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The Role of Nonlinear Gradients in Parallel Imaging: A k-Space Based Analysis

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Abstract

Sequences that encode the spatial information of an object using nonlinear gradient fields are a new frontier in MRI, with potential to provide lower peripheral nerve stimulation, windowed fields of view, tailored spatially-varying resolution, curved slices that mirror physiological geometry, and, most importantly, very fast parallel imaging with multichannel coils. The acceleration for multichannel images is generally explained by the fact that curvilinear gradient isocontours better complement the azimuthal spatial encoding provided by typical receiver arrays. However, the details of this complementarity have been more difficult to specify. We present a simple and intuitive framework for describing the mechanics of image formation with nonlinear gradients, and we use this framework to review some the main classes of nonlinear encoding schemes.

Keywords

parallel imaging; accelerated imaging; PatLoc; O-Space; nonlinear imaging

I. INTRODUCTION

Whereas standard MRI applies linear gradients to project an object onto linear isocontours, imaging with nonlinear gradients projects the sample onto curved isocontours. Interest in these techniques first reemerged due to their ability to permit faster gradient switching with minimal peripheral nerve stimulation and possibly provide anatomically adapted fields of view (1, 2). Since then, several other applications have evolved, including simple windowing to a reduced field of view (3), imaging of curved slices that can isolate certain biomedical regions of interest (4), or better preconditioning of an imaging dataset for compressed sensing (5). These fields also provide spatially varying resolution and have been shown to permit higher resolution imaging at the edge of the field of view, where signal to noise ratio (SNR) from coil arrays is highest. However, the most ambitious goal of nonlinear imaging is to achieve better under-sampled imaging with multichannel coils (6–10).

Linear gradients are limited in their ability to complement the spatial encoding provided by multi-channel receiver coils, which is why acceleration factors are typically limited to 2 or 4 in each direction, even with very large coil arrays (11–13). Increasing the number of coil elements provides spatial localization information near the surface but not internally to a structure. In principle, when N_c spatially localized coils are used to acquire the N_t time points of an acquisition, the $N_c \times N_t$ data points collected should allow us to solve up to $N_c \times N_t$ independent components of the image, or $N_c \times N_t$ pixels. However, to solve for the maximum number of unknowns, the phasor encodings would have to complement the spatial localization of the coils in a way that minimized redundancy of the information (8, 14). More precisely, all elements of the phasor-coil product basis should be as close to orthogonal as possible. For linear encodings, this would imply the coil basis should supply an N_c element Fourier basis, which is usually not physically achievable. Many nonlinear imaging techniques take the opposite approach, attempting to reduce the redundancy in multicoil datasets by modifying the geometry of the phasor basis.

A wide range of nonlinear encoding schemes have been explored to achieve better reconstruction of highly undersampled images. Early work by Patz et al. (15) explored the use of a spatially sinusoidal gradient along one dimension. More recently, Schultz and coworkers (1) introduced the phase encoded Cartesian PatLoc method, which essentially produced warped Cartesian images, but moving to a projection-reconstruction approach fueled the development of more arbitrary encoding schemes (6). This added flexibility to simultaneously play linear and nonlinear gradients in arbitrary combinations, which has helped address the low resolution that exists where the nonlinear fields are flat (10). More recently, researchers have begun to show the utility of hybrid phase encoded-projection imaging methods, such as four-dimensional (4D)-RIO, where the initial phase on each projection changes with each readout, and COGNAC, in which nonlinear spatial encoding magnetic fields (SEMs) are played as phase encodes while linear SEMs are played as readouts, or vice versa, simplifying image reconstruction (9, 16). Still, further generalizations are being explored using arbitrary field shapes (14, 17), including approaches like null space imaging, which takes arbitrary projections with shapes inspired by the null space of the coil profiles (8).

Amid this virtual explosion of encoding methods, there are too few tools available to understand the mechanics of these imaging methods, that is, how the gradients encode space on their own and exactly how they complement receiver geometry. Gallichan et al. (9) published the useful concept of local **k**-space, which helps researchers visualize the spatially varying encoding applied by a given gradient scheme. However, this metric is blind to the role of coil encoding. Layton et al. (18) published an elegant quantitative metric that does incorporate coil encoding and efficiently estimates the variance of pixels reconstructed by a given encoding scheme, but it does not provide much intuition on how a particular encoding scheme makes better use of the coil basis.

In this article, we describe a more intuitive framework for describing how nonlinear gradients can improve the efficiency with which coils multiply the available basis functions, helping us fill **k**-space from a minimum number of time points. The analysis focuses on the **k**-space version of the tenets underlying SMASH (19–22) and GRAPPA (23–25), which we

describe as “stamps” in $\mathbf{k}$ -space. We show that this “stamp” framework easily generalizes to describe the encoding provided by nonlinear gradients and then apply the analysis to some of the major classes of nonlinear encoding schemes. We also motivate how the coverage of $\mathbf{k}$ -space is related to various features of nonlinearly encoded images, including their spatially varying resolution and the utility of coil localization even in fully sampled schemes.

The stamp interpretation focuses on the (k_x, k_y) -space representation of each data point acquired in an experiment, as many aspects of image quality are well understood in this domain. Ordinarily, in linear encoding methods of MRI, each data point is acquired when the phase modulation across image space is $e^{-i(k_x x + k_y y)}$, and so this data point can be interpreted as the value on the point (k_x, k_y) in the $\mathbf{k}$ -space of the object. Put another way, each data point samples $\mathbf{k}$ -space with a sampling function that is a delta function at a particular point in $\mathbf{k}$ -space. However, whenever the modulation on the object is not purely sinusoidal, instead of sampling single points in $\mathbf{k}$ -space, we are sampling a blob or “stamp” in $\mathbf{k}$ -space. Therefore, the data point can be interpreted as a weighted sum over various $\mathbf{k}$ -space points. This is the case whether the nonsinusoidal modulation is from coil weighting, a nonlinear gradient, or both, and this “smearing” of $\mathbf{k}$ -space underlies the well-known $\mathbf{k}$ -space based parallel imaging techniques, such as SMASH, GRAPPA, and SPIRiT (19, 24, 26, 27). The stamps interpretation is different in that it focuses predominantly on the features of these smears, which we call stamps, and their relation to the final image reconstruction. This small shift in perspective is surprisingly helpful, because it naturally incorporates both coil sensitivities and nonlinear gradient encoding into the analysis, and it permits interpretation of nonlinear encoding in a traditional $\mathbf{k}$ -space manner.

II. STAMP INTERPRETATION OF LINEARLY ENCODED PARALLEL IMAGING

SMASH, GRAPPA and their variants all use linear combinations of $\mathbf{k}$ -space signals as received by different coils to synthesize neighboring $\mathbf{k}$ -space data. They differ in their methods, SMASH using measured coil sensitivities and GRAPPA relying on the use of auto-calibrating reference lines to estimate the coefficients directly from fully sampled $\mathbf{k}$ -space. However, both techniques begin from the observation that linear combinations of data from a coil-weighted acquisition of a $\mathbf{k}$ -space point (k_{xi}, k_{yj}) can be used to generate a point $(k_{xi} + \Delta k_{xi}, k_{yj} + \Delta k_{yj})$. For these well-known parallel imaging methods, stamps can be used to explain why data points can be synthesized from their seemingly independent neighbors, why only a few lines of calibration data are needed, and why each synthesized data point can use the same coefficients.

Reconstructing One Point in $\mathbf{k}$ -Space

The heart of all $\mathbf{k}$ -space based parallel imaging techniques is that if one can define composite coils such that:

$$C^{\text{comp}}(x, y) = \sum_{i=1}^{N_c} a_i C_i(x, y) = e^{in\Delta k_y y} \quad [1]$$

Those same coefficients can be used to reconstruct a data point shifted by $n \cdot k_y$ relative to the acquired $\mathbf{k}$ -space point. Typically, this is motivated in the image domain, but this relationship is equally valid in the Fourier domain:

$$F[C^{\text{comp}}(x, y)] = \sum_{i=1}^{N_c} a_i F[C_i(x, y)] = \delta(k - n\Delta k_y). \quad [2]$$

In this domain, the Fourier transform (FT) of each coil provides a sampling topology or “stamp” in $\mathbf{k}$ -space, and the same coefficients can be used to produce a linear combination of stamps that equals a particular shifted delta function in $\mathbf{k}$ -space. As established in the literature, it then follows that

$$S(k_x, k_y + n\Delta k_y) = \sum_{i=1}^{N_c} a_i S_i(k_x, k_y). \quad [3]$$

As the $\mathbf{k}$ -space version of this relationship (i.e., Eq. [2] rather than Eq. [1]) is easier to extend to nonlinear gradients, we next review Cartesian parallel imaging using the stamp interpretation. For example, consider four coils like those shown in Fig. 1 acquiring a data point associated with the $k = (0,0)$ phasor from a given object. Most commonly, the complex number received by each coil is regarded as the value of the $(0,0)$ $\mathbf{k}$ -space point for the coil-weighted image. In the stamps interpretation, we regard this data point as it relates to the $\mathbf{k}$ -space of the unweighted object. In this context, that data point represents a weighted sum of several $\mathbf{k}$ -components of the object. Recall that each data point read by a given coil at a given time is associated with an encoding function, E , which in this case is simply $C_j e^{i(k_m x + k_n y)}$. Calling a particular encoding function E_1 , we have:

$$S(E_1) = \iint \rho(x, y) E_1(x, y) dx dy. \quad [4]$$

Writing E_1 as a sum of Fourier components and assuming that it has no spatial frequencies higher than those being sampled for our reconstruction,

$E_1 = \sum_{m=k_{x,\min}}^{k_{x,\max}} \sum_{n=k_{x,\min}}^{k_{x,\max}} b_{mn} e^{i(k_m x + k_n y)}$. The set of b_{mn} is simply the FT of the encoding function. Substituting into Eq. [4], we have:

$$\begin{aligned} S(E_1) &= \iint \rho(x, y) \sum_{m=k_{x,\min}}^{k_{x,\max}} \sum_{n=k_{x,\min}}^{k_{x,\max}} b_{mn} e^{i(k_m x + k_n y)} dx dy \\ &= \sum_{m=k_{x,\min}}^{k_{x,\max}} \sum_{n=k_{x,\min}}^{k_{x,\max}} b_{mn} \left\{ \iint \rho(x, y) e^{i(k_m x + k_n y)} dx dy \right\} \end{aligned} \quad [5]$$

Each term in braces is simply the true value of the $\mathbf{k}$ -space of the object at (k_m, k_n) . Therefore, the signal associated with E_1 , the signal received at that time point by that coil, can be seen as a weighted sum of various $\mathbf{k}$ -space points of the object. The weighting on the sum, the set of b_{mn} coefficients, is simply the FT of the encoding function. This essentially restates the fact that a coil weighting “smears” the $\mathbf{k}$ -space of the object, which is what

allows us to deduce neighboring $\mathbf{k}$ -space points from each other. It then follows that there may be a weighted sum over $S(E_i)$ that isolates a particular delta-function component in the $\mathbf{k}$ -space of the object, and the coefficients on that sum are the a_i coefficients in Eqs. [1] and [2].

For a typical circumferential arrangement of four coils encoding the center of $\mathbf{k}$ -space, the real and imaginary weights of each sampling function (i.e. the FT of the coil profiles) are those shown in Fig. 1 for each coil. Each column corresponds to a coil oriented at a different corner of the field of view (FOV), and its real and imaginary parts are, respectively, shown in the second and third row. As can be seen, the main contribution to the signal does come from the nominal $\mathbf{k}$ -space point at (0,0). However, due to the coil modulation, which takes the form of a convolution in $\mathbf{k}$ -space, there are also significant contributions from neighboring $\mathbf{k}$ -space points. The magnitudes of the coil stamps have similar distributions in $\mathbf{k}$ -space as they are essentially rotated copies of each other, but their phases are very different. Thus, there may exist a linear combination that nearly cancels the sampling function at $(k_x, k_y) = (0,0)$ and maximizes the sampling at some other location in the tails of the stamps. Seeking this solution is the “stamps” interpretation of SMASH. Each stamp is simply the FT of a single row in the encoding matrix, and some linear combination of those stamps approximates a delta function in $\mathbf{k}$ -space.

Reconstructing the Rest of $\mathbf{k}$ -Space

If we know the coil profiles, we know the weightings provided by each encoding function corresponding to the (0,0) phasor. Therefore, we can solve for the linear combination of encoding functions whose weightings sum to best approximate a weight of 1 at $(k_x, k_y) = (1,0)$ and zero elsewhere. Those coefficients would be the SMASH coefficients used to generate a data point shifted by $k_x = 1$. We would simply be arriving at the coefficients by solving Eq. [2] instead of Eq. [1]. A sum of single coil signals weighted by these coefficients would then approximate the data point shifted by that amount, as in Eq. [5].

If some later point in the experiment acquires N_c points corresponding to the nominal $(k_x, k_y) = (2,0)$ phasor, the same analysis applies, and coefficients can be found to approximate the $k = (1,0)$ data point from those stamps instead, albeit with different coefficients. The stamp corresponding to the $k = (2,0)$ point will be identical to that at the (0,0) point but shifted in space, so that it is centered at $k = (2,0)$. However, a different set of coefficients will generally be needed to reconstruct a point at a different orientation from the center of the stamps. Of course, a better approximation of the $k = (1,0)$ point can be found by fitting it with both the $k = (0,0)$ and $k = (2,0)$ stamps simultaneously, or with an even larger set, which leads to the standard practice of using several $\mathbf{k}$ -space points to generate the kernel.

Finally, because the stamp in $\mathbf{k}$ -space only shifts with linear gradient encoding, this same kernel can be used to fill the $k = (3,0)$ data point with a linear sum of coil readings from the $k = (2,0)$ and $k = (4,0)$ data points, and so on. Therefore, if we skip all the odd lines of $\mathbf{k}$ -space except for one, that single odd line can be used to estimate the coefficients that best generate the shifted delta function in $\mathbf{k}$ -space. This strategy lets us to estimate the

coefficients needed to reconstruct our data without even explicitly knowing the encoding function, which is the approach taken by GRAPPA.

In standard parallel imaging, the smearing of the $\mathbf{k}$ -space sampling function caused by coil weighting can be treated separately from the translation through $\mathbf{k}$ -space caused by gradient evolution. The effect of linear gradients only changes the position of the stamp in $\mathbf{k}$ -space, as $F[E_{lmn}] = F[C_l(x, y)] \times F[e^{i(m k_y + n k_x)}]$, and the latter term in the convolution is simply a delta function in $\mathbf{k}$ -space. This is why the same kernel can be used to fill any location in $\mathbf{k}$ -space. This tidy relationship between all the points in $\mathbf{k}$ -space must be abandoned in the case of nonlinear encodings.

III. NONLINEAR ENCODINGS

General Considerations

Just as coil weightings smear $\mathbf{k}$ -space, allowing us to deduce $\mathbf{k}$ -space points different from the nominal $\mathbf{k}$ -space point being acquired, nonlinear phasor windings also smear $\mathbf{k}$ -space, but more dramatically. In the absence of coil weightings, the stamp from a pure linear phasor is a delta function, but the stamp from a nonlinear phasor is typically a broad and complex function. Adding coil weighting to these encodings actually has the opposite effect to the one described above, of smearing a delta-function. With the broad stamps of nonlinear windings, coils somewhat localize the smeared weighting function (Though this may appear counterintuitive, recall that when describing a quadratic field in $\mathbf{k}$ -space, the frequency components to be captured intimately depend on the field of view being considered. As coils effectively window the field of view, a smaller subset of frequency components can capture most of the modulation. This is also why an infinite quadratic phasor requires infinite $\mathbf{k}$ -space components to describe it, that is, why the FT of $e^{ik_1 x^2}$ has uniform magnitude through all of $\mathbf{k}$ -space), emphasizing different regions of the broadened stamp.

Furthermore, in most nonlinear encoding schemes, the nonlinear phasor changes with time. Therefore, nonlinear gradient evolution does not simply shift the position of the stamps; it changes the overall topology of the sampling function. If the nonlinear gradients are played simultaneously with linear gradients, as in O-Space, 4D-RIO, and Null Space Imaging, the stamp both spreads and translates during its coverage of $\mathbf{k}$ -space. Therefore, the sum over data points needed to generate a missing $\mathbf{k}$ -space point is different for each location in $\mathbf{k}$ -space, and it may require a sum over many encoding functions compared to the linear case.

1D Nonlinear Gradient

Before considering actual imaging schemes and their coverage of $\mathbf{k}$ -space, it is useful to consider the weighting function or stamp for the simplest model of a nonlinear phasor, $e^{-ik_1 x^2}$ which would result from evolution under a field like $B_z = Gx^2$, without coil weighting. The FT of this imaginary Gaussian is another imaginary Gaussian, with a standard deviation of $\frac{1}{4k_1}$. Therefore, the phase winding of the stamp is tightest when the gradient moment is minimal and loosens as the phase winding in image space gets tighter. Furthermore, though the image space phasor on an infinite plane would contain infinitely high frequencies, the phasor on a finite plane would have a finite magnitude in $\mathbf{k}$ -space. The

exact expression for the FT of a one-dimensional (1D) quadratic phasor $e^{ik_1x^2}$ across finite limits of $+/-d$ is:

$$\text{FFT}[e^{-ik_1x^2}] = \frac{e^{\frac{ik_x^2}{1}} \sqrt{\pi} \left((-a) \|b\| \text{erfi} \left[\frac{\sqrt{ik_1} \|a\|}{2k_1} \right] + (b) \|a\| \text{erfi} \left[\frac{\sqrt{ik_1} \|b\|}{2k_1} \right] \right)}{2 \sqrt{ik_1} \|a\| \|b\|}, \quad [6]$$

where we have defined $a = k_x - 2k_1d$ and $b = k_x + 2k_1d$ and erfi is the imaginary error function. This expression defines the $\mathbf{k}$ -space stamp across k_x for a 1D quadratic phasor with winding k_1 .

Figure 2 plots the stamp (Eq. [6]), omitting the Gaussian phase prefactor, as a function of k_x for a range of nonlinear phasor windings, k_1 . Most notably, the stamp is always an even function of k_x , so it always samples k_x and $-k_x$ simultaneously and equally, thus making it impossible to distinguish them. The figure assumes units of length equal to pixels with a FOV of 128^2 . As can be seen, the weighting function broadens considerably for even modest values of k_1 , which means that data points taken under these encodings sample a wide swath of $\mathbf{k}$ -space points. To further explore this phenomenon, we next examine the $\mathbf{k}$ -space coverage of four very different nonlinear imaging schemes: the original implementation of PatLoc, O-Space, 4D-RIO, and Null Space Imaging.

IV. PATLOC IMAGING

Arguably, the most straightforward nonlinear imaging scheme is that of phase encoded PatLoc imaging with quadrupolar fields (1, 28–30). This imaging scheme follows the sampling pattern of Cartesian phase encoded imaging, but using the gradient shapes $2xy$ and $x^2 - y^2$. As in Cartesian imaging, the combination of frequency and phase encoding resolves signal along the two frequency coordinates. In image space, this localizes the signal to voxels defined by the intersection of the isocontours of the frequency and phase encoded gradients (Fig. 3). Although many features of PatLoc imaging are intuitively explained simply by examining the encoding fields in image space, we will address how those features can also be understood by considering the stamps collected during the acquisition. This framework will later allow us to analyze imaging modalities where the spatial characteristics of the encoding are less straightforward.

Encoding Redundancy and the Role of Coils

Though PatLoc imaging with quadrupolar fields resolves signal to unique points in a two-dimensional frequency space, it does not resolve signal to a unique point in physical space. Unlike Cartesian imaging, there are always two voxels that have the same value of $(\omega_{\text{Field1}}, \omega_{\text{Field2}})$, so the mapping between frequency and image space is non-bijective and leaves a two-fold ambiguity in the image. Even in a fully sampled PatLoc acquisition, the spin density in the voxel at (r, θ) contributes the exact same signal as the spin density at $(r, \theta + \pi)$, so these pixels fold on top of each other. However, to the extent that multichannel coils can differentiate signals at (r, θ) from those at $(r, \theta + \pi)$, these signals can be unfolded, leading to a final unfolded image.

This twofold redundancy can also be seen by looking at the stamps associated with a PatLoc acquisition without coil weighting. Figure 4 shows a series of stamps associated with a PatLoc acquisition. The stamps in each row correspond to a single echo (shown at $k_x^2 - y^2 = 0, 20$, and 40% of the acquisition maximum) and the different rows correspond to different phase encode steps ($k_{xy} = 0, 20$, and 40% of the maximum k_{xy}). Overall, stamps from these phasors all sample broad squares in $\mathbf{k}$ -space. The sampled area grows with increased gradient winding and it rotates to a diamond orientation as $k_x^2 - y^2$ and k_{xy} approach equal magnitudes.

Each stamp has a twofold symmetry reflected through the origin, analogous to the even stamp basis that arose in the 1D case. Therefore, the linear combination of stamps that sums to a unit function at some (k_x, k_y) also gives a unit function at $(-k_x, -k_y)$. This is different from half-Fourier imaging, which synthesizes the signal at $(-k_x, -k_y)$ as the conjugate of the signal at (k_x, k_y) . Each symmetric stamp, and thus any linear combination of them, gives only the weighted sum of the signals at (k_x, k_y) and $(-k_x, -k_y)$, more similar to a cosine transform of the object. This inability to independently sample $\mathbf{k}$ -space points opposite of each other is directly related to the twofold redundancy in the final image.

Next, we compare the stamps acquired with encoding functions that are the product of PatLoc phasors and typical coil profiles. Figure 5 shows that the stamps for encoding functions comprise the same phasor from the central panel of Fig. 4, but now with coil weighting included. Coil weighted stamps are far more localized and tend to emphasize a region of the original phasor stamp that mimics their position in the field of view.

Furthermore, the coil weighted stamps tend to follow the trajectory of the associated phasor stamp, reaching higher values of (k_x, k_y) as the phasor grows and also following a circular trajectory in $\mathbf{k}$ -space as the ratios of k_{xy} and $k_x^2 - y^2$ change. In the coil-weighted basis, symmetry about the origin is broken, and it is possible to find linear combinations that approximate isolated unit functions at any given (k_x, k_y) . This allows a full reconstruction of $\mathbf{k}$ -space, and thus results in an unaliased image.

Approximating Delta Functions with a PatLoc Basis

With the inclusion of coils, we can generate the final stamp basis for the image reconstruction and begin to consider what subset of that basis is needed to generate a single point in $\mathbf{k}$ -space. With linearly encoded data, the coil weightings simply add “tails” around the nominal $\mathbf{k}$ -space point and mostly overlap with each other. Therefore, reconstructing a target $\mathbf{k}$ -space point requires only a linear combination of the coil-phasor stamps from adjacent nominal $\mathbf{k}$ -space points. With nonlinear encoding, the coil-phasor products that correspond to a given phasor may barely overlap, so the set of stamps needed to reconstruct a particular point in $\mathbf{k}$ -space is more complex. As suggested by Figs. 4 and 5, there is often more overlap between stamps from different phasors as seen by the same coil, than between the same phasor as seen by different coils. In addition, the diffuse nature of these stamps generally requires a larger basis, possibly the entire set, to be considered to reconstruct a particular point in $\mathbf{k}$ -space.

The last major feature of PatLoc imaging is the poor resolution in the center, which essentially leads to a hole in the image reconstruction. In image space, it is simple enough to ascribe this to the flatness of both encoding fields in this region. This can be analytically described using the Jacobian for the mapping between image space and frequency space (1). However, the phenomenon can also be understood in $\mathbf{k}$ -space by considering how well the basis can reconstruct individual points in $\mathbf{k}$ -space.

Spatially Varying Spatial Resolution

Though coil-weighting breaks the symmetry and makes opposite points in $\mathbf{k}$ -space distinguishable, this does not guarantee that every $\mathbf{k}$ -space point can be reconstructed cleanly. The diffuse stamps of nonlinear imaging may appear to sample more of $\mathbf{k}$ -space, but the tradeoff is often that it may be more difficult to completely isolate any particular unit function in $\mathbf{k}$ -space. As an example, Fig. 6 shows the best fit linear combination to reconstruct a $\mathbf{k}$ -space point in outer $\mathbf{k}$ -space. Rather than the desired basis of individual points in $\mathbf{k}$ -space, the cleanest basis that we can generate from PatLoc stamps still contains some contributions from neighboring $\mathbf{k}$ -space points. These linear combinations in $\mathbf{k}$ -space can cause regions of low resolution. For example, the best fit to a unit function at k_i might still be a linear combination like:

$$c_1 \delta(k - k_i) + c_2 \delta(k - k_{i+1}) \quad [7]$$

In image space, this means we are reconstructing the image from basis functions like:

$$c_1 e^{ik_i x} + c_2 e^{ik_{i+1} x} \quad [8]$$

In general, a basis function that is a weighted sum of complex exponentials contains a spatially varying amplitude, so such a basis function may not contribute resolution to certain regions. For example, for a two term sum with $c_1 = -c_2$, this basis function would not contribute to resolution on the center of the field of view, as:

$$c_1 e^{ik_{\text{mean}} x} (e^{-i\Delta x} - e^{i\Delta x}) = c_1 e^{ik_{\text{mean}} x} \sin(\Delta x) \quad [9]$$

where k_{mean} is the mean of the two $\mathbf{k}$ -space values and Δx is one half their difference. Thus, these spatial frequency components are absent or minimal near the center of the image, leading to lower resolution there.

It is important to note that spatially varying frequency components are not exactly equivalent to spatially varying resolution, though the concepts are closely related. Speaking more precisely, when frequency components beyond some radius in $\mathbf{k}$ -space do not contribute to a region of the image space, this region has lower resolution and we can accurately term this as spatially varying resolution. In the more general case, it is more accurate to say that certain frequency components are not captured in certain regions of the image.

Generalizing from this, the amount of spillage in a reconstructed $\mathbf{k}$ -space point often indicates that frequency component may not be contributing equally to all of the image region. However, two important exceptions should be noted. One is the case of a plane wave, e^{ik_0x} , whose frequency does not exactly lie on the transform grid, so the coefficients on each k_i in the transform axis are in the 1D case:

$$e^{i(k_i - k_0 x_{\max}/N)} \frac{\sin[\pi(Nk_i - k_0 x_{\max})]}{\sin[\pi(k_i - \frac{k_0 x_{\max}}{N})]},$$

where N is the image resolution. If the coefficients follow this particular pattern, the image domain basis function has unit magnitude over the entire image region. Second, it should be noted that experimental coil profiles often window the effective field of view to the actual bore of the coil, typically a circular field of view. With this windowing, perfect delta functions in $\mathbf{k}$ -space can never be achieved, though this can be seen as a kind of spatial modulation of the frequency components, where all frequency components have amplitude zero outside the coil radius.

However, for the linear combinations that typically sum to an approximated point in $\mathbf{k}$ -space, the side lobes typically cause a more arbitrary amplitude modulation. A useful, though not ironclad, rule of thumb is that where the stamps themselves are most diffuse, unless the region is sampled by a large number of stamps, reconstructed $\mathbf{k}$ -space points also tend to be diffuse. It follows that these resolution components may not contribute evenly over the field of view. This is often the tradeoff to the greater coverage of $\mathbf{k}$ -space from a nonlinear coil-phaser basis.

It is important to note that this rule of thumb has exceptions, even for a coil-phaser basis broadened by nonlinear windings. For example, an encoding strategy like that outlined in Witschey et al. (3) consists of smeared stamps and still allows us to reconstruct very clean delta functions over most of $\mathbf{k}$ -space. However, this basis is particularly regular (similar to those described in Section 2), as each stamp is the convolution of a delta function in $\mathbf{k}$ -space and the same nonlinear winding. Because of this regularity in the encoding strategy, there are more straightforward ways to interpret such datasets, as done in Ref. 3. The advantage of a stamps analysis is its ability to make intuitive predictions from very arbitrary encoding approaches.

Figure 6 demonstrates this to be the case in PatLoc imaging with coils. Panel (a) shows a reconstructed $\mathbf{k}$ -space point near the center of $\mathbf{k}$ -space, where much of the remaining “skirt” on the delta function comes from the coil profile being windowed to the coil sensitivity region. The corresponding basis function in image space, (b) and (c) for real and imaginary parts, shows relatively uniform magnitude within the active volume. In contrast, a $\mathbf{k}$ -space point near the edge of $\mathbf{k}$ -space shows considerably greater deviations (d), and the spatial representation of that basis function, (e) and (f) for real and imaginary parts, shows considerable magnitude modulation, with little contribution in the center of the image.

Note that this also means that stamps can be used to incorporate coil encoding into a modification of the local $\mathbf{k}$ -space formalism, albeit at considerable computational expense.

For a given gradient encoding and set of coil profiles, one can calculate the best fit at each $\mathbf{k}$ -space point. This will capture any remaining spatially varying amplitude like that shown in Figure 6 and will show which components of Cartesian $\mathbf{k}$ -space are being captured at that location. The local $\mathbf{k}$ -space can now be constructed by measuring the local amplitude of each basis function, and the $\mathbf{k}$ -space point is said to be part of the local $\mathbf{k}$ -space of that voxel if the amplitude of the corresponding basis function is above some chosen threshold at that location. Alternatively, one could include the point when the mean squared error (MSE) between the local basis function and the ideal basis function is below some threshold for that location.

V. O-SPACE IMAGING

Encoding and Stamps of O-Space

O-Space is a projection reconstruction method, rather than the phase encoded approach used in PatLoc, so the relationship between the gradients and the final image is less obvious from an analysis of the fields, making the analysis in $\mathbf{k}$ -space even more valuable (6, 31, 32). In single slice O-Space imaging, the magnetic field used for spatial encoding is defined by

$B_z = \gamma G_z \frac{1}{2} ((x-x_0)^2 + (y-y_0)^2)$ instead of $B_z = \gamma G_x x$, and this projects the object along rings in the field of view (Fig. 7). Instead of phase encodings, a full dataset is acquired by taking echoes that correspond to different (x_0, y_0) or center placements, typically at locations forming a circle with radius $\text{FOV}/2$. Each center placement is realized by playing linear gradients, oriented in the direction of the center placement, simultaneously with the nonlinear field. As the center placement changes with each scan, each echo provides a projection of the object along different circular contours, so reconstruction amounts to finding the minimum norm least squares solution that is compatible with the entire set of projections over the various circular isocontours.

The synergy between this kind of gradient encoding and multichannel receiver encoding is less straightforward. Although the unfolding action of coils is clear in PatLoc imaging, it is difficult to pinpoint how coil encoding improves the reconstruction of O-Space images (One explanation in image space is that the ring-like isocontours may provide more coil-resolved regions along the isochromat contours of each read gradient. However, this is only partially true once the vertex of the parabola is moved off-center, and it gives limited insight into the advantages of one O-Space scheme relative to another). Like undersampled radial imaging, O-Space has a complex point-spread function that does not result in a simple folding artifact that coils can obviously resolve. The analysis in image space is further complicated as the point-spread function varies across the field of view. Yet, O-Space images without coil encoding are markedly of lower quality, so the encoding contribution of coils must be significant.

The “stamps” approach helps illuminate how O-Space encoding is complemented by coil geometry; coils turn the diffuse phasor stamps into more localized, but nonoverlapping stamps, which are more suited for reconstructing individual points in $\mathbf{k}$ -space, as further explored in “The Role of Coils in O-Space” Section. However, considering the question in $\mathbf{k}$ -space helps one to visualize how smeared $\mathbf{k}$ -space sampling functions can be used to fill in

missing radial information. Similarly, O-Space creates smeared sampling functions that can help fill out desired points in $\mathbf{k}$ -space.

The $\mathbf{k}$ -space sampling of an O-Space acquisition can be understood by examining its stamps, such as those shown in Fig. 8. Across each row, we plot various time points in the echo, corresponding to 5, 50, and 75% of the maximum winding (which occurs at the edges of the O-Space readout), and different rows correspond to echoes with different (x_0, y_0) , corresponding to $(0, \text{FOV}/2)$, $(\text{FOV}/2, 0)$, and $(0, -\text{FOV}/2)$. Looking across each row, as the magnetization is increasingly modulated as a function of time $e^{-i\gamma G_z 2\frac{1}{2}((x-x_0)^2 + (y-y_0)^2)t}$, we see a broadening of the $\mathbf{k}$ -space stamp. However, as the nonlinear gradient plays out alongside linear fields, the center of the stamp simultaneously translates through the $\mathbf{k}$ -space plane as it broadens. At each moment, the center of the stamp sits at the $\mathbf{k}$ -space point associated with the linear gradient moment that has accumulated at that time point. Therefore, though we are describing a translation through $\mathbf{k}$ -space not image space, the stamp will seem to move in “the direction of the center placement.” As the center placement changes from scan to scan of a complete image acquisition, so does the direction of translation of the stamp center. The linear gradients that move the center placements to different points in a circle resemble those of a radial trajectory, and so the trajectory of the stamp center also resembles a radial trajectory. Each time, the stamp starts as a small point at the center of $\mathbf{k}$ -space and grows as it moves out to the edges of $\mathbf{k}$ -space.

The Role of Coils in O-Space

As with PatLoc, reconstructions of O-Space data acquired without coil weightings have some artifacts due to the symmetry of the encoding function. However, the resulting artifacts are considerably less pronounced than in PatLoc imaging and do not create a global symmetry in the reconstructed image. In Pat-Loc, the symmetry of the pure phasor stamp is across the origin, but in O-Space imaging, the symmetry is across the (k_x, k_y) point defined by the linear gradient moment. Therefore, the artifacts resulting from an O-Space acquisition without coil weighting may be thought of as being more similar to those discussed in Spatially Varying Spatial Resolution Section, relating to spatially varying resolution. The symmetry of the unweighted phasors only allows us to create basis functions that are linear combinations of nearby $\mathbf{k}$ -space points, rather than resolving an independent point in $\mathbf{k}$ -space. This amounts to an amplitude modulation on the resolution contribution of that $\mathbf{k}$ -space point, creating regions of low resolution rather than a completely symmetric image.

Adding coils to the encoding information improves reconstruction fidelity of high-frequency spatial components. Coil weighting breaks the symmetry of the pure phasor stamps allowing more independent $\mathbf{k}$ -space points to be reconstructed. Figure 9 shows four stamps corresponding to the phasor in the bottom right panel of Fig. 8, but including coil weightings. Coil weighting creates more localized but unique stamps, and we would expect this to improve our ability to reconstruct individual points in $\mathbf{k}$ -space from their sum.

To demonstrate this point, Fig. 10(a) shows a $\mathbf{k}$ -space point reconstructed from an O-Space basis without coil weighting. The reconstructed point shows a considerably “skirt” in $\mathbf{k}$ -space, and the corresponding spatial basis function (10b) indeed shows a defined amplitude

modulation. This frequency component is not being captured near the center of the image. In the next row, we show this same $\mathbf{k}$ -space point as reconstructed from a basis of coil-phaser stamps. This gives a much cleaner reconstruction of the single $\mathbf{k}$ -space point, and the corresponding spatial basis function now has relatively uniform amplitude across the field of view. The last column shows the overall effect on a two-dimensional (2D) image of a numerical phantom.

From this analysis, O-Space can be viewed as a strategy to fill in missing projections of an undersampled radial dataset. Near the center of $\mathbf{k}$ -space, the missing points of an undersampled radial acquisition are at small offsets from the nominally acquired $\mathbf{k}$ -space points (those defined by the linear gradient moment), and in these regions, O-Space is acquiring relatively localized stamps. Without coil weighting, the stamp moves along a radial-like spoke and becomes increasingly diffuse near the edges of $\mathbf{k}$ -space. When we add in coil weighting, these very diffuse stamps become localized blotches in $\mathbf{k}$ -space at increasing distances from the nominal spoke being traversed by the linear gradients. These blotches may sample spokes that would be missing from the under-sampled radial trajectory traced out by the linear gradients alone. Furthermore, filling a radial dataset requires the reconstruction of points at very different orientations from the stamps. In linear acquisitions, the stamps tend to have a very pronounced directionality in their sampling, but the more isotropic spreading seen in nonlinear stamps may provide a better basis to reconstruct missing data points at arbitrary orientations relative to the nominal (k_x, k_y) positions acquired.

VI. OTHER NONLINEAR IMAGING TECHNIQUES

4D-RIO

4D-RIO imaging is a projection reconstruction technique using four gradient fields: the linear x and y gradients and the nonlinear x^2-y^2 and $2xy$ gradients (9). Each gradient is played out in a radial-like trajectory, so over subsequent echoes, the x and y fields are played at amplitudes that follow cosine and sine, respectively. This moves the center of the nonlinear field to different points on a circle, similar to the ring of center place projections in O-Space. At the same time, the $2xy$ and x^2-y^2 fields also play out a cosine and sine trajectory, respectively, such that the orientation of the nonlinear field plays out at a different orientation at each echo. This sequence was found to provide the best images when the linear phasor component and nonlinear phasor component did not pass through zero at the same time.

Stamps can be used to intuitively describe how this sequence samples $\mathbf{k}$ -space. Each echo begins at $k_{\text{linear}} = (0,0)$ and $k_{\text{nonlinear}} = (k_{\text{max}}, k_{\text{max}})$, so the first stamps of the sequence will be diffuse and sample the center of $\mathbf{k}$ -space. As the echo proceeds, the stamps being collected move off-center along a radial spoke, and the stamp becomes increasingly narrow. Furthermore, the advantage of cycling through different combinations of xy and x^2-y^2 over subsequent echoes is also made intuitive from considering the $\mathbf{k}$ -space stamps. This creates a set of stamps at the center with different orientations, making it possible to form linear combinations that isolate particular points. It also keeps a constant orientation between a stamp and its spoke, which may better mimic a radial-like dataset.

Null Space Imaging

To date, the gradient fields explored have been the second-order harmonics, primarily due to the available expertise in building gradients with these field shapes. However, higher order fields are an active area of research. For example, Null Space Imaging takes the interesting approach of applying gradients derived from the null space of the imaging profiles (8, 33, 34). These shapes are not always physically achievable, and current research has only approximated them with spherical harmonics of second order. However, higher order fields would provide more degrees of freedom to create the ideal phasors.

In terms of $\mathbf{k}$ -space, higher order fields provide more degrees of freedom in shaping the stamps that will ultimately be used to deduce the $\mathbf{k}$ -space of the object. As an example, Fig. 11 shows fields from x and y , and it can be seen that these provide more directional structure to the stamp, even in the absence of coils. This directionality might be useful to match coil geometries so that phasor-coil product stamps will have minimal overlap, yet be localized enough to recreate individual $\mathbf{k}$ -space points. However, the effect on the final basis depends greatly on the specific trajectory of all gradients in the sequence.

Although stamps can be used to predict the possible advantages of higher order imaging, they can also be used to analyze the inherent challenges. For example, fourth-order fields will provide stamps with fourfold symmetry, a symmetry that can be broken by coil weighting. However, these product stamps will also be more diffuse, so it is likely that reconstructed $\mathbf{k}$ -space points will have more dramatic side lobes, which is associated with spatially modulated $\mathbf{k}$ -space components. This is also what we would expect from an analysis in the image domain. Using a fourth-order field, the coils are still able to resolve the folding, but we see greater loss of resolution at the center of the image.

VII. CONCLUSION

This article reviews some of the current acquisition strategies being researched in the field of nonlinear imaging. We review the main features of PatLoc, O-Space, 4D-RIO, and Null Space Imaging in both image space and $\mathbf{k}$ -space. Our analysis in $\mathbf{k}$ -space includes an extension of the principles underlying SMASH and GRAPPA to nonlinear gradients, and we show that the approach can be used to analyze many arbitrary nonlinear imaging schemes. In this framework, nonlinear phasors are equivalent to time-varying coil weightings, smearing the sampling function in $\mathbf{k}$ -space.

We show that examining the stamp basis for a sequence can elucidate how coils aid various nonlinear imaging techniques and certain features of the image. In PatLoc, a major role of coil weighting is to break symmetry across the origin, whereas in O-Space, coils eliminate regions of low spatial resolution. This intuitive framework may aid the rational design of future trajectories and is a useful tool for understanding the potential advantages of nonlinear imaging schemes.

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Biographies

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Jason P. Stockmann: Jason Stockmann studied applied physics at Cornell University before pursuing his Ph.D. at Yale University under the direction of R. Todd Constable. Dr. Stockmann's thesis work involved simulation and experimental realization of a novel parallel imaging method called O-Space imaging which combined, for the first time, spatial encoding from both linear and nonlinear gradient fields. His present include nonlinear gradient image reconstruction, low-field portable MRI scanners, and flexible strategies for B0 shimming. [Apart from his research, Dr. Stockmann enjoys Baroque music, bicycle commuting, and advocating for bicycle and pedestrian safety.]

Leo Tam: Leo graduated from Brown University in 2007 with a Sc. B. in physics and a M.Sc. from Yale University in 2009. His research interests include nonlinear gradient imaging, compressed sensing, and novel applications of MRI. In his spare time, he enjoys tennis, reading, and traveling.

Hemant D. Tagare: Hemant D. Tagare, is an Associate Professor in the Departments of Diagnostic Radiology, Biomedical Engineering, and Electrical Engineering at Yale University. His research is concerned with mathematical and algorithmic problems in image reconstruction, image segmentation, and image registration as applied to electron

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Dana C. Peters received a Ph.D. in Physics from University of Wisconsin, Madison, working with Dr. Charles A. Mistretta, with a thesis on undersampled radial imaging. Her postdoctoral work at the NIH focused on cardiovascular and interventional MRI. She was Assistant Professor of Medicine at Harvard Medical School in the Cardiac MR Center of Beth Israel Deaconess, where she introduced new high resolution methods of imaging the left atrium. Recently, she has joined Yale's Magnetic Resonance Research Center and the Department of Diagnostic Radiology.

R. Todd Constable was born in Winnipeg, Canada. he obtained his B.Sc. at the University of Winnipeg, his M.Sc. at the University of Manitoba, and his Ph.D. at the University of Toronto under the supervision of Dr. R. Mark Henkelman. He then moved to Yale University as a postdoctoral fellow under the supervision of Dr. John C. Gore and moved from there to a faculty position and is now a full professor of Diagnostic Radiology, Neurosurgery, and Biomedical Engineering. Along with Dr. Douglas Rothman, Dr. Constable codirects the Yale Magnetic Resonance Research Center (MRRC) and oversees MRI research at Yale. His interests are diverse and range from basic MR pulse sequence design with emphasis recently on accelerated parallel imaging through investigations of spatial encoding using nonlinear magnetic field gradients in addition to functional MRI research on methodology as well as the application of novel fMRI methods to basic problems in neuroscience as well as direct clinical applications.

Figure 1.

Coil weightings, like those shown in the top row, smear $\mathbf{k}$ -space. Signals acquired by each coil can then be thought of as a weighted sum of $\mathbf{k}$ -space samplings, where the weighting function is the FT of the coil profile. The second and third rows of the figure show the real and imaginary parts of this weighting for each coil. Though their magnitudes are similar, each coil has different phase modulation, so linear combinations can be made that isolate a particular region of the weighting.

Figure 2.

The sampling in $\mathbf{k}$ -space associated with an encoding function, also referred to as the stamp for that encoding function, is shown for a 1D nonlinear phasor without coil weighting ($e^{ik_1 x^2}$). The 1D stamp across k_x is shown for various nonlinear gradient moments, k_1 . The amplitude is shown as a three-dimensional surface, while the phase of the stamp at each point is shown on the image below. This figure shows how nonlinear winding leads to a broad sampling of $\mathbf{k}$ -space with each data point, with both magnitude and phase modulation.

Figure 3.

Quadrupolar Cartesian PatLoc is a phase encoded imaging scheme using gradients $G_1 = x^2 - y^2$ and $G_2 = xy$, whose isocontour lines are shown in panels (a) and (b), respectively. A 2D transform of PatLoc data gives the magnetization at $(\omega_{x^2-y^2}, \omega_{xy})$, which spatially translates to the intersection of the isocontours of the image (c). However, there are always two locations that correspond to each frequency pair, with symmetry across the origin, so coil weightings are needed to arrive at an unfolded image.

Figure 4.

The sampling of $\mathbf{k}$ -space (imaginary part) in a quadrupolar Cartesian PatLoc acquisition without coil weightings is shown for various time points. Each row corresponds to $k_{x^2-y^2} = 0, 20$, and 40% of the maximum winding that would be applied in a single echo. Each column corresponds to $k_{2xy} = 0, 20$, and 40% of the largest phase encode winding. As the total nonlinear moment increases, the stamps sample a broader swath of $\mathbf{k}$ -space, and the stamp rotates as $k_{x^2-y^2}$ approaches k_{xy} . The imaginary part of the stamp exhibits the symmetry of each point through the origin.

Figure 5.

With the addition of coil weighting, the diffuse stamps associated with nonlinear gradients become more localized and asymmetric. This figure shows the $\mathbf{k}$ -space stamps (magnitude) for the phasor associated with the center panel of Fig. 4 ($k_x^2 - y^2 = 20\%$, $k_{xy} = 20\%$) but with coil weighting as in Fig. 1 included in the encoding function. A magnitude image of the stamp for the same phasor without coil weighting is shown as an inset. In PatLoc imaging, coil weighting is crucial to breaking the symmetry of the stamp basis across the origin of $\mathbf{k}$ -space. This makes it possible to reconstruct sampling functions that sample (k_x, k_y) independently of $(-k_x, -k_y)$. Note the stamp becomes more localized by incorporating the coils sensitivities, which is opposite of the behavior seen with linear encodings.

Figure 6.

Using stamps from all the PatLoc phasors, it is possible to synthesize a discrete blob in $\mathbf{k}$ -space. However, the residual intensity at neighboring points in $\mathbf{k}$ -space causes this frequency component to not evenly contribute to all regions of the image. A low frequency $\mathbf{k}$ -space point, like that shown in panel (a), shows minimal spillage into neighboring points (caused primarily by the coil windowing to a circular field of view). The corresponding image representation of this basis function has uniform magnitude in the active field of view, with real part shown in (b) and imaginary part in (c). In contrast, frequency components further in $\mathbf{k}$ -space must be formed from more diffuse stamps, and the resulting $\mathbf{k}$ -space point (d) shows more spillage into neighboring pixels. This is often associated with a spatially varying amplitude of that basis function, as shown in the image representation of this basis function, (e) and (f).

Figure 7.

Panel (a) shows a frequency map for a single z-slice of the nonlinear field used in O-Space imaging. For each echo of the acquisition, this field is played simultaneously with linear gradients, which move the vertex of the field off the center of the FOV to a desired center placement, as shown in (b)–(e). Therefore, the FT of each resulting echo, shown below, projects the object along the ring-like frequency isocontours of the offset field. The image is then reconstructed by finding the minimum norm object that is consistent with all of these projections along different isocontours.

Figure 8.

Sampling functions (real part) from an O-Space acquisition without coil weighting are shown for different time points. Each column corresponds to a read gradient moment of 5, 50, and 75% of the maximum phase winding. Each row represents a different echo taken with a different (x_0, y_0) placement of the parabola vertex. The $\mathbf{k}$ -space stamp both spreads and translates during the course of the echo as both nonlinear and linear gradient moment build. As in PatLoc, there is symmetry in the stamp about its center, but in O-Space imaging, this does not amount to symmetry about the origin. Each stamp is symmetric about the $\mathbf{k}$ -space point defined by the accumulated linear gradient moment. This indistinguishability between nearby points is associated with local resolution loss, as opposed to a global symmetry in the final image.

Figure 9.

The addition of coil weighting localizes the otherwise diffuse stamps associated with O-Space imaging. Stamps are shown (magnitude images) for the phasor associated with the bottom right panel of Figure 8 but with weighting from coils like those shown in Figure 1. The magnitude of the stamp for the unweighted phasor is shown to the right. Over the course of an acquisition, these coil-weighted stamps are translated in $\mathbf{k}$ -space by the linear gradient moment, whose time course resembles a conventional radial trajectory. As the coil weights are convolved with increasingly diffuse phasor stamps, they sample points at increasing distances from the $\mathbf{k}$ -space point associated with the linear gradient moment. Thus, each coil-phasor stamp also traces out a radial-like trajectory.

Figure 10.

In O-Space imaging, coil weighting does not resolve a global folding, but instead improves spatially varying resolution. Without coil weighting, the best approximation to a point in $\mathbf{k}$ -space (a) shows a complex skirt. The corresponding frequency component in image space does not contribute evenly to the entire field of view, as seen in the real part (b), imaginary part (c), and magnitude (d) of this basis function in image space. This complex spatial variation in different frequency components results in complex artifacts in (e) the final reconstruction. Adding coil encoding makes the stamp basis less diffuse, and the same $\mathbf{k}$ -space point (f) can be better approximated. This $\mathbf{k}$ -space component is captured in all regions of the field of view (g–i), and the final image quality (j) is improved.

Figure 11.

Stamps from higher order phasors have strange and increasingly structured shapes. This might improve their ability to form a basis that maximizes orthogonality in a coil-phasor product basis. Shown here are simulated stamps for phasors derived from third-order spherical harmonic gradients (Shapes that reduce to second-order shapes for a plane in z are excluded).