^{-2}(r)$ as $r \rightarrow r_s$. The location of the outer interface occurs at an r where this local stress-energy $\propto M^{-4}\varepsilon^{-2}$, becomes large enough to affect the classical Schwarzschild curvature $\sim M^{-2}$, i.e., when

$$\varepsilon \sim \frac{M_{\text{Pl}}}{M} \approx 10^{-38} \left(\frac{M_{\odot}}{M} \right), \quad [18]$$

where M_{Pl} is the Planck mass $\sqrt{\hbar c/G} \approx 2 \times 10^{-5}$ g. Thus, ε is indeed very small for a stellar mass object, justifying the approximation *a posteriori*. With this semiclassical estimate for ε we find

$$\ell \sim \sqrt{L_{\text{Pl}} r_s} \approx 3 \times 10^{-14} \left(\frac{M}{M_{\odot}} \right)^{\frac{1}{2}} \text{ cm}. \quad [19]$$

Although still microscopic, the thickness of the shell is very much larger than the Planck scale $L_{\text{Pl}} \approx 2 \times 10^{-33}$ cm. The energy density and pressure in the shell are of the order M^{-2} and far below Planckian for $M \gg M_{\text{Pl}}$, so that the geometry can be described reliably by Einstein's equations in both regions I and II.

Although $f(r)$ is continuous across the interfaces at r_1 and r_2 , the discontinuity in the equations of state does lead to discontinuities in $h(r)$ and the first derivative of $f(r)$, in general. Defining the outwardly directed unit normal vector to the interfaces, $n^b = \delta_r^b \sqrt{h(r)}$, and the extrinsic curvature $K_a^b = \nabla_a n^b$, the Israel junction conditions determine the surface stress energy η and surface tension σ on the interfaces to be given by the discontinuities in the extrinsic curvature through (29):

$$[K_t^t] = \left[\frac{\sqrt{h}}{2f} \frac{df}{dr} \right] = 4\pi G(\eta - 2\sigma), \quad [20]$$

$$[K_\theta^\theta] = [K_\phi^\phi] = \left[\frac{\sqrt{h}}{r} \right] = -4\pi G\eta. \quad [21]$$

Since h and its discontinuities are of the order ε , the energy density in the surfaces $\eta \sim \sqrt{\varepsilon}$, whereas the surface tensions are of the order $\varepsilon^{-1/2}$. The simplest possibility for matching the regions is to require that the surface energy densities on each interface vanish. From Eq. 21 this condition implies that $h(r)$ is also continuous across the interfaces, which yields the relations,

$$h(r_1) = 1 - H_0^2 r_1^2 \approx \varepsilon \frac{(1 + w_1)^2}{w_1}, \quad [22]$$

$$h(r_2) = 1 - \frac{2GM}{r_2} \approx \varepsilon \frac{(1 + w_2)^2}{w_2}, \quad [23]$$

$$\frac{f(r_2)}{h(r_2)} = 1 \approx \frac{w_1}{w_2} \frac{f(r_1)}{h(r_1)} = C \left(\frac{1 + w_1}{1 + w_2} \right)^2. \quad [24]$$

From Eq. 13 $dw/dr < 0$, so that $w_2 < w_1$ and $C < 1$. In this case of vanishing surface energies $\eta = 0$, the surface tensions are determined by Eqs. 20 and 21 to be

$$\sigma_1 \approx -\frac{1}{32\pi G^2 M} \frac{(3 + w_1)(w_1)}{(1 + w_1)^2} \frac{1}{\varepsilon}, \quad [25]$$

$$\sigma_2 \approx \frac{1}{32\pi G^2 M} \frac{w_2}{(1 + w_2)} \left(\frac{w_2}{\varepsilon} \right)^{\frac{1}{2}}. \quad [26]$$

to leading order in ε at r_1 and r_2 , respectively. The negative surface tension at the inner interface is equivalent to a positive tangential pressure, which implies an outwardly directed force on the thin shell from the repulsive vacuum within. The positive surface tension on the outer interfacial boundary corresponds to the more familiar case of an inwardly directed force exerted on the thin shell from without.

The entropy of the configuration may be obtained from the Gibbs relation, $p + \rho = sT + n\mu$, if the chemical potential μ is known in each region. In the interior region I, $p + \rho = 0$ and the excitations are the usual transverse gravitational waves of the Einstein theory in de Sitter space. Hence, the chemical potential μ may be taken to vanish and the interior has zero entropy density $s = 0$, consistent with a single macroscopic condensate state, $S = k_B \ln \Omega = 0$ for $\Omega = 1$. In region II there are several alternatives depending on the nature of the fundamental excitations there. The $p = \rho$ equation of state may come from thermal excitations with negligible μ or it may come from a conserved number density n of gravitational quanta at zero temperature. Let us consider the limiting case of vanishing μ first.

If the chemical potential can be neglected in region II, then the entropy of the shell is obtained from the equation of state, $p = \rho = (a^2/8\pi G)(k_B T/\hbar)^2$. The T^2 temperature dependence follows from the Gibbs relation with $\mu = 0$, together with the local form of the first law $d\rho = Tds$. The Newtonian constant G has been introduced for dimensional reasons, and a is a dimensionless constant. Using the Gibbs relation again the local specific entropy density $s(r) = a^2 k_B^2 T(r)/4\pi\hbar^2 G = a(k_B/\hbar)(p/2\pi G)^{1/2}$ for local temperature $T(r)$. Converting to our previous variable w , we find $s = (ak_B/4\pi\hbar Gr) w^{1/2}$ and the entropy of the fluid within the shell is

$$S = 4\pi \int_{r_1}^{r_2} s r^2 \frac{dr}{\sqrt{h}} \approx \frac{ak_B r_s^2}{\hbar G} \sqrt{\varepsilon} \ln \left(\frac{w_1}{w_2} \right), \quad [27]$$

to leading order in ε . By using 16 and 19, this is

$$S \sim ak_B \frac{M\ell}{\hbar} \sim 10^{57} ak_B \left(\frac{M}{M_{\odot}} \right)^{\frac{3}{2}} \ll S_{\text{BH}}. \quad [28]$$

The maximum entropy of the shell and therefore of the entire configuration is some 20 orders of magnitude smaller than the Bekenstein–Hawking entropy for a solar mass object and is of the same order of magnitude as a typical progenitor of a few solar masses. The scaling of Eq. 28 with $M^{3/2}$ is also the same as that for supermassive stars with $M > 100 M_{\odot}$, whose pressure is dominated by radiation pressure (30). Thus, the formation of the GBEC star from either a solar mass or supermassive stellar progenitor does not require an enormous generation or removal of entropy, and, unlike a black hole, a GBEC star does not suffer from any information paradox.

Because of the absence of an event horizon, the GBEC star does not emit Hawking radiation. Since w is of the order unity in the shell whereas $r \approx r_s$, the *local* temperature of the fluid within the shell is of the order $T_H \sim \hbar/k_B GM$. The strongly red-shifted temperature observed at infinity is of the order $\sqrt{\varepsilon} T_H$, which is very small indeed. Hence, the rate of any thermal emission from the shell is negligible.

If we do allow for a positive chemical potential within the shell, $\mu > 0$, then the temperature and entropy estimates just given become upper bounds, and it is possible to approach a zero-temperature ground state with zero entropy. This nonsingular

final state of gravitational collapse is a cold, completely dark object sustained against any further collapse solely by quantum zero-point pressure.

Stability

To be a physically realizable endpoint of gravitational collapse, any quasi-black hole candidate must be stable (17). Since only the region II is nonvacuum, with a “normal” fluid and a positive heat capacity, it is clear that the solution is thermodynamically stable. The most direct way to demonstrate this stability is to work in the microcanonical ensemble (in the case of zero chemical potential) with fixed total M and to show that the entropy functional,

$$S = \frac{ak_B}{\hbar G} \int_{r_1}^{r_2} r dr \left(2 \frac{dm}{dr} \right)^{\frac{1}{2}} \left(1 - \frac{2m(r)}{r} \right)^{-\frac{1}{2}}, \quad [29]$$

is maximized under all variations of $m(r)$ in region II with the endpoints r_1 and r_2 , or equivalently w_1 and w_2 fixed.

The first variation of this functional with the endpoints r_1 and r_2 fixed vanishes, i.e., $\delta S = 0$ by the Einstein Eqs. 4 and 5 for a static, spherically symmetric star. Thus, any solution of Eqs. 4–6 is guaranteed to be an extremum of S (31). This is also consistent with regarding Einstein’s equations as a form of hydrodynamics, strictly valid only for the long-wavelength, gapless excitations in gravity. In the context of a hydrodynamic treatment, thermodynamic stability is also a necessary and sufficient condition for the dynamical stability of a static, spherically symmetric solution of Einstein’s equations (31).

The second variation of Eq. 29 is

$$\delta^2 S = \frac{ak_B}{\hbar G} \int_{r_1}^{r_2} r dr \left(2 \frac{dm}{dr} \right)^{-\frac{3}{2}} h^{-\frac{1}{2}} \left\{ - \left[\frac{d(\delta m)}{dr} \right]^2 + 2 \frac{(\delta m)^2}{r^2 h^2} \frac{dm}{dr} \left(1 + 2 \frac{dm}{dr} \right) \right\}, \quad [30]$$

when evaluated on the solution. Associated with this quadratic form in δm is a second-order linear differential operator $\mathcal{L}$ of the Sturm–Liouville type, namely,

$$\mathcal{L}\chi \equiv \frac{d}{dr} \left\{ rh^{-\frac{1}{2}} \left(\frac{dm}{dr} \right)^{-\frac{3}{2}} \frac{d\chi}{dr} \right\} + \frac{2}{r} h^{-\frac{5}{2}} \left(\frac{dm}{dr} \right)^{-\frac{1}{2}} \left(1 + 2 \frac{dm}{dr} \right) \chi. \quad [31]$$

This operator possesses two solutions satisfying $\mathcal{L}\chi_0 = 0$, obtained by variation of the classical solution, $m(r; r_1, r_2)$ with respect to the parameters (r_1, r_2) . Since these correspond to varying the positions of the interfaces, χ_0 does not vanish at (r_1, r_2) and neither function is a true zero mode. For example, it is easily verified that one solution is $\chi_0 = 1 - w$, from which the second linearly independent solution $(1 - w) \ln w + 4$ may be obtained. For any linear combination of these we may set $\delta m \equiv \chi_0 \psi$, where ψ does vanish at the endpoints and insert this into the second variation (Eq. 30). Integrating by parts, using the vanishing of δm at the endpoints, and $\mathcal{L}\chi_0 = 0$ gives

$$\delta^2 S = - \frac{ak_B}{\hbar G} \int_{r_1}^{r_2} r dr h^{-\frac{1}{2}} \left(2 \frac{dm}{dr} \right)^{-\frac{3}{2}} \chi_0^2 \left(\frac{d\psi}{dr} \right)^2 < 0. \quad [32]$$

Thus, the entropy of the solution is maximized with respect to radial variations that vanish at the endpoints, i.e., those that do not vary the positions of the interfaces. Perturbations of the fluid

in region II that are not radially symmetric decrease the entropy even further than Eq. 32, which demonstrates that the solution is stable to all small perturbations keeping the endpoints fixed.

Allowing for endpoint variations as well requires the inclusion of the vacuum stress, $\langle T_a^b \rangle$ in the vicinity of the interfaces, which fixed ε by the estimate (Eq. 18). It is clear that the vacuum $\langle T_a^b \rangle$ must be included in a more complete model for another reason. The general stress-energy in a spherically symmetric, static space-time has three components, namely, ρ , p , and $p_\perp$. We have set $p_\perp = p$ and restricted ourselves to only two isotropic equations of state $p = -\rho$ and $p = +\rho$ only for simplicity, to illustrate the general features of a nonsingular solution to the gravitational collapse problem in a concrete example. For any static solution, we must expect also the stress-energy tensor of vacuum polarization in the Boulware vacuum to contribute. This stress-energy satisfies $p = \rho/3 < 0$ near the horizon (3). The addition of such a negative pressure equation of state in the thin outer edge obviates the need for the positive pressure discontinuity from negative pressure inside to positive pressure outside. Hence, a completely smooth matching of h and df/dr is possible and the surface tensions (Eqs. 25 and 26) can be made to vanish identically. A full analysis of dynamical stability without restriction on the interface boundaries will be possible in the framework of a more detailed model which leads to these vacuum stresses in the boundary layer. Such an investigation can be carried out without reference to thermodynamics or entropy and would apply then even in the case of a configuration at absolute zero.

Conclusions

A compact, nonsingular solution of Einstein’s equations has been presented here as a possible stable alternative to black holes for the endpoint of gravitational collapse. Realizing this alternative requires that a quantum gravitational vacuum phase transition intervene before the classical event horizon can form. Since the entropy of these objects is of the same order as that of a typical stellar progenitor, even for $M > 100M_\odot$, no entropy paradox and no significant entropy shedding are needed to produce a cold gravitational vacuum or “gravastar” remnant.

Since the exterior space-time is Schwarzschild until distances of the order of the diameter of an atomic nucleus from $r = r_s$, a gravastar cannot be distinguished from a black hole by present observations of x-ray bursts (32). However, the shell with its maximally stiff equation of state $p = \rho$, where the speed of sound is equal to the speed of light, could be expected to produce explosive outgoing shock fronts in the process of formation. Active dynamics of the shell may produce other effects that would distinguish gravastars from black holes observationally, possibly providing a more efficient particle accelerator and central engine for energetic astrophysical sources. The spectrum of gravitational radiation from a gravastar should bear the imprint of its fundamental frequencies of vibration and, hence, also be quite different from a classical black hole.

Quantitative predictions of such astrophysical signatures will require an investigation of several of the assumptions and extension of the simple model presented in this article in several directions. Although the equation of state $p = \rho$ is strongly suggested both by the limit of causality characteristic of a relativistic phase transition and by the correspondence of the fluid entropy with S_{BH} when the inner GBEC region is shrunk to zero, this equation of state has been assumed here, not derived from first principles. Knowledge of the effective excitations in the shell is necessary to determine the chemical potential μ , and whether the entropy estimate (Eq. 28) is accurate or more properly to be regarded as an upper bound on the entropy of a GBEC star. The neglect of the $p = \rho/3 < 0$ vacuum polarization in our model leads to some freedom in matching at the two interfaces and the surface tensions (25 and 26), which may be

different in detail or not present at all in a more complete treatment. A full analysis of the dynamical stability of the object, including the motion of the interfaces or the boundary layer(s) that replace them, requires at least a consistent mean field description of quantum effects in this transition region. Although general theoretical considerations indicate that nonlocal quantum effects may be present in the vicinity of classical event horizons, a detailed discussion of how these effects can alter the classical picture of gravitational collapse to a black hole has not been attempted in this article. Last, distinguishing the signatures of gravastars from classical black holes in realistic astrophysical environments, such as in the presence of nearby masses or accretion disks, will depend on the details of the dynamical surface modes and the extension of the spherically symmetric static model presented here to include rotation and magnetic fields.

One may regard the model presented in this article as a proof of principle, the simplest example of a physical alternative to the

formation of a classical black hole, consistent with quantum principles, which is free of any interior singularity or information paradox. Additional theoretical and observational effort will be required to establish the cold, dark, compact objects proposed in this article as the stable final states of gravitational collapse.

Finally, the interior de Sitter region with $p = -\rho$ may be interpreted also as a cosmological space-time, with the horizon of the expanding universe replaced by a quantum phase interface. The possibility that the value of the vacuum energy density in the effective low-energy theory can depend dynamically on the state of a gravitational condensate may provide a new paradigm for cosmological dark energy in the universe. The proposal that other parameters in the standard model of particle physics may depend on the vacuum energy density within a gravastar has been discussed by Bjorken (33).

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