A test of deep water Rytov theory at 284 Hz and 107 km in the Philippine Sea

Rex K. Andrew,^{a)} Andrew W. White, and James A. Mercer

Applied Physics Laboratory, University of Washington, Seattle, Washington 98105, USA

Matthew A. Dzieciuch and Peter F. Worcester

Scripps Institution of Oceanography, La Jolla, California 92037, USA

John A. Colosi

Naval Postgraduate School, Monterey, California 93943, USA

(Received 9 January 2015; revised 13 July 2015; accepted 14 August 2015; published online 9 October 2015)

Predictions of log-amplitude variance are compared against sample log-amplitude variances reported by White, Andrew, Mercer, Worcester, Dzieciuch, and Colosi [J. Acoust. Soc. Am. **134**, 3347–3358 (2013)] for measurements acquired during the 2009 Philippine Sea experiment and associated Monte Carlo computations. The predictions here utilize the theory of Munk and Zachariasen [J. Acoust. Soc. Am. **59**, 818–838 (1976)]. The scattering mechanism is the Garrett–Munk internal wave spectrum scaled by metrics based on measured environmental profiles. The transmitter was at 1000 m depth and the receivers at nominal range 107 km and depths 600–1600 m. The signal was a broadband *m*-sequence centered at 284 Hz. Four classes of propagation paths are examined: the first class has a single upper turning point at about 60 m depth; the second and third classes each have two upper turning points at roughly 250 m; the fourth class has three upper turning points at about 450 m. Log-amplitude variance for all paths is predicted to be 0.04–0.09, well within the regime of validity of either Born or Rytov scattering. The predictions are roughly consistent with the measured and Monte Carlo log-amplitude variances, although biased slightly low. Paths turning in the extreme upper ocean (near the mixed layer) seem to incorporate additional scattering mechanisms not included in the original theory.

© 2015 Acoustical Society of America. [<http://dx.doi.org/10.1121/1.4929900>]

[TFD]

Pages: 2015–2023

I. INTRODUCTION

In a previous paper, White *et al.*¹ reported on acoustical fluctuations that were observed in broadband signals centered at 284 Hz and transmitted for over 60 h during a 2009 experiment in the Philippine Sea. The low-latitude sound speed profile and the range of 107 km supported several distinct wholly refracted paths having one, two, or three upper turning points. Paths confined to and below the main thermocline had small log-amplitude variances of 0.04–0.09. The propagation regime suggests that a theory advanced by Munk and Zachariasen² (hereafter, referred to as MZ), which itself is an extension of an underlying theory that assumes weak fluctuations, might accurately model the observations. This paper shows that MZ calculations of log-amplitude variance are roughly consistent within observation error with the actual log-amplitude variance for these paths, although biased slightly low. A path turning very near the surface had twice as much log-amplitude variance and appears to be in a different propagation and/or scattering regime.

The underlying theory belongs to Rytov.³ When the index of refraction inhomogeneities along a propagation

path are weak and smoothly varying in space, Rytov *et al.*³ found an approximate solution to the fundamental scattering equations called the method of smooth perturbations. Conditions for validity of this approximation occur in a number of practical cases, and Rytov theory is used today in characterizing scattering in, for example, terrestrial microwave links and infrared and optical paths.⁴

Adaptation of Rytov theory to underwater acoustics, however, has not been straightforward. In contexts where the propagation path is more or less straight, Rytov theory has proven accurate enough. These configurations generally involve short ranges and high frequencies, e.g., probing turbulent scales of mixing in a coastal underwater channel⁵ over 670 m at 67 kHz, or using ultrasound in a fluid test cell⁶ over 30 cm at 100 kHz.

For longer ranges, the depth dependence of the sound speed causes the propagation path to refract upward, or downward, or both. In order to accommodate these additional propagation features, Munk and Zachariasen applied Rytov theory locally, invoking the local depth and angle of the unperturbed ray (i.e., the ray theoretic solution in an ocean having only a non-stochastic background sound-speed profile.) Statistics were computed by integrating local scattering contributions along the unperturbed ray path from transmitter to receiver.

^{a)}Electronic mail: rex@apl.washington.edu

Predictions from MZ theory (and others) were compared to observations made during several experiments conducted in the 1970s. In the first set of experiments, called here the Cobb Seamount experiments,^{7,8} Ewart⁷ used signals from 2 to 8 kHz over a range of 17.2 km with lower paths characterized by upward turning depths of about 975–1500 m. Desaubies⁹ showed that MZ theory incorporating an appropriately parameterized Garrett–Munk (GM) internal wave spectrum yielded reasonably accurate measures of log-amplitude variance at 4 kHz, but not at 8 kHz, and showed significant discrepancies explaining the log-amplitude auto-spectra for both frequencies.

In a second experiment, Worcester¹⁰ analyzed signals at 2250 Hz transmitted over 25 km via both an upper and lower path. The lower path turned upward at a depth of 1471 m, and the upper path turned downward at 185 m. The analysis showed that fluctuations along both paths were not inconsistent with MZ theory using a version of the GM spectrum. Comparisons of intensity auto- and cross-correlation functions were, however, ambiguous because the experiment duration spanned only 15 h.

It is important to note that in both of these experiments, the acoustic paths can be characterized by eigenrays having shallow launch angles $|\theta_0| \leq 4^\circ$ (with respect to horizontal). We will return to this issue in Sec. VI.

Following these experiments, the focus of deep water research shifted for more than a decade to much longer ranges and much lower frequencies. Rytov theory, in general, and the deep water adaptation of MZ theory in particular, were not expected to be valid in these configurations. Recently, however, Colosi *et al.*¹¹ demonstrated that MZ predictions had reasonable agreement with several signal statistics for an 87 km path at 75 Hz at a mid-latitude site in the North Pacific. This regime is longer in range than the earlier Ewart and Worcester experiments, but at a much lower frequency.

White *et al.*¹ describe a scattering experiment in the Philippine Sea with a similar range—107 km—but a signal frequency (284 Hz) nearly two octaves higher. Corresponding *a posteriori* theoretical Monte Carlo calculations were remarkably accurate for most of the reception configurations. (The calculations are *a posteriori* in the sense that the calculations are made after the experiment using oceanographic measurements obtained during the experiment.) This paper provides associated *a posteriori* MZ theory predictions, completing a three-way comparison between at-sea measurements, Monte Carlo simulations, and theoretical predictions. MZ theory represents an analytical approach to predicting acoustic field statistics, and can provide greater physical insight to mechanisms of randomization than can Monte Carlo simulations. In addition, the MZ calculations are much faster to obtain compared to Monte Carlo results. [In the case presented here, a single MZ calculation required about 1 min, and a Monte Carlo parabolic equation (MCPE) simulation with 226 realizations required several weeks.] This effort therefore serves to further delineate the regions where MZ theory makes reasonably accurate predictions and where the theory develops some discrepancies.

This report is organized as follows. Section II outlines the structure of the calculations used here, including a Fresnel tube computation that is an alternative to that used in Colosi *et al.*¹¹ Section III summarizes the key features of the experiment. Section IV summarizes the theoretical Monte Carlo calculations reported by White *et al.*¹ Section V provides the comparison between the observed acoustical log-amplitude variance and the corresponding *a posteriori* MZ predictions. The theoretical expressions require several environmental profiles and a parameter—the scattering strength—which must be determined from oceanographic measurements. Provenance and construction of these are described in this section. Some of these are common and some are not. Estimation of the scattering strength parameter is not well-known: Sec. V also describes this calculation and, importantly, the error in the estimate. A discussion of the apparent regimes of validity of MZ theory and directions for further investigations are provided in Sec. VI.

II. MUNK-ZACHARIASEN THEORY

There are three key constructs in MZ theory: the eigenray, the associated Fresnel tube, and the space-time spectrum of sound-speed anomalies. Eigenrays have been thoroughly treated in the literature and will not be reviewed here. The theory is, in principle, agnostic with respect to the exact form of the sound-speed anomaly field, but here, as in most previous applications to the open ocean, the sound-speed anomalies are assumed to be due to fluid displacements caused by internal waves, with the space-time internal wave displacement field given by the GM model. This derivation is briefly summarized in Sec. II A. Construction of the Fresnel tube is described in Sec. II B, and Sec. II C describes the numerical calculation required for this study.

MZ theory writes the field $\psi = \psi_0 \exp(\chi + i\phi)$, where ψ_0 is the field in the absence of sound-speed anomalies, and χ is the (zero-mean) log-amplitude fluctuation and ϕ is the zero-mean phase fluctuation. Write $X = \chi + i\phi$. Then

$$\text{var}(\chi) = \langle \chi^2 \rangle = \frac{1}{2} \left(\langle |X|^2 \rangle + \text{Re}\{\langle X^2 \rangle\} \right). \quad (1)$$

Using MZ Eq. (66) for $\langle |X|^2 \rangle$ and MZ Eq. (74) for $\langle X^2 \rangle$ [note that this equation has a sign error; see MZ Eq. (75)] one can obtain

$$\langle \chi^2 \rangle = \frac{2}{\pi} k_0^2 N_0 b \int_0^R dx \sum_{j=1}^{\infty} \frac{1}{j} \int_{\omega_L}^{\omega} d\omega \frac{S_{\mu\mu}(\omega, j)}{\sqrt{\omega^2 - \omega_L^2}} \frac{\sin^2 \gamma}{\cos^2 \theta}. \quad (2)$$

Quantities θ , ω_L , and γ are all functions of the unperturbed eigenray path depth $z_{\text{ray}}(x)$, which in turn is parameterized by range x from 0 to R . Wave number $k_0 = 2\pi f/c_0$, where f is the acoustical frequency in Hertz and c_0 is a reference sound speed. The product $N_0 b$ incorporates the Brunt–Väisälä frequency scale N_0 and a depth scale b , and is $\int N(z) dz$ where $N(z)$ is the Brunt–Väisälä or buoyancy frequency profile; however, N_0 and b always appear together, and therefore $N_0 b$ will always be treated as a single

quantity.¹² Angle θ is the path inclination relative to the horizontal. The term $\omega_L^2 = f_c^2 + N(z)^2 \tan^2 \theta$, where f_c is the local Coriolis parameter (here, in radians/s). $S_{\mu\mu}(\omega, j)$ is the spectrum (described more fully in Sec. II A) of fractional sound-speed anomalies μ ($\equiv \delta c/c$) as a function of internal wave frequency ω and mode number j . The term γ is given in MZ by

$$\gamma = \frac{\pi}{4} \frac{j^2}{(N_0 b)^2} \left((\omega^2 - \omega_L^2) R_{Fy}^2 + N(z)^2 R_{Fz}^2 \right), \quad (3)$$

where $R_{Fy}^2 = 2\pi x(R-x)/k_0 R$ is the horizontal Fresnel radius, and R_{Fz}^2 is the vertical Fresnel radius. Since the horizontal correlation length of the sound-speed anomalies is much greater than the vertical correlation length, the first term is generally much smaller than the second and is neglected. This leaves

$$\gamma \approx \frac{1}{4\pi} \left(\frac{\pi j N(z)}{N_0 b} \right)^2 R_{Fz}^2 = \pi \left(\frac{j N(z) R_{Fz}}{2 N_0 b} \right)^2. \quad (4)$$

A. Spectrum of sound-speed anomalies

The spectrum of fractional sound-speed anomalies $S_{\mu\mu}(\omega, j)$ due to internal wave displacements requires a mapping $\mu = T\zeta$ between fractional sound-speed anomaly μ ($= \delta c/c$) and vertical fluid displacement ζ . The standard transformation is given by MZ as

$$\mathcal{T}_c = (1/\bar{c})(\partial_z \bar{c}_P). \quad (5)$$

Here and below, the term $\bar{c}$ denotes the depth-dependent background sound-speed profile; explicit depth dependence will be omitted. The potential sound-speed gradient is

$$\partial_z \bar{c}_P = \partial_z \bar{c} - \partial_z c_A. \quad (6)$$

Here, c_A is the adiabatic sound speed, such that $-\bar{c}^{-1} \partial_z c_A \equiv \gamma_A = 1.14 \times 10^{-5} \text{ m}^{-1}$ [Flatté *et al.*,¹³ Eq. (1.1.10)].

Given this mapping, the spectrum of fractional sound-speed anomalies is

$$S_{\mu\mu}(\omega, j) = \mathcal{T}_c^2 S_{\zeta\zeta}(\omega, j). \quad (7)$$

The most mature version¹⁴ of the GM spectrum of vertical fluid displacements can be written as¹²

$$S_{\zeta\zeta}(\omega, j) = b E_0 \frac{N_0 b}{N(z)} \frac{\omega^2 - f_c^2}{\omega^2} H(j) B(\omega). \quad (8)$$

Here $H(j) = H_0^{-1}(j^2 + j_*^2)^{-1}$ is the modal spectrum and H_0 is such that $\sum_0^\infty H(j) = 1$, and $B(\omega)$ is

$$B(\omega) = \frac{2f_c}{\pi \omega} \frac{1}{\sqrt{\omega^2 - f_c^2}} \quad (9)$$

and is normalized so that $\int_{f_c}^{N(z)} B(\omega) d\omega \approx 1$ for $N(z)/f_c \gg 1$. The product $b E_0$ incorporates the energy scale E_0 and the depth scale b ; however, these two terms always also appear

together, and like $N_0 b$ will be considered a single quantity.¹² Combining Eqs. (7)–(9) yields

$$S_{\mu\mu}(\omega, j) = \mathcal{T}_c^2 b E_0 \frac{N_0 b}{N(z)} H(j) D(\omega), \quad (10)$$

where the function $D(\omega)$ is

$$D(\omega) = \frac{2f_c}{\pi} \frac{\sqrt{\omega^2 - f_c^2}}{\omega^3}. \quad (11)$$

The normalization of $B(\omega)$ provides $\int_{f_c}^{N(z)} D(\omega) d\omega \approx \frac{1}{2}$ for $N(z)/f_c \gg 1$.

B. Fresnel tubes

The line of sight in ocean acoustics is the eigenray, which is generally not straight over any typical range. Let the eigenray be the parametric curve $z_{\text{ray}}(x)$; there is no loss of generality in restricting the path to the two-dimensional system with range x and depth z . The transmitter is at point $\mathcal{S} = (0, z_{\text{ray}}(0))$ and the receiver at point $\mathcal{R} = (R, z_{\text{ray}}(R))$. The Fresnel tube is defined as that volume around the eigenray whose boundary points $\mathcal{B}$ satisfy

$$\frac{1}{2}T = |[\tau(\mathcal{B}, \mathcal{S}) + \tau(\mathcal{B}, \mathcal{R}) - \tau(\mathcal{R}, \mathcal{S})]|, \quad (12)$$

where $\tau(\mathcal{B}, \mathcal{A})$ is the travel time from point $\mathcal{A}$ to point $\mathcal{B}$. The eigenray does not formally have an associated frequency, but the Fresnel concept is frequency-dependent, and therefore T is the period of the propagating field, assumed here to be monochromatic. Equation (12) says that the Fresnel tube boundary is defined when the difference in travel time between the travel time along the eigenray and the travel time along an alternate nearby path to and from a boundary point is $\pm$ half a wave period. (Some authors use $\pm$ quarter wave period.¹⁵)

Flatté *et al.*¹³ give a prescription for computing the boundary of the Fresnel tube. An alternate approach is taken here, utilizing dynamic ray tracing theory.¹⁶ In addition to producing Fresnel tubes for simple propagation paths such as those defined here, dynamic ray tracing can handle more complex situations such as range-dependent media and reflections from interfaces. The approach uses a local coordinate system in which one axis (say, the third axis, $\hat{\mathbf{q}}_3$) is aligned along the local tangent to the ray. The first axis, $\hat{\mathbf{q}}_1$, is then located in the (x, z) plane (i.e., in the vertical plane), and the second axis, $\hat{\mathbf{q}}_2$, is directed out-of-plane.

The derivation of the Fresnel tube is thoroughly described in the dynamic ray tracing literature¹⁷; only key points are mentioned here. In the paraxial approximation, one obtains

$$\frac{1}{2}T = \frac{1}{2} \mathbf{u}^T [\mathbf{M}(\mathcal{O}, \mathcal{S}) + \mathbf{M}(\mathcal{O}, \mathcal{R})] \mathbf{u}, \quad (13)$$

where $\mathbf{u} = [u_1 \hat{\mathbf{q}}_1 + u_2 \hat{\mathbf{q}}_2]$. $\mathbf{M}$ is a 2×2 propagator matrix. Equation (13) is solved for (u_1, u_2) at every point $\mathcal{O}$ on the eigenray. Due to the anisotropic nature of the sound-speed

anomaly inhomogeneities, the influence of u_1 dominates that of u_2 , and therefore Eq. (13) can be simplified to

$$\frac{1}{2}T = \frac{1}{2}u_1^2([\mathbf{M}(\mathcal{O}, \mathcal{S})]_{1,1} + [\mathbf{M}(\mathcal{O}, \mathcal{R})]_{1,1}), \quad (14)$$

which only requires the first element of the propagator matrix. The boundary of the Fresnel tube (under the paraxial approximation) in the $\hat{\mathbf{q}}_1$ direction is then given by

$$u_1^{-1} = \sqrt{f} \sqrt{[\mathbf{M}(\mathcal{O}, \mathcal{S})]_{1,1} + [\mathbf{M}(\mathcal{O}, \mathcal{R})]_{1,1}} \quad (15)$$

for an acoustical frequency f .

For a cylindrically symmetric ocean model, the propagator matrix is particularly simple and is completely described in this case by Porter and Bucker.¹⁸ The computations are implemented in the propagation code BELLHOP. In practice, there are two complications. BELLHOP uses an adaptive step integrator and therefore solutions for the propagator elements and, hence, solutions u_1 do not occur on a regular x -axis grid. Second, the boundary point $\mathcal{B}$ for each solution $u_1(x)$ does not occur on a regular x -axis grid either, because the local coordinate system is generally rotated relative to the local Cartesian (x, y, z) system. Both problems are easily circumvented via interpolation.

An example tube calculation (for one of the paths described later) is shown in Fig. 1.

C. Numerical details

Equation (2) is the general MZ expression. For the specific case involving the GM spectrum,¹⁴ $S_{\mu\mu}(\omega, j)$ is separable and the integration over ω can be done analytically. The numerical evaluation of Eq. (2) proceeds by writing

$$\langle \chi^2 \rangle = A \sum_{j=1}^J j^{-1} H(j) I_j, \quad (16)$$

where the summation upper limit has been truncated at $J, A = (4/\pi^2) b E_0 (N_0 b)^2 f_C k_0^2$ and I_j is the one-dimensional integral

FIG. 1. Fresnel tube (vertical slice), 284 Hz, Philippine Sea 2009 experiment. The eigenray is the solid black line, the tube is represented by the shaded region.

$$I_j = \int_0^R dx Q_j(x) = R \int_0^1 d\alpha Q_j(\alpha R), \quad (17)$$

with integrand

$$Q_j(x) = Q(x; j) = \frac{T_c^2 \sin^2 \gamma}{N \cos^2 \theta} W. \quad (18)$$

All terms are implicit functions of range via a dependence on $z_{\text{ray}}(x)$. The term $W(x)$ is

$$W(x) = \int_{\omega_L}^{N(z)} \frac{d\omega \sqrt{\omega^2 - f_C^2}}{\omega^3 \sqrt{\omega^2 - \omega_L^2}}. \quad (19)$$

This integral has branch points at $|\omega| = \omega_L$ (which are also singular points.) Generally, $\omega_L > f_C$ except for $\theta = 0$. Using the branch cut $[-\omega_L, \omega_L]$, a closed form solution was found for $\theta \neq 0$ by Colosi *et al.*¹¹ An equivalent expression for $\theta \neq 0$ is

$$W(x) = \frac{\sqrt{1 - f_C^2/N(z)^2} \sqrt{1 - \omega_L^2/N(z)^2}}{2\omega_L^2} + \frac{1}{2\omega_L^2} \left(\frac{\omega_L}{f_C} - \frac{f_C}{\omega_L} \right) \coth^{-1} \left(\frac{\omega_L \sqrt{1 - f_C^2/N(z)^2}}{f_C \sqrt{1 - \omega_L^2/N(z)^2}} \right) \quad (20)$$

and at $\theta = 0$ the solution follows directly from Eq. (19) and is

$$W(x) = [1 - f_C^2/N(z)^2]/2f_C^2.$$

Romberg quadrature is used for the numerical estimation of the integrals of Eq. (17).

III. EXPERIMENTAL MEASUREMENTS

The at-sea data were collected during the 2009 Philippine Sea experiment (PhilSea09). Worcester *et al.*¹⁹ has given an overview of the entire Philippine Sea effort. White *et al.*¹ provided a comprehensive description of this experiment, so only a summary will be provided here.

An electro-acoustical transmitter was suspended at 22°16.95'N, 126°19.77'E to a depth of 1000 m from the R/V *Melville*. The transducer has sharp resonances at about 210 Hz and 320 Hz. The transmitted signal was a 1023-bit *m*-sequence²⁰ with a carrier frequency of 284 Hz and 2 cycles per bit. The transducer drive signal was equalized to mitigate the distortion induced by the doubly peaked system transfer function, yielding a reasonable facsimile of an *m*-sequence with a 130 Hz bandwidth and pulse resolution of 7.7 ms. The root-mean-square (RMS) source level was roughly 183 dB re 1 μ Pa at 1 m.

The receiver was located at 21°21.90'N, 126°01.02'E and involved a 5 km mooring suspended from a subsurface float at a range of 107 km and a bearing of 198° from the transmitter. The mooring included two autonomous subarrays of hydrophones, each consisting of 30 hydrophones and

one controller.²¹ The upper subarray had an inner interelement spacing of 25 m for 25 hydrophones and of 75 m for two hydrophones above and three hydrophones below the inner hydrophones. It spanned the deep sound channel (centered around about 1040 m). Wavefield statistics from the lower subarray were not reported by White *et al.*¹ and therefore the lower subarray will not be described further. The mooring was also outfitted with Sea-Bird SBE37-SMP pumped conductivity-temperature-pressure microCAT conductivity-temperature-depth (CTD) recorders,²² Sea-Bird SBE39 temperature recorders,²³ and a long-baseline acoustical navigation system.

The *m*-sequences were transmitted beginning roughly at 10:00 UTC on yearday 117 and continued for 60 h. Each *m*-sequence was 7.2 s in length. Sequences were transmitted continuously in a schedule of 10 min on, 5 min off, 10 min on, 5 min off, 15 min on, and off for the remainder of each hour. Data records heavily contaminated by in-water signals from unidentifiable sources were removed.

After pulse-compressing the raw data, White *et al.*¹ mapped eigenray path identifiers to individual branches of the acoustical time front. Subsections of four branches could be resolved, each involving roughly a dozen receiver depths. In every receiver time series where a particular branch could be isolated, a short time segment containing the complex pulse waveform was extracted for that branch. This yielded sample sets over the duration of the experiment of $\sim 15\,700$ pulses for every depth and branch combination, excluding combinations where branches overlapped or crossed. Each pulse was transformed into the Fourier domain. The data record for each available sensor depth and time front branch combination then consisted of the time series of squared magnitudes (i.e., the intensity) of the Fourier coefficient corresponding to the carrier frequency. White *et al.*¹ do not give the sample variances of these records themselves, but rather the span of the corresponding 95% confidence interval; see their paper for a discussion of how this confidence interval was estimated. The log-amplitude variance spans used here are taken directly from their Fig. 8 (where they give RMS log-intensity in decibels.)

White *et al.*¹ analyzed four time front branch subsections. Eigenray paths for each subsection were similar: (a) three turning points initially launched downward, denoted ID-3; (b) four turning points launched upward (ID+4); (c) four turning points launched downward (ID-4); (d) five turning points launched upward (ID+5). Representative paths are shown in Fig. 2.

IV. MONTE CARLO CALCULATIONS

The commonly accepted standard for wave-theoretic solutions in the paraxial approximation for weakly range-dependent channels is the parabolic equation, particularly for long-range, deep-water ocean acoustic scenarios where bathymetric interaction is minimal or absent, and only fields propagating near horizontal are important. This approach integrates well with the Monte Carlo method for simulating the distribution of wavefield statistics. White *et al.*¹ used this MCPE approach in an *a posteriori* prediction of

FIG. 2. Example eigenrays. The transmitter was at depth of 1000 m. Paths are shown to sensor HM028 at a nominal depth of 1550 m at range 107 km. The thick solid line at the bottom is the Smith-Sandwell bathymetry (Ref. 24). Note that paths ID+4 and ID-4 have very similar characteristics—numbers of upper and lower turns, upper turning depth—but path ID-3 is quite different.

log-amplitude variances. Details on the parabolic equation computation are given by White *et al.*¹ Each Monte Carlo realization built an acoustic channel on the background sound speed $\bar{c}(z)$ by adding a random range-depth realization of sound-speed anomalies constructed using the Henryey-Reynolds algorithm.²⁵ White *et al.*¹ set the strength of the sound-speed anomalies by analyzing the original receiver mooring microCAT CTD data to obtain a GM strength of ≈ 1.6 . (More on this in Sec. V.) This value is similar to the value of ≈ 1.4 found by Colosi *et al.*²⁶ Since the MCPE method is computationally intensive, only 226 realizations were obtained.

V. ANALYSIS

In an *a posteriori* prediction, there are several environmental quantities that must be estimated and/or constructed from measurements made during the experiment prior to the primary calculation. The background sound speed profile $\bar{c}(z)$, Brunt-Väisälä frequency profile $N(z)$, and $\partial_z \bar{c}_p$ profile were the same profiles used by White *et al.*¹ in their MCPE simulations.

Estimation of the product bE_0 , required in Eq. (10), is less straightforward. Colosi *et al.*²⁶ provide a thorough discussion of the random oceanographic fields during PhilSea09, but do not provide an explicit estimate of bE_0 . Alternatively, White *et al.*¹ describe the estimation of the GM strength coefficient in terms of a ratio $\hat{r}$ of observed integrated spectral energy $\hat{s}$ to integrated spectral energy s_{GM1} based on the standard GM79 model at unity strength. Under the paradigm that $s_{\text{GM1}} = \frac{1}{2}(bE_{\text{GM1}})(N_0b)$ [using their Eq. (1) and our notation], one can define

$$\hat{r} = \widehat{bE_0} / bE_{\text{GM1}}. \quad (21)$$

As with bE_0 , the product bE_{GM1} is considered a single variable. Under the standard model, one has²⁵

$$\frac{1}{2}(bE_{\text{GM1}})(N_0b) = 0.277 \text{ m}^2\text{rads/sec} \quad (22)$$

and hence $bE_{\text{GM1}} = 0.056 \text{ m}$ for the value of N_0b in effect here. White *et al.*¹ describe depth-averaged lower and upper bounds on $\hat{r}$, derived by examining one month records of

sound-speed time-series as measured by the 12 pumped microCATs on the receiver mooring. A lower bound of 1.6 is derived by ignoring tidal peaks in the spectra, under the assumption that tidal processes have insignificant effects on intensity fluctuations. This assumption is commonly made because tidal effects are considered to be large vertical scale perturbations, and large scale perturbations affect primarily the travel time. The intensity, however, is affected mainly by the small scale perturbations. Using their Fig. 8, the apparent standard error on this lower bound is 0.12 (assuming 12 independent measurements.) From Eq. (21), this implies $\widehat{bE_0} = 0.089 \pm 0.007$ m, ignoring tidal energy. An upper bound of 3.33 is also given, which includes tidal energy. The lower bound is used by White *et al.*¹ in reporting their MCPE calculations (Sec. IV). We address both bounds in the discussion below.

With this value of $\widehat{bE_0}$ and the other parameters shown in Table I, the log-amplitude variance equation (2) can now be computed and compared to the actual measurements. The integrals of Eq. (17) were estimated numerically with Romberg quadrature. Romberg quadrature is an adaptive algorithm whose result will be interpreted below as a random variable with a uniform distribution of $U_{[I_* - \delta, I_* + \delta]}$, where the estimated integral (i.e., at a given mode number j) is I_* and $\delta = \epsilon I_*$. Here, ϵ is the relative estimated error, which is preset in the algorithm to 1 part in 10^6 . The variance of a random variable with distribution $U_{[a,b]}$ is $(b-a)^2/12$; hence, the RMS error is $\delta/\sqrt{3}$. This value is considered the standard error. In order to compare to the 95% confidence interval error bars used by White *et al.*,¹ each integration was assigned an error of twice this RMS value. Errors for the entire calculation (i.e., over all modes) used a quadrature sum of the individual errors. The final error was obtained by the quadrature sum of the total quadrature estimation errors and the error for $\widehat{bE_0}$. In all cases, the error on $\widehat{bE_0}$ dominated.

The ranges of MZ log-amplitude variance for each class of path are shown in Table II. Also given are the corresponding values for the scintillation index and the RMS log-intensity. The full comparison of at-sea measurements, Monte Carlo calculations, and MZ predictions versus depth for all four path classes is shown in Fig. 3.

TABLE I. Parameters used in the calculation. The term $N_0 b = \int N(z) dz$ and is computed to be roughly 9.98 rads-m/s. (The integral is over the entire water column.)

Parameter	Symbol	Value
Latitude		21.5°
Buoyancy frequency	$N(z)$	See text
	$N_0 b$	See below
Coriolis parameter	f_c	5.33×10^{-5} rad/s
Modal bandwidth	j_*	3
Maximum modes	J	120
Reference sound speed	c_0	1480 m/s
Background sound speed	$\bar{c}(z)$	See text
Acoustical frequency	f	284 Hz
Strength	$\widehat{bE_0}$	0.089 ± 0.007 m

TABLE II. Comparison of typical log-amplitude and log-intensity fluctuation statistics from the MZ computations. The scintillation index (SI) is approximately $4 \text{ var } \chi$ for $\text{var } \chi \ll 1$. The RMS log-intensity (RMS I) is given in units of decibels, following Colosi *et al.* (Ref. 11); these are the same units as those used in Fig. 3.

Path	$\text{var } \chi$	SI	RMS I
ID-3	0.038–0.043	0.15–0.17	1.7–1.8
ID+4	0.067–0.074	0.27–0.30	2.3–2.4
ID-4	0.067–0.076	0.27–0.30	2.3–2.4
ID+5	0.080–0.090	0.32–0.36	2.5–2.6

VI. DISCUSSION

These comparisons indicate that *a posteriori* MZ theory is largely consistent with at-sea measurements and MCPE calculations of log-amplitude variance at the 5% level of significance for paths ID+4, ID-4, and ID+5 at 284 Hz over 107 km. This conclusion is reinforced by the power of the three measurement campaigns:

- The duration (60 h) and pulse repetition rate (500/h) guaranteed a formidable at-sea dataset; assuming an intensity correlation time of 5 m and a 50% gap loss, White *et al.*¹ still obtained roughly 360 independent samples.
- The MCPE calculations used 226 independent samples; while more would be preferable, this is commensurate with the actual data.
- The numerical estimation of integrals equation (17) continually refines its estimate of the integral until it reaches the pre-defined error bound 1 part in 10^6 .

The three-way comparison reported here is remarkably consistent. Even though the actual measurements were mediated by a mixture of oceanographic processes and the MCPE predictions were driven solely by GM-style internal waves, the correspondence between the actual measurements and the MCPE predictions is excellent. The correspondence between the MCPE and MZ predictions is also very good, which should follow because both calculations used the same scattering model. The results also suggest that MCPE calculations are more accurate predictions of the measurements than MZ predictions (using the scattering strength lower bound), with MZ slightly underpredicting the log-amplitude variance. More stringent error bounds would be needed to firmly support this conclusion.

The results for paths ID+4 and ID-4 are similar, which should be expected since the two paths themselves have similar characteristics. Agreement between observed data and either MZ predictions or MCPE calculations is not as good for path ID-3, but this is not unexpected. Figure 2 shows that this path has an upper turning depth of roughly 60 m. This near-surface region experiences considerable oceanographic variability (mixed layer dynamics, convective turbulence, etc.) not included in the GM model. This variability can cause a variety of effects not represented by the simplistic eigenray solution shown in Fig. 2, such as a surface duct or even surface reflections. White *et al.*¹ consider possible hypotheses in more comprehensive detail; here, it should suffice to note that inspection of the intensity record for path

FIG. 3. Comparison of log-intensity variances (presented as RMS log-intensity i) for the Philippine Sea 2009 experiment, 107 km, 284 Hz. Panels left to right correspond to ray path geometries ID-3, ID+4, ID-4, and ID+5. Error bar spans for at-sea measurements (PS2009) and Monte Carlo estimates (MCPE) are taken from White *et al.* (Ref. 1) and are two standard errors. The at-sea and Monte Carlo error span center points are merely the span midpoints and are only provided for reference; they are not given *per se* by White *et al.* See the text for a discussion of the solid MZ prediction error bars. The dashed MZ error bars reflect the lower (left endpoint) and upper (right endpoint) bounds given by White *et al.* MCPE results offset downward by 10 m, MZ results by 20 m for clarity. Note that the left panel has a different abscissa scale to accommodate the at-sea measurements below about 1000 m. Since $\text{var } i = 4 \text{ var } \chi$, $\text{RMS } i = (10/\ln 10)\sqrt{\text{var } i} = (20/\ln 10)\sqrt{\text{var } \chi}$ in decibels.

ID-3 (their Fig. 4) seems to indicate that the intensity fluctuations were modulated by a strong non-GM diurnal process. MZ theory incorporating the GM model has no mechanism to include this.

White *et al.*¹ allowed for internal wave strength to be between GM strength parameter 1.6 and 3.33, largely because it is not well understood how tidal spectral energy, a significant component in the PhilSea09 environment, influences acoustic signal fluctuations. Tidally forced sound-speed fluctuations are considerably unidirectional and have no representation in the GM model, which assumes horizontally isotropic internal waves. MZ predictions using GM strengths [1.6,3.33] span all measurements for paths ID+4, ID-4, and ID+5: this level of uncertainty supports only the broadest of conclusions on the model. White *et al.* speculated that the lower bound (vice the upper bound) was a better estimate of scattering strength for intensity fluctuations because it provided a better spectral fit at high frequencies, away from the strong tidal peaks. Indeed, the results presented here can be used in a rudimentary inverse problem: the MZ predictions suggest that the scattering strength is at or perhaps slightly higher than the lower bound provided by

White *et al.*¹ The scattering strength upper bound is clearly not appropriate. These acoustic results can therefore be used to reduce the uncertainty in the ocean model.

In a more general sense, questions surrounding the validity of the Rytov approximation have had a durable longevity. This issue has been debated in the western literature long after Pisareva showed in the Russian literature that the issue depended on a wave parameter D , and that when this wave parameter was small, the Rytov approximation was valid when the log-amplitude variance was less than unity.^{27,28} The wave parameter was $2\lambda R/\pi L^2$, where λ is the acoustic wavelength, R is the propagation distance, and L is a correlation length scale (e.g., in a spherical isotropic medium.) For PhilSea09, the wave parameter is $\mathcal{O}(10^{-2})$ using order-of-magnitude estimates. This suggests that this experiment involved only the Fresnel scattering regime and, therefore, only the bound on log-amplitude variance is relevant. Since the observed log-amplitude variance is less than unity, the Rytov approximation would be valid.

Scattering in the deep ocean is much more complex than the scenarios assumed in the early Russian literature due to the refraction of unperturbed paths and the anisotropy of the

sound speed inhomogeneities and, therefore, the wave parameter used by Pisareva should be inappropriate. Correlation length scales in the horizontal and vertical differ considerably. Flatté–Dashen theory,¹³ which develops moment expressions starting with the path integral formalism, was built specifically to accommodate the additional complexities of the ocean scenario. In Flatté–Dashen theory, the scattering regime is categorized by the product $\Lambda\Phi^2$ involving a diffraction parameter Λ and a strength parameter Φ . Under Flatté–Dashen theory, the Rytov approximation is valid when $\Lambda\Phi^2 < 1$. (See, e.g., Fig. 8.6 in Flatté *et al.*¹³) For the PhilSea09 experiment, calculations from CAFI^{29,30} (which implements Flatté–Dashen expressions) show that $\Lambda\Phi^2$ ranges from 0.3 to 0.6 using the parameter values of Table I and an effective GM strength of 1.6 for the four rays considered here. Hence, under Flatté–Dashen theory, the Rytov approximation should be valid. (As an aside, the Pisareva condition is therefore not misleading.)

Even though Flatté–Dashen theory is a more complete theory¹¹ and supercedes MZ theory, the literature^{31,32} indicates that MZ theory and Flatté–Dashen theory are the same when $\Lambda\Phi^2 < 1$, at least to first order in the Rytov series. Flatté–Dashen theory dictates that the log-amplitude variance in the geometrical-optics approximation is given approximately by $\frac{1}{4}\Lambda\Phi^2$ (Flatté *et al.*¹³, Eq. 8.3.2). Table III shows the depth-averaged values of log-amplitude variance predictions from MZ theory and from Flatté–Dashen theory (implemented via CAFI) for the four eigenrays studied here. These are compared to MCPE calculations, which here are considered the benchmark result. This comparison shows that MZ theory slightly underpredicts the MCPE result. While it is not the purpose of this paper to compare Flatté–Dashen theory or CAFI against MCPE calculations, Table III shows that CAFI overpredicts the MCPE result, and that the CAFI result for $\frac{1}{4}\Lambda\Phi^2$ is almost twice the MZ result within the observed error. [This is, however, consistent with comments offered by Flatté *et al.*:¹³ the log-amplitude variance in the geometric region is $\mathcal{O}(\frac{1}{4}\Lambda\Phi^2)$, an assertion which proves true here.] The value of MZ theory is therefore that it provides Rytov-based expressions for the log-amplitude variance and, by extension, the scintillation index, which, within the limits of its validity, appear to be more accurate than the CAFI-based Flatté–Dashen theory results.

Calculations of $\Lambda\Phi^2$ for the Worcester experiment¹⁰ at 2250 Hz and the Cobb Seamount experiments¹³ at 4 and

8 kHz dictate that the Rytov approximation (hence, MZ theory) remains valid for these scenarios. This is consistent with the lower frequency Worcester measurements,¹⁰ but not the higher frequency Ewart measurements.^{8,9} Both of these experiments analyzed near-horizontal propagation paths. Thus, measurements covering a sparse set of scenarios from high angle, low frequency, hundred kilometer range (Colosi *et al.*,¹¹ White *et al.*¹) to low angle, mid frequency, ten kilometer range may have delineated the boundaries of valid Rytov and MZ theory application. In an effort to probe the sensitivity of the phase and log-amplitude variance and spectral predictions of MZ theory versus frequency, range, and propagation path steepness, Tombul conducted an unpublished numerical study to compare MZ predictions against direct MCPE simulation.³³ Individual eigenrays were simulated using beams with launch angles of 0°, 5°, 10°, and 14°. Ranges varied from 50 to 200 km. Frequencies were 75, 200, and 400 Hz. The medium profiles were analytic models scaled for good correspondence with typical oceanographic data. For shallow launch angles, he found good agreement for log-amplitude variance. For steeper launch angles, he observed discrepancies. The MZ results failed completely at 400 Hz and 14° ray angles. At 200 Hz and 10°, which compares closely with these data, he found that MZ underpredicted MCPE results to the extent that he concluded that MZ theory was no longer valid for these and higher frequencies at this angle. Tombul’s results are generally consistent with the results presented here: Fig. 3 and Table III show that MZ calculations are underpredicting the MCPE results. Tombul’s conclusion may be too severe; MZ theory is largely consistent with measured and MCPE results, but it does appear that a higher order correction is required to eliminate the prediction bias. Simulations were not conducted at the frequencies used by Worcester¹⁰ and Ewart.^{7,8}

The transition region from Rytov/MZ theory to the more comprehensive Flatté–Dashen theory seems to be bounded by a couple of experiments and simulations performed over the past several decades, but several gaps remain in these boundaries. In order to eliminate these gaps, MCPE simulations should be pushed to higher frequencies, and careful ocean experiments pursued at low mid-frequencies for steeper propagation paths.

ACKNOWLEDGMENTS

The authors are grateful for discussions throughout the development of this work with Dr. F. Henyey. L. Buck repeated the calculations over ray ID and sensor depth every time the author invented a new scenario. This work was supported by the Long Range/DeepWater Propagation thrust area of the Ocean Acoustics Program at the Office of Naval Research under APL-UW Grant Nos. N00014-08-1-0843 and N00014-13-1-0009.

¹A. W. White, R. K. Andrew, J. A. Mercer, P. F. Worcester, M. A. Dzieciuch, and J. A. Colosi, “Wavefront intensity statistics for 284-hz broadband transmissions to 107-km range in the Philippine Sea: Observations and modeling,” *J. Acoustic. Soc. Am.* **134**, 3347–3358 (2013).

TABLE III. Comparison of Rytov-based predictions of log-amplitude variance from MZ and Flatté–Dashen theory against MCPE calculations for the four eigenrays studied here. The MCPE, MZ, and $\frac{1}{4}\Lambda\Phi^2$ columns are the depth-averaged values. The final row provides the eigenray-averaged ratios of MZ predictions against the MCPE calculations and the CAFI predictions against the MCPE calculations.

Ray ID	MCPE	MZ	MZ/MCPE	$\frac{1}{4}\Lambda\Phi^2$	$\frac{1}{4}\Lambda\Phi^2$ /MCPE
−3	0.0473	0.0389	0.82	0.075	1.59
+4	0.0961	0.0705	0.73	0.133	1.38
−4	0.0832	0.0725	0.87	0.148	1.78
+5	0.0940	0.0830	0.88	0.113	1.20
		Average	0.83	Average	1.49

- ²W. H. Munk and F. Zachariassen, "Sound propagation through a fluctuating stratified ocean: Theory and observation," *J. Acoust. Soc. Am.* **59**, 818–838 (1976).
- ³S. M. Rytov, Y. A. Kravtsov, and V. I. Tatarskii, *Principles of Statistical Radiophysics 4, Wave Propagation Through Random Media* (Springer, Berlin Germany, 1989), pp. 49–50.
- ⁴A. D. Wheelon, *Electromagnetic Scintillation 2, Weak Scattering* (Cambridge University Press, Cambridge, UK, 2003), pp. 1–9.
- ⁵D. Di Iorio and D. M. Farmer, "Path-averaged turbulent dissipation measurements using high frequency acoustical scintillation analysis," *J. Acoust. Soc. Am.* **96**, 1056–1069 (1994).
- ⁶T. A. Andreeva and W. W. Durgin, "Experimental investigation of ultrasound propagation in turbulent, diffractive media," *J. Acoust. Soc. Am.* **115**, 1532–1536 (2004).
- ⁷T. E. Ewart, "Acoustic fluctuations in the open ocean—A measurement using a fixed refracted path," *J. Acoust. Soc. Am.* **60**, 46–59 (1976).
- ⁸T. E. Ewart and S. A. Reynolds, "The mid-ocean acoustic transmission experiment—MATE," *J. Acoust. Soc. Am.* **75**, 785–802 (1984).
- ⁹Y. J. F. Desaubies, "On the scattering of sound by internal waves in the ocean," *J. Acoust. Soc. Am.* **64**, 1460–1469 (1978).
- ¹⁰P. F. Worcester, "Reciprocal acoustic transmission in a midocean environment: Fluctuations," *J. Acoust. Soc. Am.* **66**, 1173–1181 (1979).
- ¹¹J. A. Colosi, J. Xu, P. F. Worcester, M. A. Dzieciuch, B. M. Howe, and J. A. Mercer, "Temporal and vertical scales of acoustic fluctuations for 75-Hz, broadband transmissions to 87-km range in the eastern North Pacific Ocean," *J. Acoust. Soc. Am.* **126**, 1069–1083 (2009).
- ¹²F. S. Henyey, D. Rouseff, J. M. Grochocinski, S. A. Reynolds, K. L. Williams, and T. E. Ewart, "Effects of internal waves and turbulence on a horizontal aperture sonar," *J. Ocean. Eng.* **22**, 270–280 (1997).
- ¹³S. M. Flatté, R. Dashen, W. H. Munk, K. Watson, and F. Zachariassen, *Sound Transmission through a Fluctuating Ocean* (Cambridge University Press, Cambridge, UK, 1979), pp. 6–7.
- ¹⁴W. Munk, "Internal waves and small-scale processes," in *The Evolution of Physical Oceanography*, edited by C. Wunsch and B. Warren (MIT Press, Cambridge, MA, 1981), pp. 264–291.
- ¹⁵I. I. Rypina and M. G. Brown, "On the width of a ray," *J. Acoust. Soc. Am.* **122**, 1440–1448 (2007).
- ¹⁶V. Červený, *Seismic Ray Theory* (Cambridge University Press, Cambridge, UK, 2001), pp. 102–235.
- ¹⁷V. Červený and J. E. P. Soares, "Fresnel volume ray tracing," *Geophysics* **57**, 902–915 (1992).
- ¹⁸M. B. Porter and H. P. Buckner, "Gaussian beam tracing for computing ocean acoustic fields," *J. Acoust. Soc. Am.* **82**, 1349–1359 (1987).
- ¹⁹P. F. Worcester, M. A. Dzieciuch, J. A. Mercer, R. K. Andrew, B. D. Dushaw, A. B. Baggeroer, K. D. Heaney, G. D. D'Spain, J. A. Colosi, R. A. Stephen, J. N. Kemp, B. M. Howe, L. J. V. Uffelen, and K. E. Wage, "The North Pacific Acoustic Laboratory deep-water acoustic propagation experiments in the Philippine Sea," *J. Acoust. Soc. Am.* **134**, 3359–3375 (2013).
- ²⁰W. H. Munk, P. F. Worcester, and C. Wunsch, *Ocean Acoustic Tomography* (Cambridge University Press, Cambridge, UK, 1995), pp. 190–195.
- ²¹P. F. Worcester, S. Carey, M. A. Dzieciuch, L. L. Green, D. Horwitt, J. C. Lemire, and M. Norenberg, "Distributed vertical line array (DVLA) acoustic receiver," in *Proc. 3rd International Conf. on Underwater Acoustic Measurements*, edited by J. S. Papadakis and L. Bjørnø (Foundation for Research and Technology Hellas, Nafplion, Greece, 2009), pp. 113–118.
- ²²Sea-Bird Electronics, Inc., "MicroCAT C-T (P optional) recorder SBE 37-SMP," Technical Report, Sea-Bird Electronics, Inc., 13431 NE 20th Street, Bellevue WA (2012), available at http://www.seabird.com/products/spec_sheets/37smpdata.htm (Last viewed 6/11/2012).
- ²³Sea-Bird Electronics, Inc., "Temperature and pressure recorder SBE 39," Technical Report, Sea-Bird Electronics, Inc., 13431 NE 20th Street, Bellevue WA (2012), available at http://www.seabird.com/products/spec_sheets/39data.htm (Last viewed 6/11/2012).
- ²⁴W. H. F. Smith and D. Sandwell, "Global seafloor topography from satellite altimetry and ship depth soundings," *Science* **277**, 1956–1962 (1997).
- ²⁵F. S. Henyey and S. A. Reynolds, "A numerical simulator of ocean internal waves for long-range acoustics," Technical Report No. APL-UW TM 1-13, Applied Physics Laboratory, University of Washington, Seattle, WA (2013).
- ²⁶J. A. Colosi, L. J. van Uffelen, B. D. Cornuelle, M. A. Dzieciuch, P. F. Worcester, B. D. Dushaw, and S. R. Ramp, "Observations of sound-speed fluctuations in the western Philippine Sea in the spring of 2009," *J. Acoust. Soc. Am.* **134**, 3185–3200 (2013).
- ²⁷V. V. Pisareva, "O granitsakh primenimosti metoda 'plavnykh' ozmushcheniĭ v zadache o rasprostraneniĭ izlucheniya cherez sredy neodnorodnostyami" ("Limits of the applicability of the method of 'smooth' perturbations in the problem of radiation propagation through a medium containing inhomogeneities"), *Akust. Zh.* **6**, 87–91 (1960).
- ²⁸A. D. Wheelon, *Electromagnetic Scintillation 2, Weak Scattering* (Cambridge University Press, Cambridge, UK, 2003), pp. 23–30.
- ²⁹S. M. Flatté and G. Rovner, "User's guide to CAFI" (1998), available at http://oalib.hlsresearch.com/Other/cafi/CAFI_Guide.pdf (Last viewed 3/29/2015).
- ³⁰S. M. Flatté and G. Rovner, "Calculations of internal-wave-induced fluctuations in oceanacoustic propagation," *J. Acoust. Soc. Am.* **108**, 526–534 (2000).
- ³¹R. Dashen, "Path integrals for waves in random media," *J. Math. Phys.* **20**, 894–920 (1979).
- ³²D. R. Palmer, "An introduction to the application of Feynman path integrals to sound propagation in the ocean," Naval Research Laboratory Report No. 8148, Naval Electronic Systems Command, Washington, D.C. 20360 (1978).
- ³³S. Tombul, "A numerical study of the validity regimes of weak fluctuation theory for ocean acoustic propagation through random internal wave sound speed fields," Master's thesis, Naval Postgraduate School, Monterey, CA, 2007.