Published in final edited form as:

IEEE Trans Nucl Sci. 2015 February ; 62(1): 42–56. doi:10.1109/TNS.2014.2379620.

Impact of the Fano Factor on Position and Energy Estimation in Scintillation Detectors

Vaibhav Bora [Member, IEEE]

Department of Medical Imaging and College of Optical Sciences, University of Arizona, Tucson, AZ 85225 USA (bora.vaibhav@gmail.com)

Harrison H. Barrett [Fellow, IEEE]

Department of Medical Imaging and College of Optical Sciences, University of Arizona, Tucson, AZ 85225 USA

Abhinav K. Jha [Member, IEEE]

Division of Medical Imaging Physics, Department of Radiology and Radiological Sciences, Johns Hopkins University, Baltimore, MD 21287 USA

Eric Clarkson [Member, IEEE]

Department of Medical Imaging and College of Optical Sciences, University of Arizona, Tucson, AZ 85225 USA

Abstract

The Fano factor for an integer-valued random variable is defined as the ratio of its variance to its mean. Light from various scintillation crystals have been reported to have Fano factors from sub-Poisson (Fano factor < 1) to super-Poisson (Fano factor > 1). For a given mean, a smaller Fano factor implies a smaller variance and thus less noise. We investigated if lower noise in the scintillation light will result in better spatial and energy resolutions. The impact of Fano factor on the estimation of position of interaction and energy deposited in simple gamma-camera geometries is estimated by two methods - calculating the Cramér-Rao bound and estimating the variance of a maximum likelihood estimator. The methods are consistent with each other and indicate that when estimating the position of interaction and energy deposited by a gamma-ray photon, the Fano factor of a scintillator does not affect the spatial resolution. A smaller Fano factor results in a better energy resolution.

Keywords

Cramér-Rao bound; energy estimation; energy resolution; Fano factor; Fisher information matrix; maximum-likelihood estimator; position estimation; scintillators; spatial resolution

I. Introduction

GAMMA-RAY imaging detectors are used in a number of different applications ranging from astronomy and medical imaging to national security. Gamma-ray scintillation detectors are widely used because they are relatively inexpensive, have high density and are a mature technology [1].

A gamma-ray scintillation imaging detector (gamma-ray scintillation camera) has two main components: a scintillation crystal and an array of optical detectors. When a scintillation crystal is excited by gamma rays, it emits optical photons. These optical photons are then detected by an array of optical detectors whose outputs are used to estimate the position of interaction (x, y, z) and the energy deposited (E) .

The key difference between gamma-ray spectroscopy detectors and gamma-ray imaging detectors is that the gamma-ray spectroscopy detectors estimate only the energy of gamma rays. Therefore, they are designed to make the scintillation-light collection independent of the position of interaction. However, in a gamma-ray imaging detector, both the energy and position of interaction of the gamma rays are estimated. The estimates of position and energy influence each other, and the position of interaction (x, y, z) helps account for any position dependence of the light collection.

The Fano factor for an integer-valued random variable is defined as the ratio of its variance to its mean. When a gamma-ray photon deposits energy E in a scintillator, it produces a random number of optical photons N . The Fano factor for the optical scintillation photons is defined as

$$F_N = \frac{\sigma_N^2}{\overline{N}}, \quad (1)$$

where $\overline{N}$ and σ_N^2 are the mean and variance, respectively, of the number of optical photons emitted.

Based on the Fano factor, light sources can be classified into three categories: sub-Poisson ($F_N < 1$), Poisson ($F_N = 1$), and super-Poisson ($F_N > 1$). Light from scintillation crystals has been reported to have Fano factors ranging from sub-Poisson to super-Poisson [2], [3].

In a scintillation gamma-ray camera, the various parameters that describe the interaction of the gamma-ray photon with the detector, such as the position of interaction and energy deposited by a detected gamma-ray photon, are estimated using the detector outputs. Since a reduction in the Fano factor results in a smaller variance in the number of emitted optical photons and consequently a smaller variance in the detector outputs, we would expect that this should also lead to a reduction in variance of the parameters estimated from the low-variance detector outputs. Thus, a variation in the Fano factor could potentially affect the energy and spatial resolution of a gamma-ray imaging system.

We used two approaches to study the impact of the Fano factor on the spatial and energy resolution: calculating the Cramér-Rao Bound (CRB) and estimating the variance of a

maximum likelihood (ML) estimator [4], [5]. CRB is the theoretical lower bound on the variance of an unbiased estimator.

An unbiased estimator is efficient if it achieves the CRB [6]. If an efficient estimator exists, the ML estimator will be efficient. We do not directly prove the existence of an efficient estimator for our problem. However, if the estimates of the variance of the ML estimator are unbiased and approach the CRB, then the results are consistent with the hypothesis that an efficient estimator exists and our ML estimator is efficient. We can then quantitatively validate both of our approaches.

The use of ML estimation methods for position estimation in scintillation gamma-ray detectors was first proposed by Gray and Macovski [7], and then demonstrated on modular gamma cameras [8], [9], [10], [11], [12]. The use of ML position estimation in SPECT imaging systems was demonstrated by Rowe *et al.* [13]. The availability of faster computing, advances in calibration, and faster algorithms have made ML position estimation very fast and inexpensive to implement [14], [15]. The ML estimators have significant advantages over the traditional Anger arithmetic—no bias, lower mean-squared error, and the ability to achieve the CRB [16].

The ability of the ML position estimators to approach the CRB in scintillation gamma-ray detectors has made the CRB a very useful tool. The CRB has been widely used for evaluating the performance of gamma-ray detectors [17], [18]. The CRB has also been used to optimize gamma-camera design [19], [20], evaluate different readout strategies [21], and calculate the theoretical bound on timing resolution [22].

The paper is organized as follows: In Section II-A, we briefly introduce the likelihood function, Fisher information matrix, and Cramér-Rao bound. In Section III, we discuss our model of production and transport of scintillation light. We also discuss the various implementation details, including assumptions and simulation parameters. We introduce two geometries in Section IV—with 3×1 and 3×3 optical detector-elements. For the 3×1 geometry, we analytically calculate the CRB for two special values of the Fano factor ($F_N = 0$ and $F_N = 1$), and use a more general model to numerically calculate the CRB for Fano factors other than zero. We use Monte-Carlo simulations to estimate the variance of the ML estimator for the 3×1 and the 3×3 geometries. The results of the analytical and numerical calculations of the CRB and the variance of the ML estimator for the 3×1 geometry and the 3×3 geometries are discussed in Section V. We evaluate the impact of Fano factor for a simple Anger camera in Appendix B.

II. Theory

A. Likelihood Function

If we intend to estimate a parameter vector $(\vec{\theta})$ from acquired data $(\vec{g})$, we can define the likelihood function as

$$l(\vec{\theta}|\vec{g}) = Pr(\vec{g}|\vec{\theta}). \quad (2)$$

Here, $Pr(\vec{g}|\vec{\theta})$ is the probability of the parameters $(\vec{\theta})$ resulting in data outputs $(\vec{g})$. In a gamma-ray scintillation camera, $\vec{g}$ is a vector of detector outputs for a gamma-ray interaction, and because we are estimating the position of interaction and gamma-ray energy, $\vec{\theta} = (x, y, z, E)$. The likelihood function $l(\vec{\theta}|\vec{g})$ gives us the likelihood of measuring $\vec{g}$ given a gamma-ray photon that deposits energy E at location (x,y,z) in the scintillation crystal [6].

B. Score, Fisher Information Matrix, and Cramér-Rao Bound

The sensitivity of the likelihood function to changes in the parameter vector $\vec{\theta}$ is given by the score $(\vec{s})$. The score is defined as the gradient of the logarithm of the likelihood function (log-likelihood) of the acquired data

$$s_i(\vec{g}|\theta) = \frac{\frac{\partial}{\partial \theta_i} Pr(\vec{g}|\vec{\theta})}{Pr(\vec{g}|\vec{\theta})} = \frac{\partial}{\partial \theta_i} \log(Pr(\vec{g}|\vec{\theta})). \quad (3)$$

The Fisher information matrix (I) is the covariance matrix of the score. Because the mean value of the score is zero [6], the $(i, k)^{\text{th}}$ element of the Fisher information matrix is given by

$$I_{ik} = \langle s_i s_k \rangle_{\vec{g}|\vec{\theta}}. \quad (4)$$

The angle brackets here indicate the expectation value, which involves multiplying s_i, s_k by the probability $Pr(\vec{g}|\vec{\theta})$ and integrating over all the detector outputs $\vec{g}$ for a given value of $\vec{\theta} = (x, y, z, E)$. The diagonal elements of the inverse of the Fisher information matrix give us the CRB

$$CRB_i = (I^{-1})_{ii}. \quad (5)$$

Here, the CRB_i denotes the Cramér-Rao bound for the i^{th} parameter. If we use an unbiased estimator to estimate a parameter $\hat{\theta}_i$, then the variance of $\hat{\theta}_i$ ($Var(\hat{\theta}_{iUB})$) cannot be lower than the CRB of the i^{th} parameter

$$Var(\hat{\theta}_{iUB}) \geq CRB_i. \quad (6)$$

In a typical gamma-ray scintillation camera, four parameters (x,y,z,E) are estimated. Therefore, the complete Fisher information matrix is a 4×4 matrix. However, if we know the value of some of these parameters, then the Fisher information dimensionality reduces. For example, if we place a thin lead slit perpendicular to the y-axis above the scintillator crystal then the slit localizes the y interactions position, and we can treat y as a known parameter and only estimate (x,z,E) , reducing the Fisher information to a 3×3 matrix. In

this scenario, because we assume that the exact value of y is known, the uncertainty in the $\hat{y}$ estimates does not add uncertainty to the $\hat{x}$, $\hat{z}$, and $\hat{E}$ estimates. In fact, it can be mathematically shown that, for any estimation model, the CRB on the i^{th} parameter calculated from a Fisher information matrix of dimensions $a \times a$, denoted by $\text{CRB}_{(a)_i}$, will always be greater than or equal to the CRB calculated from a smaller square sub-matrix of the $a \times a$ Fisher information matrix (see Appendix A for proof)

$$\text{CRB}_{(a)_i} \geq \text{CRB}_{(a-1)_i}. \quad (7)$$

C. Variance of ML Estimator

The variance of an unbiased estimator is a good metric for the resolution of a system. For an unbiased estimator, a smaller variance enables a system to resolve closer values of the parameters, giving the system better resolution.

The ML estimator maximizes the likelihood function to yield the most likely parameter vector $\vec{\theta}$ that would result in output data $\vec{g}$. In our study, detector outputs were generated for a position of interaction (x,y,z) and gamma-ray energy deposited (E). The ML estimator was applied on each sets of detector outputs to estimate $\hat{y}$, $\hat{y}$, $\hat{z}$, and $\hat{E}$. The operator $\arg \max_{\vec{\theta}}$ returns the values of the arguments of the likelihood function at its maximum value

$$\vec{\hat{\theta}} = \arg \max_{\vec{\theta}} \left(l \left(\vec{\theta} | \vec{g} \right) \right). \quad (8)$$

The variance of the ML estimator is estimated by computing the variance of the estimates.

III. Model

A. Modeling the Scintillation Process

When excited by gamma rays, a scintillator crystal de-excites through a complicated cascade process and emits optical photons [1], [23]. In this study, all the scintillation light is assumed to be emitted from the point of interaction. This is an approximation because the energy deposited by the gamma-ray photon produces a high-energy electron, which travels at a high velocity, depositing energy and creating electron-hole pairs along its path. Some of these electron-hole pairs (excitons) recombine radiatively to emit optical scintillation photons, not just at the point of interaction, but along the path of the high-energy electron.

The excitons that de-excite to emit the optical photons have no memory of the direction of the incident gamma ray or the high-energy electron. Hence, it is reasonable to assume that the scintillation photons are emitted isotropically from the point of interaction.

Consider a gamma-ray interaction that deposits energy E in the scintillator. The scintillator de-excites by producing a random number of optical scintillation photons (N). For simplicity, we assume that the gamma-ray energy deposited and the mean number of photons emitted have a linear relationship. Therefore, for a given gamma-ray energy deposited, the mean number of optical photons emitted is given by

$$\overline{N} = Q \cdot E. \quad (9)$$

Here, Q is the average number of optical photons emitted per unit energy deposited. Scintillator non-proportionality can result in a non-linear relationship between E and $\overline{N}$, and make Q a function of deposited energy [24].

If the Fano factor of the scintillator is denoted by F_N , using (1) the variance in the number of optical photons is given by $\sigma_N^2 = F_N \cdot \overline{N}$. The probability of producing N optical scintillation photons given energy E deposited is modeled as a discrete normal distribution with mean $\overline{N}$ and variance $F_N \cdot \overline{N}$. In the simulations, because the mean and variance of this discrete normal distribution are relatively large, the probability of N is approximated by a sampled continuous normal distribution [25] given by

$$\begin{aligned} Pr(N|F_N, E) \\ = \frac{1}{\sqrt{(2\pi F_N Q E)}} \exp\left(-\frac{(N - QE)^2}{2F_N Q E}\right). \end{aligned} \quad (10)$$

Even if Q is a function of the deposited gamma-ray energy, the relationship between the deposited gamma-ray energy and average number of scintillation photons emitted is a monotonically increasing function; as we increase the energy of the gamma-ray photons, on average, a larger number of scintillation photons are emitted. Therefore, instead of estimating the position of interaction and the deposited gamma-ray photon energy (x, y, z, E) , we can estimate the position of interaction and the mean number of scintillation photons emitted $(x, y, z, \overline{N})$.

We use (9) to rewrite the statistical model for scintillation light emission in (10) as a function $\overline{N}$

$$\begin{aligned} Pr(N|F_N, \overline{N}) \\ = \frac{1}{\sqrt{2\pi F_N \overline{N}}} \exp\left(-\frac{(N - \overline{N})^2}{2F_N \overline{N}}\right). \end{aligned} \quad (11)$$

B. Modeling the Optical Photon Transport

If N optical scintillation photons are produced from a gamma-ray interaction at (x, y, z) , the number of detected optical photons on a J -element optical detector follows a multinomial distribution with $J + 1$ outcomes. J of the outcomes are due to the photons detected at detector elements $j=1, 2 \dots J$, with probability of detection at the j^{th} element, α_j . In our model α_j is given by

$$\alpha_j(x, y, z) = \eta \frac{\Omega_j(x, y, z)}{4\pi}. \quad (12)$$

Here, η , the quantum efficiency of the optical detector elements, is assumed to be independent of the angle of incidence, and Ω_j is the effective solid angle subtended by the j^{th}

detector element from the point of interaction (x,y,z) . Specular or Lambertian reflectors can be used to increase the effective solid angle. We have also ignored all scattering processes—only optical photons directly impinging on the detector are considered. Thus, α_j is equal to the product of quantum efficiency and geometrical efficiency of the j^{th} detector element. The $(J+1)^{\text{th}}$ outcome contains all the optical photons not detected by any of the J detector

elements. The probability of an optical photon not being detected is $\left(1 - \sum_{j=1}^J \alpha_j\right)$ [6].

The detector array is assumed to be photon counting and noiseless. These two assumptions ensure that the data outputs are integer-valued and reduce the Fisher information matrix calculation from an integral to a summation. For each gamma-ray event, the detector-array outputs $\vec{g}$ is a J -dimensional integer vector whose j^{th} element is the number of optical photons detected on the j^{th} detector element. The probability of measuring $\vec{g}$ for a gamma-ray interaction which produces N optical photons at (x,y,z) is given by the multinomial distribution

$$Pr(\vec{g}|x, y, z, N) = \frac{N! \left(\prod_{j=1}^J \frac{\alpha_j^{g_j}}{g_j!} \right) \left(1 - \sum_{j=1}^J \alpha_j \right)^{N - \sum_{j=1}^J g_j}}{\left(N - \sum_{j=1}^J g_j \right)!} \quad (13)$$

The complete probability of $\vec{g}$ for an interaction at position (x,y,z) , the Fano factor F_N , and deposited energy E producing on average $\bar{N}$ optical scintillation photons is given by marginalizing (13) over N

$$Pr(\vec{g}|x, y, z, F_N, \bar{N}) = \sum_{N=1}^{\infty} Pr(\vec{g}|x, y, z, N) \times Pr(N|F_N, \bar{N}) \quad (14)$$

We substitute (14) in (2) to obtain an expression for the likelihood function $l(x, y, z, \bar{N}|\vec{g})$ of a gamma-ray interaction at x,y,z with the mean number of optical photons emitted $\bar{N}$, resulting in detector output vector $\vec{g}$.

$$\begin{aligned} l(x, y, z, \bar{N}|F_N, \vec{g}) &= Pr(\vec{g}|x, y, z, F_N, \bar{N}) \\ &= \sum_{N=1}^{\infty} Pr(\vec{g}|x, y, z, N) \times Pr(N|F_N, \bar{N}) \end{aligned} \quad (15)$$

IV. Implementation

A. Assumptions for maximizing the impact of the Fano factor

All practical detectors only convert a fraction of the incident optical photons to photoelectrons, which then are amplified and recorded. The Fano factor of the photoelectrons on the j^{th} detector element (F_{nj}) is given by [26]

$$F_{nj} = 1 + \alpha_j (F_N - 1). \quad (16)$$

Here, α_j is the fraction of emitted optical photons detected at the j^{th} detector element. In the photon transport model described in Section III-B, α_j is the product of the quantum and geometrical efficiencies of the j^{th} detector element. If each detector element captures a very small fraction of scintillation light (small α_j), then irrespective of the Fano factor of the scintillator, all the detectors elements will have Poisson statistics and a Fano factor of one (See (16)). Our initial studies conducted with practical geometries and quantum efficiency of 40% found no impact of the Fano factor on the spatial resolution. To ensure that our simulation results are not an artifact due to low optical photon-collection efficiency, we maximized the impact of the Fano factor on the detector outputs by maximizing the geometrical and quantum efficiencies of the detector elements.

The geometrical efficiency was maximized by using a large-area optical detector divided into a small number of detector elements. Using a 100% reflecting retro-reflector, the scintillation light that is emitted in a direction away from the detector is reflected back onto the detector. The retro-reflector effectively doubles the geometrical efficiency of the detector. The quantum efficiency of the detector is set to one; thus, all optical photons incident on the detector array are detected and counted. Sources of noise which will add variance are assumed to be zero—the photodetector is assumed to be noiseless, and the scintillator crystal is assumed not to scatter or absorb the scintillation light.

B. Computation Limitations for Calculating the Cramér-Rao Bound

To calculate the CRB, we first need to compute the score and its covariance matrix. The computation required to calculate the score can be clearly seen by using (14) to expand the angle brackets in (4)

$$I_{ik} = \sum_{N=1}^{\infty} \sum_{\vec{g}} (s_i s_k) Pr(\vec{g}|x, y, z, N) Pr(N|F_N, \bar{N}). \quad (17)$$

For an optical detector with J detector elements, calculating one matrix element of the Fisher information matrix for one set of $\vec{\theta} = (x, y, z, \bar{N})$ requires $(J + 2)$ dimensional summations. Equation (17) requires $(J + 1)$ dimensional summations, J -dimensional summations over all the detector outputs and a summation over N . In addition, the expression for

$Pr(\vec{g}|x, y, z, \bar{N})$ from (14) has a summation over N . Due to the large computation time required to calculate the Fisher information matrix, we computed the Fisher information matrix only for the geometry with a 3×1 array of detector elements.

C. Geometry of the 3×1 Detector

In one study, the scintillator crystal is a single crystal with dimensions of $20 \text{ cm} \times 20 \text{ cm} \times 2 \text{ cm}$. The scintillator crystal is sandwiched between a $20 \text{ cm} \times 20 \text{ cm} \times 1 \text{ cm}$ light guide of the same refractive index as the scintillator and a retro-reflector on the opposite face (see Fig. 1). Light from the scintillator crystal travels through the light guide onto the optical

detector. Reflectivities at the interfaces between the crystal, light guide, and optical detector are assumed to be zero, and the interface between the crystal and the retro-reflector is assumed to be 100% reflecting. The four other faces of the crystal are blackened and assumed to be 100% absorptive. The surfaces of the light guide, not in contact with the scintillator or the photodetector, are also blackened and assumed to be 100% absorbing. The 3×1 detector geometry has three $5 \text{ cm} \times 15 \text{ cm}$ optical detector elements.

In this 3×1 geometry, as we lose nearly all information about y ; we estimate only x , z , and $\overline{N}$. To minimize computation, y is treated as a known parameter. We used the symmetry of the system and computed the Fisher information matrix for only one side of the detector, as shown in Fig. 1.

The mean detector response function (MDRF) is the average detector response for a given position of interaction and gamma-ray energy (x, y, z, E) . The quantum efficiency is assumed to be one ($\eta = 1$), and the 100% reflecting retroreflector doubles the effective solid angle subtended by each detector element. The expression of the normalized MDRF of j^{th} detector elements is

$$\text{MDRF}_j(x, y, z) = 2 \times \frac{\Omega_j(x, y, z)}{4\pi}. \quad (18)$$

The value of the MDRF and its derivative are very important for the calculation of the Fisher information matrix as well as for ML estimations. At $(x = 0 \text{ cm}, y = 0 \text{ cm}, z = 2 \text{ cm})$, on average 76% of the total emitted scintillation light is collected. As the point of interaction moves towards the edge of the detector at $(x = 5 \text{ cm}, y = 0 \text{ cm}, z = 2 \text{ cm})$, the total average light collection drops marginally to 67%. At the edge of the photodetector $(x = 7.5 \text{ cm}, y = 0 \text{ cm}, z = 2 \text{ cm})$, the total light collection drops to 40%.

D. Geometry of the 3×3 Detector

To ensure that the results from our analysis do not suffer from artifacts due to the one-dimensional geometry of our 3×1 detector or from treating y as a known parameter, the effect of the Fano factor in a 3×3 detector is investigated. The geometry of the scintillator crystal and the retroreflector is identical to the geometry described in Section IV-C. The photodetector is divided into nine $5 \text{ cm} \times \text{cm}$ detector elements in a 3×3 configuration, for a total area of $15 \text{ cm} \times 15 \text{ cm}$.

Computational limitations described in Section IV-B prevent us from computing the CRB for a detector with more than three detector elements. Instead, we perform a Monte-Carlo simulation to estimate the variance of the ML estimator.

E. Analytical Solution

The expression for the score for x for the model described in Section III involves taking the logarithm of a sum of an expression containing a number of factorials (see (13, 19)). To calculate the elements of the Fisher information matrix, covariance of the score must be averaged over all the values of $(\vec{g})$ for the given value of $(x, y, z, \overline{N})$. As a result, a general

analytical solution to (19) with an expression for the Fisher information matrix as a function of the Fano factor is extremely challenging, if not impossible.

$$s_x(\vec{g}|x, y, z, F_N, \bar{N}) = \frac{\partial}{\partial x} \log \left(\sum_{N=1}^{\infty} Pr(\vec{g}|x, y, z, N) \times Pr(N|F_N, \bar{N}) \right). \quad (19)$$

However, we can analytically calculate the elements of the Fisher information matrix for two special cases: $F_N = 0$ and $F_N = 1$. Both calculations were done for the 3×1 geometry shown in Fig. 1. Because we cannot estimate four independent parameters from three detector outputs, we treat y as a known parameter and calculate a 3×3 Fisher information matrix.

Multinomial Case ($F_N = 0$)—When $F_N = 0$, there is no uncertainty in the value of N and

$N = \bar{N}$ ($\bar{N}$ can only take integer values). Therefore, $Pr(\vec{g}|x, y, z, \bar{N})$ reduces to the multinomial distribution $Pr(\vec{g}|x, y, z, N)$ in (13). However, we could not simplify the expressions for scores involving N , which limited us to calculating a 2×2 Fisher information matrix with x and z as unknown parameters and y and $\bar{N}$ as known parameters.

Using the mean, variance, and covariance of the multinomial distribution and some arithmetic, we derived the expressions for the elements of the Fisher information matrix for

the $F_N = 0$, $I_{xx}^{(mn)}$, $I_{xz}^{(mn)}$ and $I_{zz}^{(mn)}$ case as

$$I_{xx}^{(mn)} = \frac{N}{(1-\alpha_1-\alpha_2-\alpha_3)} \times \left\{ \left(\frac{\partial \alpha_1}{\partial x} \right)^2 \left(\frac{1-\alpha_2-\alpha_3}{\alpha_1} \right) + \left(\frac{\partial \alpha_2}{\partial x} \right)^2 \left(\frac{1-\alpha_1-\alpha_3}{\alpha_2} \right) + \left(\frac{\partial \alpha_3}{\partial x} \right)^2 \left(\frac{1-\alpha_1-\alpha_2}{\alpha_3} \right) + 2 \frac{\partial \alpha_1}{\partial x} \frac{\partial \alpha_2}{\partial x} + 2 \frac{\partial \alpha_2}{\partial x} \frac{\partial \alpha_3}{\partial x} + 2 \frac{\partial \alpha_1}{\partial x} \frac{\partial \alpha_3}{\partial x} \right\}, \quad (20)$$

$$I_{zz}^{(mn)} = \frac{N}{(1-\alpha_1-\alpha_2-\alpha_3)} \times \left\{ \left(\frac{\partial \alpha_1}{\partial z} \right)^2 \left(\frac{1-\alpha_2-\alpha_3}{\alpha_1} \right) + \left(\frac{\partial \alpha_2}{\partial z} \right)^2 \left(\frac{1-\alpha_1-\alpha_3}{\alpha_2} \right) + \left(\frac{\partial \alpha_3}{\partial z} \right)^2 \left(\frac{1-\alpha_1-\alpha_2}{\alpha_3} \right) + 2 \frac{\partial \alpha_1}{\partial z} \frac{\partial \alpha_2}{\partial z} + 2 \frac{\partial \alpha_2}{\partial z} \frac{\partial \alpha_3}{\partial z} + 2 \frac{\partial \alpha_1}{\partial z} \frac{\partial \alpha_3}{\partial z} \right\}, \quad (21)$$

$$\begin{aligned}
I_{xz}^{(mn)} &= I_{zx}^{(mn)} = \frac{N}{(1-\alpha_1-\alpha_2-\alpha_3)} \\
&\times \left\{ \left(\frac{\partial \alpha_1}{\partial x} \frac{\partial \alpha_1}{\partial z} \right) \left(\frac{1-\alpha_2-\alpha_3}{\alpha_1} \right) \right. \\
&+ \left(\frac{\partial \alpha_2}{\partial x} \frac{\partial \alpha_2}{\partial z} \right) \left(\frac{1-\alpha_1-\alpha_3}{\alpha_2} \right) \\
&+ \left(\frac{\partial \alpha_3}{\partial x} \frac{\partial \alpha_3}{\partial z} \right)^2 \left(\frac{1-\alpha_1-\alpha_2}{\alpha_3} \right) \\
&+ \frac{\partial \alpha_1}{\partial x} \frac{\partial \alpha_2}{\partial z} + \frac{\partial \alpha_1}{\partial z} \frac{\partial \alpha_2}{\partial x} + \frac{\partial \alpha_2}{\partial x} \frac{\partial \alpha_3}{\partial z} \\
&\left. + \frac{\partial \alpha_2}{\partial z} \frac{\partial \alpha_3}{\partial x} + \frac{\partial \alpha_1}{\partial x} \frac{\partial \alpha_3}{\partial z} + \frac{\partial \alpha_1}{\partial z} \frac{\partial \alpha_3}{\partial x} \right\}. \quad (22)
\end{aligned}$$

We use (20-22) to construct a 2×2 Fisher information matrix for the 3×1 geometry in Section IV-C and calculate the Cramér-Rao bound for the multinomial case ($F_N = 0$).

Poisson Case ($F_N = 1$)—Let us consider the Poisson case ($F_N = 1$) applicable for any geometry with 3 detector-elements. Here, $Pr(N|\bar{N})$ is a Poisson distribution with mean $\bar{N}$

$$Pr(N|\bar{N}) = \frac{\bar{N}^N e^{-\bar{N}}}{N!}. \quad (23)$$

For the case of 3 detector-elements with Poisson-distributed scintillation photons, we treated $x, z, \bar{N}$ as unknown parameters and calculated a 3×3 Fisher information matrix. To compare the analytical results from the $F_N = 0$ and $F_N = 1$ cases, we also calculated a reduced CRB from the 2×2 section of the Fisher information matrix which yields the CRB for the x and z estimates.

Multinomial sampling with three outcomes of the Poisson-distributed scintillation light results in the three Poisson-distributed detector outputs, with means $\alpha_1 \bar{N}$, $\alpha_2 \bar{N}$ and $\alpha_3 \bar{N}$ [6]

$$Pr(\vec{g}|x, y, z, \bar{N}) = \prod_{i=1}^3 \frac{(\alpha_i \bar{N})^{g_i} e^{-(\alpha_i \bar{N})}}{g_i!}. \quad (24)$$

Using the expressions for the mean, variance and covariance of the Poisson distribution we derived the expression for the $I_{xx}^{(poiss)}$, $I_{zz}^{(poiss)}$, $I_{\bar{N}\bar{N}}^{(poiss)}$, $I_{xz}^{(poiss)}$, $I_{x\bar{N}}^{(poiss)}$ and $I_{z\bar{N}}^{(poiss)}$ elements of the Fisher information matrix for $F_N = 1$ case as

$$\begin{aligned}
I_{xx}^{(poiss)} &= \bar{N} \left\{ \frac{1}{\alpha_1} \left(\frac{\partial \alpha_1}{\partial x} \right)^2 \right. \\
&+ \frac{1}{\alpha_2} \left(\frac{\partial \alpha_2}{\partial x} \right)^2 + \frac{1}{\alpha_3} \left(\frac{\partial \alpha_3}{\partial x} \right)^2 \left. \right\}, \quad (25)
\end{aligned}$$

$$\begin{aligned}
I_{zz}^{(poiss)} &= \bar{N} \left\{ \frac{1}{\alpha_1} \left(\frac{\partial \alpha_1}{\partial z} \right)^2 \right. \\
&+ \frac{1}{\alpha_2} \left(\frac{\partial \alpha_2}{\partial z} \right)^2 + \frac{1}{\alpha_3} \left(\frac{\partial \alpha_3}{\partial z} \right)^2 \left. \right\}, \quad (26)
\end{aligned}$$

$$I_{\overline{N}\overline{N}}^{(poiss)} = \frac{\alpha_1 + \alpha_2 + \alpha_3}{\overline{N}}, \quad (27)$$

$$I_{xz}^{(poiss)} = I_{zx}^{(poiss)} = \overline{N} \left\{ \frac{1}{\alpha_1} \left(\frac{\partial \alpha_1}{\partial x} \right) \left(\frac{\partial \alpha_1}{\partial z} \right) + \frac{1}{\alpha_2} \left(\frac{\partial \alpha_2}{\partial x} \right) \left(\frac{\partial \alpha_2}{\partial z} \right) + \frac{1}{\alpha_3} \left(\frac{\partial \alpha_3}{\partial x} \right) \left(\frac{\partial \alpha_3}{\partial z} \right) \right\}, \quad (28)$$

$$I_{x\overline{N}}^{(poiss)} = I_{\overline{N}x}^{(poiss)} = \frac{1}{\overline{N}} \left\{ \left(\frac{\partial \alpha_1}{\partial x} \right) + \frac{1}{\alpha_2} \left(\frac{\partial \alpha_2}{\partial x} \right) + \frac{1}{\alpha_3} \left(\frac{\partial \alpha_3}{\partial x} \right) \right\}. \quad (29)$$

$$I_{z\overline{N}}^{(poiss)} = I_{\overline{N}z}^{(poiss)} = \frac{1}{\overline{N}} \left\{ \left(\frac{\partial \alpha_1}{\partial z} \right) + \frac{1}{\alpha_2} \left(\frac{\partial \alpha_2}{\partial z} \right) + \frac{1}{\alpha_3} \left(\frac{\partial \alpha_3}{\partial z} \right) \right\}. \quad (30)$$

We use (25-30) to construct a 2×2 and 3×3 Fisher information matrix for the 3×1 geometry in Section IV-C and calculate the respective Cramér-Rao bounds for the Poisson case ($F_N = 1$).

F. Calculation of the Fisher Information Matrix and the Cramér-Rao Bound

Analytical Solution for $F_N = 0$ and $F_N = 1$ —The expressions for the elements of the Fisher information matrix described in Section IV-E for the multinomial ($F_N = 0$) and Poisson ($F_N = 1$) cases, along with the derivatives of the MDRF curves shown in Figs. 2 and 3, are used to calculate the elements of the Fisher information matrix for various points of interaction (x, y, z) and the mean number of optical photons emitted $(\overline{N})$.

For the multinomial ($F_N = 0$) case, we are limited to calculating a 2×2 Fisher information matrix, while for the Poisson ($F_N = 1$) case we calculated a 3×3 Fisher information matrix. The Fisher information matrices were numerically inverted to obtain the CRBs.

Numerical Computation of the Cramér-Rao Bound—The numerical computation of the Fisher information matrix for the different Fano factors, at a given position of interaction (x, y, z) and mean number of scintillation photons emitted $(\overline{N})$, requires us to calculate the score for the different parameters which are being estimated. The score for x was calculated numerically using the following expression

$$s_x(\vec{g}) = \frac{\partial}{\partial x} \log \left(\Pr(\vec{g} | x, y, z, F_N, \overline{N}) \right) \approx \frac{\log(\Pr(\vec{g} | x + \Delta x, y, z, F_N, \overline{N})) - \log(\Pr(\vec{g} | x, y, z, F_N, \overline{N}))}{\Delta x}. \quad (31)$$

The convergence of the numerical derivative was verified by using different values of x . Scores for z and $\overline{N}$, denoted by s_z , and $s_{\overline{N}}$, respectively, have similar expressions. For the 3×1 geometry described in Section IV-C, y is treated as a known parameter. Thus, the Fisher information matrix is a 3×3 matrix, and we estimated only three parameters, x , z , and $\overline{N}$.

The elements of the Fisher information matrix for each set of $(x, y, z, \bar{N})$ were calculated by a 5-dimensional summation, summing $\pm 5\sigma$ about the respective mean values in each of the five dimensions. The Fisher information matrix was inverted, and the diagonal elements of the inverse of the Fisher information matrix were the CRB of the respective estimators.

G. Estimating the Variance of the ML Estimator

Generating Data—To estimate the variance of the ML estimator, we used the forward model described in Section III to generate the detector output data for a given position of interaction, energy deposited, and Fano factor. For a given scintillation photon Fano factor (F_N) and the mean number of optical photons emitted ($\bar{N}$), the probability distribution in (11) was sampled to obtain the number of optical photons generated (N) from the scintillation process. Using the geometries of the scintillator and the detector array, the probabilities (a_j) of an optical photon emitted at the point of interaction (x, y, z) creating a photoelectron at the j^{th} detector element were computed. The number of optical photons detected at each detector element were generated using the multinomial statistics in (13) with the total number of optical photons, N , and the probability of detection at j^{th} detector element, a_j . Thus, detector output vectors were generated for a gamma ray with energy E which interacts with a scintillator having a Fano factor F_N , producing on an average $\bar{N}$ optical photons at the point of interaction (x, y, z).

Maximizing the Log-Likelihood—Because the logarithm is a monotonically increasing function, the logarithm of the likelihood function achieves its maximum value at the same points as the likelihood function itself. Instead of maximizing the likelihood function, it is often more convenient to maximize the log-likelihood.

In this study, for a given data vector $\vec{g}$, the negative of log-likelihood function given in (15) was minimized using the Nelder-Mead method to estimate the position of interaction

$(\hat{x}, \hat{y}, \hat{z})$ and the mean number of scintillation photons emitted ($\hat{\bar{N}}$) [27]. This is equivalent to maximizing the log-likelihood of the observed data to obtain the ML estimates

$$(\hat{x}, \hat{y}, \hat{z}, \hat{\bar{N}})_{ML} = \arg \max_{(x, y, z, \bar{N})} \left(\log \left(l(x, y, z, \bar{N} | \vec{g}) \right) \right). \quad (32)$$

V. Results

All the CRB calculations and ML estimations for both the 3×1 geometry shown in Fig. 1 and the 3×3 geometry shown in Fig. 4 were computed for different values of the x coordinate of the point of interaction at $y = 0$ cm and $z = 2$ cm. The gamma-ray energy was arbitrarily assumed to be 70 KeV. We assumed a scintillator yield of 50,000 optical photons per MeV to get on average 3500 optical photons per gamma-ray interaction.

The CRB calculations were only performed for the 3×1 detector geometry given in Fig. 1. The analytically calculated CRB from the 2×2 and 3×3 Fisher information matrices are denoted by CRB_{2A} and CRB_{3A} , respectively, where A denotes analytic. The numerically

computed CRB from the 3×3 Fisher information matrix is denoted by CRB_{3C} , where C denotes computed.

A. Resolution and Variance of the Estimator

The spatial resolution is often defined as the full width at half maximum (FWHM) of the distribution of a position estimator. If the position estimates are assumed to be normally distributed, then the relationship between the variance of the position estimator and FWHM is given by $\text{FWHM} = 2.35\sigma$. Therefore, the spatial resolution of the x ML estimates denoted by δ_{ML_x} is given by

$$\delta_{ML_x} = 2.35\hat{\sigma}_{ML_x}. \quad (33)$$

Because the CRB is a lower bound on the variance of the unbiased estimator, δ_{CRB} is the lower bound on the spatial resolution of an unbiased estimator

$$\delta_{CRB_x} = 2.35\sqrt{(\text{CRB})_x}. \quad (34)$$

A gamma-ray interaction in which all the energy from the gamma ray is deposited in the crystal is a photopeak event. Thousands of these photopeak events are collected to make a histogram. Energy resolution is defined as the ratio of the FWHM and the mean of the photopeak. It is usually expressed as a percentage

$$\delta_E = \frac{\text{FWHM of Photopeak}}{\text{Mean of Photopeak}} \times 100. \quad (35)$$

If we assume that the energy estimates are normally distributed, then the FWHM of the photopeak is given by $\text{FWHM} = 2.35\sigma_E$. Using the relationship between E and $\bar{N}$, $E = Q\bar{N}$ we get $Q\sigma_{\bar{N}}$. Thus, the energy resolution computed using the ML estimates, denoted by δ_{ML_E} , is given by

$$\delta_{ML_E} = \frac{2.35Q\hat{\sigma}_{\bar{N}}}{Q\bar{N}}100 = \frac{2.35\hat{\sigma}_{\bar{N}}}{\bar{N}}100. \quad (36)$$

Similarly, the lower bound on the energy resolution from an unbiased estimator, denoted by δ_{CRB_E} , is given by

$$\delta_{CRB_E} = \frac{2.35\sqrt{\text{CRB}_E}}{\bar{N}}100. \quad (37)$$

B. Results for the 3×1 Detector Geometry

Analytical Solution for the Cramér-Rao Bound for 3×1 Geometry for $F_N = 0$ and $F_N = 1$ —As we could only analytically calculate a 2×2 Fisher information matrix for $F_N = 0$, we compared the bounds on x and z resolution with the resolution bound from the corresponding reduced Fisher information matrix for the $F_N = 1$ case. The CRB computed with the reduced 2×2 Fisher information matrix gives us the bounds for the x and z

resolution, which are applicable when the true values of y and $\overline{N}$ are known. We also analytically calculated CRB_{3A} for the $F_N = 1$ case by using the 3×3 Fisher information matrix. In Fig. 5, the x resolution bounds calculated from the 2×2 and 3×3 Fisher information matrix are plotted. We observe that, for $F_N = 1$, CRB_{3A} values are higher than CRB_{2A} values because, in addition to x and z , $\overline{N}$ is also an unknown parameter. The uncertainty in the estimate of $\hat{\overline{N}}$ and the interaction between the estimators results in increased CRB_{3A} and resolution bounds of the x and z estimators.

The x resolutions bounds calculated from the 2×2 Fisher information matrix in Fig. 5 indicate that, if we only estimate x and z , and know the true values of y and $\overline{N}$, a scintillator with a Fano factor of zero outperforms a scintillator with a Fano factor of one. The dip in all the x resolution bound curves at $x \approx \pm 25$ mm corresponds to the boundary between two detector elements. The resolution bounds for CRB_{2A} for $F_N = 0$, CRB_{2A} for $F_N = 1$ and CRB_{3A} for $F_N = 1$ are very close to each other at $x = 0$ and $x \approx \pm 25$ mm. The knowledge of the true value of $\overline{N}$ results in a dip in the CRB_{2A} bounds for the CRB_{2A} for higher values of x . This is because, a lower total detector output can only be caused by a higher value of x , and not by a lower $\overline{N}$. Due to the uncertainty in $\overline{N}$, the bounds calculated from CRB_{3A} do not dip with increase in x .

The energy estimates and the z estimates are tightly coupled. This is because the number of detected optical photons varies strongly with the energy of the gamma-ray photon as well as with the depth of interaction. In comparison, the variation in the number of detected optical photons with the x or y position of interaction is not as significant. Hence, in Fig. 6, CRB_{2A} , CRB_{2A} , $\hat{z}$ bound for the low-noise multinomial model ($F_N = 0$) is substantially smaller than CRB_{2A} , $\hat{z}$ bound for the Poisson model ($F_N = 1$). When the interactions with the energy estimator are included to calculate the CRB_{3A} for the Poisson model, the CRB_{3A} for the z estimator increases significantly and a large part of the graph is beyond the y -axis of the graph.

Numerical Results of CRB for 3×1 Geometry—The Fisher information matrix was numerically computed for the 3×1 detector geometry described in Section IV-C for Fano factors x from 0.0-1.8 at 34 equally spaced values of from 0.00-4.95 cm, at $y = 0$ cm, and $z = 2$ cm.

The bounds on the x , z , and energy resolutions were numerically computed using a 3×3 Fisher information matrix for different Fano factors and plotted in Figs. 7 and 8. For comparison, the analytically calculated bounds on the x , z and energy resolutions using the 3×3 Fisher information matrix for the Poisson case [from equations (25-30)] are also plotted in Figs. 7 and 8. The numerically computed and the analytically calculated resolution bounds for the $F_N = 1$ case on x and x estimators are in agreement with each other. Despite the assumptions made in Section IV-A to maximize the impact of the Fano factor on detector outputs, CRB_{3A} and CRB_{3C} for the x , as well as z estimators are observed to be independent of the Fano factor.

When the point of interaction is over the center of the detector ($x = 0$ cm, $y = 0$ cm, $z = 2$ cm), we observed that the $x - z$ and $x - \overline{N}$ off-diagonal elements of the Fisher information

matrix are orders of magnitude smaller than the diagonal elements. As the point of interaction is moved away from $x = 0$ cm, we observed that the off-diagonal elements increase by 4-5 orders of magnitude. This indicates that, at the center, the x estimate is independent of the z and $\bar{N}$ estimates, but the same is not true when the point of interaction occurs off-center.

At the boundary between two detector elements, the relatively large MDRF slopes along x make the detector outputs relatively more sensitive to changes in x (see (25)). Therefore, the bound on the x resolution is the smallest over the boundary between two detector elements. As the point of interaction is moved further away from the center of the detector, a smaller fraction of the emitted optical photons are collected, and the CRB_{3C} increases for both the x and z estimators.

The numerically computed energy resolution bounds for different Fano factors and the analytically calculated energy resolution bound for $F_N = 1$ are plotted as a function of x in Fig. 8. As the point of interaction moves away from the center of the detector, the geometrical efficiency of the optical detector reduces and, as per (16), the effect of the Fano factor diminishes. Thus, as the point of interaction shifts away from the center, the spacing between the CRB_{3C} curves of the energy estimator for different Fano factors reduces. The numerically computed energy resolution bound for the $F_N = 1$ case is in agreement with the analytically calculated Poisson case.

ML Estimator for the 3×1 Detector Geometry—The method described in Section IV-G was used to simulate five hundred gamma-ray photopeak events 3×1 for the geometry shown in Fig. 4 to generate detector outputs for equally spaced values of x , $y = 0$ cm, $z = 2$ cm, $\bar{N} = 3500$, and different values of the scintillation Fano factors. To compare the variance of the ML estimator with the CRB_{3C} , we chose the same points of interaction, energy deposited, and Fano factors for which the CRB_{3C} was calculated in Section V-B. The position of interaction ($y = 0$ cm, z) and mean number of optical photons emitted $\bar{N}$ were simultaneously estimated using an ML estimator. The mean and variance of the ML estimator was estimated from the 500 simulated photopeak events. The estimates of the x and z resolutions of the ML x and z estimators are plotted in Fig. 9, and the estimate of the energy resolution is plotted in Fig. 10.

The bias of the position estimate results in distortion of the image (see [16]). The mean of the ML estimates was used to estimate the bias of the ML estimators (see Figs. 11 and 12). The bias of the energy estimator was calculated in KeV using (9) from the bias in the estimate of the mean number of scintillation photons. At $x = \pm 50$ cm, where the bias seems to be maximum, the estimates of the biases of the x , z and E estimators are less than 3% of their true values. Using a larger number of gamma-ray interactions will yield better estimates of the resolution as well as the bias.

Comparison of the Cramér-Rao Bound and the Variance of the ML Estimator—In this section, the CRB_{3C} and the variance of the ML estimator are compared. If the spatial and energy resolution bounds calculated from the CRB_{3C} and the estimates of spatial and

energy resolutions from an unbiased ML estimator are very close, then it is consistent with the hypothesis that an efficient estimator exists, and our ML estimator is efficient.

The spatial and energy resolution bounds calculated from the CRB_{3C} (Section V-B2) and the spatial and energy resolution estimates from the ML estimator (Section V-B3) were compared with each other. All the resolution bound curves for CRB_{3C} and resolution estimates from the ML estimators for all Fano factors considered were found to be in agreement with each other. However, for clarity, only plots for the Fano factor of 0.2 are plotted in Figs. 13 and 14.

The estimate of the variances of the ML estimators becomes more accurate as more gamma-ray events are used for estimation task. The variance of the ML estimator was estimated from 5,000 gamma-ray interactions and compared to the CRB. Computation time limited us to estimating the variance of the ML estimator at one point on the detector. The CRB_{3C} and the variance of the ML estimator were compared at ($x = 0$ cm, $y = 0$ cm, $z = 2$ cm, $\overline{N} = 3500$).

We observe in Figs. 13–16 that the calculations and the estimates of the variance of the ML estimators are in agreement with each other. These observations strongly support our hypothesis that an efficient estimator exists and that our implementation of the ML estimator is efficient.

C. Results for the Detector Geometry

The method described in Section IV-G was used to simulate five hundred gamma-ray photopeak events for the geometry in Fig. 4 to generate detector outputs for equally spaced values of x , $y = 0$ cm, $z = 2$ cm, $\overline{N} = 3500$ and different values of Fano factors. The position of interaction (x, y, z) and mean number of optical photons emitted $\overline{N}$ were simultaneously estimated using an ML estimator. The spatial and energy resolutions of the ML estimator were estimated from each of these 500 interactions. The estimates of the x resolution, as well as the z resolution of the ML estimator as a function of x are plotted in Fig. 17. The estimate of the y resolution as a function of x is plotted in Fig. 18, and the energy resolution of the ML estimator as a function of x is plotted in Fig. 19.

The ML estimates of the x , y , and z resolutions are independent of the Fano factor (see Figs. 17–18). The estimates of the energy resolution for the 3×3 detector elements are plotted in Fig. 19.

Another interesting observation from our simulation is that the variances of the ML estimators at the center of the detector $x = 0$ cm, $y = 0$ cm, $z = 2$ cm in the 3×3 geometry are marginally smaller than the variances of the ML estimator (and CRB) in the 3×1 geometry. However, as we move away from the center of 3×3 the detector, the variances of the ML estimator for the detector geometry are much smaller than the variance of the ML estimator for the 3×1 detector. Thus, the extra information from the 3×3 detector elements not only enables us to estimate the y coordinate of the position of interaction, but also improves the estimates of x , z , and $\overline{N}$. In the 3×1 geometry, we used a three-element data vector to estimate three unknown parameters. In the 3×3 case, we used a nine-element data

vector to estimate four unknown parameters. The improved resolution in the 3×3 geometry is most likely due to a better ratio of data elements to unknown parameters.

VI. Analysis of Results and Conclusions

A. Spatial Resolution and the Fano Factor

Analytical calculation of a reduced 2×2 Fisher information matrix for the 3×1 geometry indicates that, if the correct values of y and $\overline{N}$ are known, then the spatial resolution is better with $F_N = 0$ than with $F_N = 1$.

However, when x , z , and $\overline{N}$ were estimated for the same geometry, the variance of the ML estimator was found to be in agreement with the CRB and independent of the Fano factor. The ML estimates of the spatial resolution from the 3×3 detector geometry (when we are simultaneously estimating the 3-D position of interaction and $\overline{N}$) also indicates that Fano factor does not have any impact on position estimation. Thus, we conclude that, when estimating position and energy simultaneously, the Fano factor does not have any impact on the spatial resolution for the idealized detector configuration that we have considered.

The assumptions made in Section IV-A, namely, ideal detectors with 100% quantum efficiency of detectors, 100% reflecting retro-reflectors, no gain or electronic noise in detectors, and large detector elements, were made to maximize the impact of the Fano factor on the detector outputs. Since, even in this idealized case with assumptions to maximize the impact of Fano factor, the Fano factor has no impact on spatial resolution, we can infer that, for a practical detector with lower quantum and geometrical efficiency, the Fano factor will not impact the spatial resolution.

The reason for lack of impact of Fano factor for Anger gamma-ray cameras are discussed in the appendix Appendix B.

B. Energy Resolution and the Fano Factor

Our results indicate that a smaller Fano factor results in a better energy resolution. Let us

consider a practical detector with light detection efficiency η , gain G , and gain noise $\beta = \frac{\sigma_G^2}{G^2}$. Assuming that the photopeak is normally distributed, the relationship between the Fano factor and energy resolution is given by

$$\delta_E = 2.35 \sqrt{\frac{1 + \beta + \eta(F_N - 1)}{\eta \overline{N}}}. \quad (38)$$

A smaller energy resolution results in a narrower photopeak, making it easier to distinguish between different gamma-ray energies. The width of the photopeak can be calculated if we know the Fano factor and a few geometrical and detector parameters, such as quantum efficiency and gain variance. The knowledge of the width of the photopeak can be used as a prior to constrain an estimation algorithm to further improve the capability of the system to resolve energies. However, exploiting the better energy resolution does not require prior

knowledge of the underlying scintillator Fano factor. The photopeak width can be experimentally measured and used as a prior in the same way described above.

Acknowledgment

The authors would like to thank H. B. Barber and L. R. Furenlid for the insights and discussions which made this work possible. We would also like to thank L. Caucci for setting up and managing our computing cluster.

This work was supported in part by the National Institutes of Health through Grants P41 EB002035, RD1 EB000803, and in part by the Science Foundation Arizona.

Appendix A

In this section, we prove that the diagonal elements of the inverse of a sub-matrix of the Fisher information matrix are less than or equal to the corresponding diagonal elements of the inverse of the complete Fisher information matrix.

Let us consider a Fisher information matrix M . By definition, M is square, symmetric and positive-definite matrix. We write M as a block matrix consisting of four sub-matrices of dimensions given by their indices

$$M_{p \times p} = \begin{bmatrix} A_{m \times m} & B_{m \times n} \\ C_{n \times m} & D_{n \times n} \end{bmatrix}. \quad (39)$$

The inverse of the block matrix M is given by

$$M^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C) & (D - CA^{-1}B)^{-1} \end{bmatrix}. \quad (40)$$

As M is a symmetric matrix, sub-matrix C is a transpose of the sub-matrix B ($C = B^\dagger$). Thus we need to prove that for an arbitrary vector $\vec{V}$

$$\vec{V}^\dagger (A - BD^{-1}B^\dagger)^{-1} \vec{V} \geq \vec{V}^\dagger A^{-1} \vec{V} \quad (41)$$

By definition, matrices A and D are also positive definite. As the inverse of a positive definite is also positive definite, D^{-1} is also positive definite. Therefore the inequality below is satisfied

$$\vec{V}^\dagger (A - BD^{-1}B^\dagger) \vec{V} = \vec{V}^\dagger A \vec{V}. \quad (42)$$

For arbitrary, invertible matrices M_1, M_2 of the same dimensions if the inequality

$$\vec{V}^\dagger (M_1) \vec{V} \leq \vec{V}^\dagger (M_2) \vec{V} \text{ is true then it can be shown that } \vec{V}^\dagger (M_1)^{-1} \vec{V} \geq \vec{V}^\dagger (M_2)^{-1} \vec{V} \text{ [28].}$$

Applying this result to (42) gives us the inequality

$$\vec{V}^\dagger (A - BD^{-1}B^\dagger)^{-1} \vec{V} \geq \vec{V}^\dagger A^{-1} \vec{V} \quad (43)$$

proving that the CRB calculated from a sub-matrix of the Fisher information matrix is less than or equal to the CRB calculated from the complete Fisher information matrix.

Appendix B

Anger arithmetic is a widely used technique for estimating the position of interaction in gamma-ray detectors [29]. Anger arithmetic is fast and easily implemented in hardware. But, if the MDRF is nonlinear, the Anger arithmetic is biased. Therefore, as Anger arithmetic is not an optimal method for position estimation, any impact, or the absence of impact, of the Fano factor on position estimation in Anger arithmetic will not be conclusive, since the results can be attributed to the nature of the estimator. However, due to widespread use of Anger arithmetic, the impact of Fano factor on Anger position estimation is studied.

A. Geometry and Model

Let us again consider the Anger-camera geometry described in VI-A. For our analysis, we chose a geometry given in Fig. 20. As the center of the two detector elements are at $\pm \frac{L}{4} = \pm 3.75$ cm from the center of the detector, the Anger estimate is given by

$$\hat{X} = \frac{L}{4} \times \frac{g_R - g_L}{g_R + g_L}. \quad (44)$$

The optical photon-production and transport models described in Section III were used for the study. The x resolution of the Anger estimator was studied for Fano factors from 0.2–1.8 at 101 equally spaced values of x from -7.5 cm to 7.5 $y = 0$ cm, at $y = 0$ cm, and $z = 2$ cm with $\overline{N} = 3500$.

B. Analytical treatment of Anger Arithmetic

Let us assume that $Pr(g_L|x, y, z, F_N, \overline{N})$ and $Pr(g_R|x, y, z, F_N, \overline{N})$ are normal distributions. This assumption is valid as long as the mean detector outputs are reasonably large. The means, the variances and covariance of the detector outputs and are given by [3]

$$\overline{g_L} = \alpha_L \overline{N}, \quad (45a)$$

$$\overline{g_R} = \alpha_R \overline{N}, \quad (45b)$$

$$\sigma_L^2 = \overline{N} (\alpha_L + \alpha_L^2 (F_N - 1)), \quad (45c)$$

$$\sigma_R^2 = \overline{N} (\alpha_R + \alpha_R^2 (F_N - 1)), \quad (45d)$$

$$\rho_{(LR)} = \overline{N} \left(\frac{\alpha_R \alpha_L (F_N - 1)}{\sigma_L \sigma_R} \right). \quad (45e)$$

Here, α_L and α_R are the probability of a optical photon emitted at (x,y,z) being detected at the left and right detector respectively, and $\rho_{(LR)}$ is the correlation coefficient between the left and right detectors.

Let us define two new random variables, $P = \frac{L}{4} (g_R - g_L)$ and $Q = g_R + g_L$. As P and Q are differences and sums of normally distributed random variables, they are also normally distributed with means, variances and covariance given by

$$\overline{P} = \frac{L}{4} \times (\overline{g_R} - \overline{g_L}), \quad (46a)$$

$$\overline{Q} = \overline{g_R} + \overline{g_L}, \quad (46b)$$

$$\sigma_P = \frac{L}{4} \times \sqrt{\sigma_R^2 + \sigma_L^2 - 2\rho_{(LR)} \sigma_R \sigma_L}, \quad (46c)$$

$$\sigma_Q = \sqrt{\sigma_R^2 + \sigma_L^2 + 2\rho_{(LR)} \sigma_R \sigma_L}, \quad (46d)$$

$$\rho_{(PQ)} = \frac{L}{4} \times (\sigma_R^2 - \sigma_L^2). \quad (46e)$$

The Anger position estimate is a ratio of two normally distributed random variables. The probability density function of the ratio of two dependent, normally-distributed random variables with non-zero means is [30]

$$\begin{aligned} pr(\hat{X}) &= \frac{\sigma_P \sigma_Q \sqrt{1 - \rho_{(PQ)}^2}}{\pi \left(\sigma_Q^2 \hat{X}^2 - 2\rho_{(PQ)} \sigma_P \sigma_Q \hat{X} + \sigma_P^2 \right)} \\ &\times \left[\exp\left(-\frac{1}{2} \cdot \sup R^2\right) + \sqrt{2\pi} R \cdot \Phi(R) \exp\left(-\frac{1}{2} (\sup R^2 - R^2)\right) \right]. \end{aligned} \quad (47)$$

Here, Φ is the error function and the expressions for R and R are

$$\begin{aligned} R(\hat{X}) &= \frac{1}{\sqrt{(1 - \rho_{(PQ)}^2)} \sqrt{\hat{X}^2 - 2\rho_{(PQ)} \frac{\sigma_P}{\sigma_Q} \hat{X} + \left(\frac{\sigma_P}{\sigma_Q}\right)^2}} \\ &\times \left(\frac{\overline{P}}{\sigma_P} - \rho_{(PQ)} \frac{\overline{Q}}{\sigma_Q} \right) \hat{X} - \left(\rho_{(PQ)} \frac{\overline{P}}{\sigma_P} - \frac{\overline{Q}}{\sigma_Q} \right) \frac{\sigma_P}{\sigma_Q}, \end{aligned} \quad (48)$$

$$\sup R^2 = \frac{\left(\frac{\bar{P}}{\sigma_P}\right)^2 - 2\rho_{(PQ)} \frac{\bar{P}\bar{Q}}{\sigma_P\sigma_Q} + \left(\frac{\bar{Q}}{\sigma_Q}\right)^2}{1 - \rho_{(PQ)}^2}. \quad (49)$$

We numerically calculated the expected value of the mean and variance of the $\hat{X}$ estimator using the probability density function

$$\bar{\hat{X}} = \int_{\hat{X}=-\infty}^{\infty} \hat{X} pr(\hat{X}) d\hat{X}, \quad (50a)$$

$$\sigma_{\hat{X}}^2 = \int_{\hat{X}=-\infty}^{\infty} \hat{X}^2 pr(\hat{X}) d\hat{X} - \bar{\hat{X}}^2. \quad (50b)$$

The bias of an estimator is given by

$$bias(x, \bar{\hat{X}}) = \bar{\hat{X}} - x. \quad (51)$$

As the Anger estimator is biased, the variance of the estimator is not a good figure of merit for it. For example, an estimator which has a very high error, but is very precise, will have a low variance. Let us consider an estimator, which independent of the actual value of the parameter, always estimates the value of parameter as the number three. This estimator has a large error and a large bias, but zero variance. The root-mean-squared error (RMSE) is a better metric of estimator performance. The expression for the RMSE is

$$RMSE_{\hat{X}} = \sqrt{\frac{1}{n} \sum_{i=1}^n (\hat{X}_i - x)^2}. \quad (52)$$

Simplifying a little gives us an expression for RMSE as the sum of variance and square of the bias of the estimator

$$RMSE_{\hat{X}} = \sqrt{\sigma_{\hat{X}}^2 + (x - \bar{\hat{X}})^2} = \sqrt{\sigma_{\hat{X}}^2 + bias(x, \bar{\hat{X}})^2}. \quad (53)$$

C. Monte Carlo Simulations of Anger Arithmetic

The optical photon-production and transport models described in Section III were used to generate ten thousand detector outputs (g_R, g_L) for Fano factors from 0.2–1.8 at 101 equally spaced values of x from -7.5 cm to 7.5 cm, at $y = 0$ cm and $z = 2$ cm with $\bar{N}=3500$. For every point of interaction, the mean, variance, bias and mean square error of the x Anger estimator for different Fano factors were computed to evaluate the impact of Fano factor on our Anger camera.

D. Results

The expectation values of the x resolution of the Anger x estimator for the position of interactions, x between -7.5 cm and 7.5 cm, $y = 0$, $z = 2$ cm, and mean number of optical photons, $\overline{N}=3500$, for Fano factors from 0.2 – 1.8 are plotted in Fig. 21. The bias of the Anger estimator as function of x is plotted in Fig. 22. The bias of the Anger estimator does not appear to depend on the Fano factor of optical photons. The bias of the Anger estimator is many orders of magnitude larger than its variance. Hence, the RMSE is dominated by the bias of the Anger estimator. The bias goes to zero in the vicinity of $x = \pm 26$ mm and results in a dip in the RMSE curve.

To validate the analytical results, for each Fano factor, 10,000 detector outputs were generated and Anger x estimates were calculated for the same position of interactions and energy deposited as above. The sample mean and variance of these 10,000 estimates were used to calculate the x resolution, bias and RMSE. The results of the analytical calculations and the Monte Carlo simulations of the Anger arithmetic for the 2×1 geometry are in agreement with each other.

E. Discussion on Spatial Resolution and the Fano Factor in Anger Arithmetic

In this section, the absence of impact of Fano factor is discussed. Most algorithms estimate position by comparing signals on different detectors. Let us consider the simplest position estimating algorithm—a two detector-element Anger camera with the center of the detector-elements at ± 1 units [29]

$$\hat{X} = \frac{g_R - g_L}{g_R + g_L}. \quad (54)$$

Here g_R and g_L are the number of photons detected at the right and left detector elements, respectively.

For the same mean number of optical photons emitted, a scintillator with a larger Fano factor has a higher variance in the number of emitted optical photons and results in a larger range of N about $\overline{N}$, while a scintillator with a smaller Fano factor will have a smaller range of N about $\overline{N}$. As discussed above position estimation is not sensitive to N and therefore, not sensitive to the optical photon Fano factor.

If the number of optical photons emitted changes from N to $N \pm \Delta N$, on average both the detector-elements outputs g_R and g_L change. But on average, the numerator of (53), i.e. the difference between g_R and g_L , does not change. As ΔN is usually much smaller than N , the denominator too does not vary much with changes in N . Thus, Anger arithmetic and the variance of the Anger position estimator are not very sensitive to N .

This effect is also evident from the flat graph of the y resolution of the ML estimator for the 3×3 geometry as a function of x at $y = 0$ cm and $z = 2$ cm in Fig. 18. As the point of interaction moves away from the center of the detector along x at $y = 0$ cm and $z = 2$ cm, the

fraction of emitted light collected by the detector reduces marginally from 76% at $x = 0$ cm to 67% at $x = 5$ cm resulting in substantial degrading of x as well as z resolution, but it does not degrade y resolution. This is because although the total average light collected by all the detector elements to the left and the right of the $y = 0$ cm axis change a little, the difference in the number of optical photons between them or division of light between them is not sensitive to the shift in the point of interaction along the x axis.

References

- [1]. Knoll, GF. Radiation Detection and Measurement. Wiley; Hoboken, NJ, USA: 2010.
- [2]. Bousselham A, Barrett HH, Bora V, Shah K. Photoelectron anticorrelations and sub-Poisson statistics in scintillation detectors. Nucl. Instrum. Meth. A. 2010; 620:359–362.
- [3]. Bora, V.; Barrett, HH.; Shah, KS.; Glodo, J. Estimation of Fano factors in inorganic scintillators. 2011. presented at the IEEE Nuclear Science Symp.
- [4]. Cramér, H. Mathematical Methods of Statistics. Princeton Univ. Press; Princeton, NJ, USA: 1946.
- [5]. Rao CR. Information and the accuracy attainable in the estimation of statistical parameters. Bull. Calcutta Math. Soc. 1945; 37:81–91.
- [6]. Barrett, HH.; Myers, KJ. Foundations of Image Science. Wiley; Hoboken, NJ, USA: 2004.
- [7]. Gray RM, Macovski A. Maximum a posteriori estimation of position in scintillation cameras. IEEE Trans. Nucl. Sci. Feb.1976 NS-23(1):849–852.
- [8]. Milster TD, Selberg LA, Barrett HH, Easton RL, Rossi GR, Arendt J, Simpson RG. A modular scintillation camera for use in nuclear medicine. IEEE Trans. Nucl. Sci. Feb.1984 NS-31(1):578–580.
- [9]. Milster TD, Selberg LA, Barrett H, Landesman A, Seacat R. Digital position estimation for the modular scintillation camera. IEEE Trans. Nucl. Sci. Feb.1985 NS-32:748–752.
- [10]. Clinthorne NH, Rogers WL, Shao L, Koral KF. A hybrid maximum likelihood position computer for scintillation cameras. IEEE Trans. Nucl. Sci. Feb.1987 34(1):97–101.
- [11]. Milster TD, Aarsvold JN, Barrett HH, Landesman AL, Mar LS, Patton DD, Roney TJ, Rowe RK, Seacat RH III. A full-field modular gamma camera. J. Nucl. Med. 1990; 31:632–639.
- [12]. Aarsvold, JN.; Mintzer, RA.; Yasillo, NJ.; abd, CEO.; Chen, CT. Implementations of maximum-likelihood position estimation in a four-PMT scintillation detector. 1995. K. L. M. , I presented at the Nuclear Science Symp. Medical Imaging
- [13]. Rowe RK, Aarsvold JN, Barrett HH, Chen JC, Klein WP, Moore BA, Pang IW, Patton DD, White TA. A stationary hemispherical SPECT imager for three-dimensional brain imaging. J. Nucl. Med. 1993; 34:474–480.
- [14]. Chen, YC.; Furenlid, LR.; Wilson, DW.; Barrett, HH. Small-Animal SPECT Imaging. Springer; New York, NY, USA: 2005. p. 195-201.ch. 12
- [15]. Hesterman JY, Caucci L, Kupinski MA, Barrett HH, Furenlid LR. Maximum-likelihood estimation with a contracting-grid search algorithm. IEEE Trans. Nucl. Sci. Jun.2010 57(3): 1077–1084. [PubMed: 20824155]
- [16]. Barrett HH, Hunter WCJ, Miller BW, Moore SK, Chen Y, Furenlid LR. Maximum-likelihood methods for processing signals from gamma-ray detectors. IEEE Trans. Nucl. Sci. Jun.2009 56(3):725–735. [PubMed: 20107527]
- [17]. Li X, Hunter WCJ, Lewellen TK, Miyaoka RS. Use of Cramer-Rao lower bound for performance evaluation of different monolithic crystal PET detector designs. IEEE Trans. Nucl. Sci. Feb.2012 59(1):3–12. [PubMed: 22685349]
- [18]. Moore, SK. ModPET: Novel applications of scintillation cameras to preclinical PET. Univ. Arizona; Tucson, AZ, USA: 2011. Ph.D. dissertation
- [19]. Korevaar MA, Goorden MC, Beekman FJ. Cramer-Rao lower bound optimization of an EM-CCD-based scintillation gamma camera. Phys. Med. Biol. 2013; 58

- [20]. van der Laan DJ, Maas MC, Schaart DR, Bruyndonckx P, Lonard S, van Eijk CWE. Using Cramer-Rao theory combined with Monte Carlo simulations for the optimization of monolithic scintillator PET detectors. *IEEE Trans. Nucl. Sci.* Jun.2006 53(3):1063–1070.
- [21]. Salcin E, Barber HB, Furenlid LR. Design considerations for the next-generation MAPMT-based monolithic scintillation camera. *Proc. SPIE.* 2011
- [22]. Seifert S, van Dam HT, Schaart DR. The lower bound on the timing resolution of scintillation detectors. *Phys. Med. Biol.* 2012; 57:1797–1814. [PubMed: 22410975]
- [23]. Rodnyi PA. Physical processes in inorganic scintillators. *Laser Opt. Technol.* 1997
- [24]. Moses WW, Bizarri GA, Williams RT, Payne SA, Vasilev AN, Singh J, Grim J, Choong WS. The origins of scintillator non-proportionality. *IEEE Trans. Nucl. Sci.* Oct.2012 59(5):2038–2044.
- [25]. Szablowski PJ. Discrete normal distribution and its relationship with Jacobi theta functions. *Statist. Prob. Lett.* 2001:289–299.
- [26]. Barrett, HH.; Swindell, W. Radiological Imaging. Academic; New York, NY, USA: 1981.
- [27]. Nelder JA, Mead R. A simplex method for function minimization. *The Computer J.* 1965; 7.4:308–313.
- [28]. Horn, RA.; Johnson, CR. Matrix Analysis. Cambridge Univ. Press; Cambridge, U.K.: 1990.
- [29]. Anger HO. Scintillation camera. *Rev. Sci. Instrum.* 1958; 29:27–33.
- [30]. Cedilnik A, Kosmelj K, Blejec A. The distribution of the ratio of jointly normal variables. *Metodoloki zvezki.* 2004; 1:99–108.

Fig. 1.

The 3×1 detector array has three detector elements of $5 \text{ cm} \times 15 \text{ cm}$ each to ensure high collection efficiency of optical photons. The origin of the z-axis is at the surface of the optical detector. The red dots indicate the points of interaction in the crystal at which the CRB and variance of ML estimator were estimated. The top view is shown without the retroreflector.

Fig. 2. Mean detector response function (MDRF) of the 3×1 detector geometry is plotted as a function of x at $y = 0$ cm and $z = 2$ cm.

Fig. 3.

Mean detector response function (MDRF) of the 3×1 detector geometry is plotted as a function of z at $x = 1$ cm and $y = 0$ cm.

Fig. 4.

Detector geometry used in the nine detector-element simulations. The 3×3 detector array has nine-detector elements of dimensions $5 \text{ cm} \times 5 \text{ cm}$ each. The total area of the optical detector is $15 \text{ cm} \times 15 \text{ cm}$. The origin of the z-axis is at the surface of the optical detector. The red dots indicate the points of interaction in the crystal at which the variance of the ML estimator was estimated. The top view is shown without the retroreflector.

Fig. 5.

The x resolution bounds calculated from CRB_{2A} for the multinomial case

($\text{CRB}_{2A} \hat{x}, F_N=0$), from CRB_{2A} for the Poisson case ($\text{CRB}_{2A} \hat{x}, F_N=1$), and from CRB_{3A} for the Poisson case ($\text{CRB}_{3A} \hat{x}, F_N=1$) are plotted as a function of x at $y = 0$ cm, $z = 2$ cm and $\overline{N}=3500$. The CRB_{2A} bound was analytically calculated for the 3×1 detector geometry from a 2×2 Fisher information matrix which treated x and z as parameters to be estimated with y and $\overline{N}$ known. The CRB_{3A} bound was analytically calculated for the same 3×1 detector geometry from 3×3 a Fisher information matrix which treated x and z and $\overline{N}$ as parameters to be estimated and y as a known parameter.

Fig. 6.

The z resolution bounds calculated from CRB_{2A} for the multinomial case

($\text{CRB}_{2A} \hat{z}, F_N=0$), from CRB_{2A} for the Poisson case ($\text{CRB}_{2A} \hat{z}, F_N=1$), and from CRB_{3A} for the Poisson case ($\text{CRB}_{3A} \hat{z}, F_N=1$) are plotted as a function of x at $y = 0$ cm, $z = 2$ cm and $\overline{N}=3500$. The CRB_{2A} bound was analytically calculated for the 3×1 detector geometry from a 2×2 Fisher information matrix which treated x and z as parameters to be estimated with y and $\overline{N}$ known. The CRB_{3A} bound was analytically calculated for the same 3×1 detector geometry from a 3×3 Fisher information matrix which treated x and z and $\overline{N}$ as parameters to be estimated and y as a known parameter.

Fig. 7.

The numerically computed lower bounds on the x resolution ($\text{CRB}_{3C} \hat{x}$) and the z resolution ($\text{CRB}_{3C} \hat{z}$) computed from a 3×3 Fisher information matrix for different Fano factors and the analytically calculated lower bound on the x resolution ($\text{CRB}_{3A} \hat{x}$, $F_N=1$) and the z resolution ($\text{CRB}_{3A} \hat{z}$, $F_N=1$) calculated from a 3×3 Fisher information matrix are plotted as a function of x at $y = 0$ cm, $y = 2$ cm, and $\overline{N}=3500$. The CRB_{3A} and CRB_{3C} bounds were calculated for the 3×1 detector geometry and are applicable when x , z , and $\overline{N}$ are simultaneously estimated, and y is a known parameter. For every value of x , the lower bounds of x and z resolutions for different Fano factors are almost equal and, therefore, on top of each other.

Fig. 8.

The numerically computed energy resolution bounds ($\text{CRB}_{3C} \hat{\bar{N}}$) for different Fano factors and the analytically calculated energy resolution bound ($\text{CRB}_{3A} \hat{\bar{N}}, F_N = 1$) for the Poisson case are plotted as a function of x at $y = 0$ cm, $z = 0$ cm, and $\bar{N} = 3500$. The CRB_{3A} and CRB_{3C} bounds were calculated for the detector geometry and are applicable when x , z , and $\bar{N}$ are simultaneously estimated, and y is a known parameter. The energy resolution bound gets larger as the Fano factor increases. All the above energy bounds were calculated from a 3×3 Fisher information matrix.

Fig. 9.

Estimates of the x resolution ($\text{ML } \hat{x}$) and z resolution ($\text{ML } \hat{z}$) from the ML estimator for Fano factors from 0.2–1.8 are plotted as a function of x at $y = 0$ cm, $z = 2$ cm, and $\overline{N}=3500$. The estimates of the x and z resolutions for the 3×1 geometry were obtained by using the known value $y = 0$ of and simultaneously estimating x , z , and $\overline{N}$.

Fig. 10.

Estimates of the energy resolution from ML estimator $(\text{ML } \hat{N})$ for Fano factors from 0.2–1.8 are plotted as a function of x at $y = 0$ cm, $y = 2$ cm, and $\bar{N}=3500$. The estimates of the 3×1 energy resolution for the geometry were obtained by using the known value of $y = 0$ and simultaneously estimating x , z , and $\bar{N}$.

Fig. 11.

Estimates of the bias of the x estimator $\text{ML}(\hat{x})$ and the z estimator $\text{ML}(\hat{z})$ from the ML estimator for Fano factors from 0.2–1.8 are plotted as a function of x at $y = 0$ cm, $z = 2$ cm, and $\overline{N}=3500$. The estimates of the bias of the x and z estimators for the 3×1 geometry were obtained by using the known value of $y = 0$ and simultaneously estimating x , z , and $\overline{N}$.

Fig. 12.

Estimates of the bias of the energy estimator as a percentage of the true value of energy

$\left(\text{ML } \hat{N} \right)$ from the ML estimator for Fano factors from 0.2–1.8 are plotted as a function of x at $y = 0$ cm, $y = 2$ cm, and $\overline{N}=3500$. The estimates of the bias of the energy estimators for the 3×1 geometry were obtained by using the known value of $y = 0$ and simultaneously estimating x , y , and $\overline{N}$.

Fig. 13.

The numerically calculated x and z resolution bounds from 3×3 a Fisher information matrix ($\text{CRB}_{3C} \hat{x}$ and $\text{CRB}_{3C} \hat{z}$, respectively) and the estimates of the x and z resolutions from the ML estimator ($\text{ML} \hat{x}$ and $\text{ML} \hat{z}$, respectively) for the Fano factor of 0.2 are plotted as a function of x at $y = 0$ cm, $z = 0$ cm, and $\overline{N}$. Both the CRB_{3C} calculations and the ML estimation were performed for the 3×1 detector geometry and are applicable for the problem of estimating x , z , and $\overline{N}$ with a known value of y . The x and z resolution bounds from the CRB_{3C} calculations the x and z resolution estimates from the ML estimators are in agreement with each other.

Fig. 14.

The energy resolution bound calculated from the $(\text{CRB}_{3A} \hat{N})$ and the estimate of the energy resolution from the ML estimator $(\text{ML} \hat{N})$ for the Fano factor = 0.2 are plotted as a function of x at $y = 0$ cm, $z = 2$ cm, and $\overline{N} = 3500$. Both the CRB_{3C} calculations and the ML estimation were performed for the 3×1 detector geometry and are applicable for the problem of estimating x , z , and $\overline{N}$ with a known value of y . The bound on the energy estimator from the CRB calculations and the energy resolution estimate from the ML estimator are in agreement with each other.

Fig. 15.

The bounds on the x and z resolutions ($\text{CRB}_{3C} \hat{x}$ and $\text{CRB}_{3C} \hat{z}$ respectively) and the estimates of the x and z resolutions from the ML estimator ($\text{ML} \hat{x}$ and $\text{ML} \hat{z}$ respectively) are plotted as a function of the Fano factor at $x = 0$ cm, $y = 0$ cm, $z = 2$ cm, and $\overline{N} = 3500$. Both the CRB_{3C} calculations and the ML estimation were performed for the 3×1 detector geometry and are applicable for the problem of estimating x , z , and $\overline{N}$ with a known value of y . The x and z resolution bounds from the CRB_{3C} calculations and the estimates of the and resolutions from ML estimator are in good agreement with each other and independent of the Fano factor.

Fig. 16.

The bound on energy resolution ($\text{CRB}_{3C} \hat{N}$) and the estimate of energy resolution of the ML estimator ($\text{ML} \hat{N}$) are plotted as a function of the Fano factor at $x = 0$ cm, $y = 0$ cm, $z = 2$ cm, and $\bar{N} = 3500$. Both the CRB_{3C} calculations and the ML estimation were performed for the 3×1 detector geometry and are applicable for the problem of estimating x , z , and $\bar{N}$ with a known value of y . The bound of the energy resolution from the CRB calculations and the estimate of the energy resolution of the ML energy estimator are in good agreement with each other.

Fig. 17.

The estimates of the x and z resolutions ($ML \hat{x}$ and $ML \hat{z}$, respectively) for the 3×3 geometry for different Fano factors are plotted as a function of x at $y = 0$ cm, $z = 2$ cm, and $\overline{N}=3500$. The ML estimator simultaneously estimated x , y , z , and $\overline{N}$. The estimates of the and resolutions of the ML estimator appear to be independent of the Fano factor.

Fig. 18.

The estimates of the y resolution of the ML estimator for the 3×3 geometry for different Fano factors are plotted as a function of x at $y = 0$ cm, $z = 2$ cm, and $\overline{N} = 3500$. The ML estimator simultaneously estimated x , y , z , and $\overline{N}$. The estimate of the y resolution of the ML estimator appears to be independent of the Fano factor as well as x .

Fig. 19.

The energy resolutions of the ML estimator for the 3×3 geometry for different Fano factors are plotted as a function of x at $y = 0$ cm, $y = 2$ cm, and $\overline{N}=3500$. The ML estimations performed are applicable for the problem of estimating x , y , z , and $\overline{N}$.

Fig. 20.

Detector geometry used in 2 detector-element simulations. The 2×1 detector array, with each detector-element of dimensions $7.5 \text{ cm} \times 15 \text{ cm}$ ensures high collection efficiency of optical photons. The origin of the z-axis is at the surface of the optical detector. The red dots indicate the points of interaction in the crystal at which the Anger estimator was studied. The top view is shown without the retroreflector.

Fig. 21.

The x resolutions of the Anger $\hat{x}$ estimators for various Fano factors calculated analytically (Anger Anal. $\hat{x}$) and estimated from the Monte Carlo simulations (Anger MC $\hat{x}$) are plotted as a function of x at $y = 0$ cm, $z = 2$ cm and $\bar{N} = 3500$. For all the values of F_N , the Anger resolutions for the different values of Fano factor appear to be very close to each other.

Fig. 22.

The biases of the x Anger estimator from the analytical calculation (Anger Anal. $\hat{x}$) and estimated from the Monte Carlo simulations (Anger MC $\hat{x}$) for various Fano factors are plotted as a function of x at $y = 0$ cm, $z = 2$ cm and $\bar{N} = 3500$. For all the values of F_N , the bias of the Anger estimates for the different values of Fano factor appear to be very close to each other.

Fig. 23.

The RMSE of the x Anger estimator from the analytical calculation (Anger Anal. $\hat{x}$) and estimated from the Monte Carlo simulations (Anger MC $\hat{x}$) for various Fano factors are plotted as a function of x at $y = 0$ cm, $z = 2$ cm and $\overline{N} = 3500$. For all the values of F_N , the RMSE of the Anger estimates for the different values of Fano factor appear to be very close to each other.