Robust synchronization of coupled neural oscillators using the derivative-free nonlinear Kalman Filter

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Abstract A synchronizing control scheme for coupled neural oscillators of the FitzHugh–Nagumo type is proposed. Using differential flatness theory the dynamical model of two coupled neural oscillators is transformed into an equivalent model in the linear canonical (Brunovsky) form. A similar linearized description is succeeded using differential geometry methods and the computation of Lie derivatives. For such a model it becomes possible to design a state feedback controller that assures the synchronization of the membrane's voltage variations for the two neurons. To compensate for disturbances that affect the neurons' model as well as for parametric uncertainties and variations a disturbance observer is designed based on Kalman Filtering. This consists of implementation of the standard Kalman Filter recursion on the linearized equivalent model of the coupled neurons and computation of state and disturbance estimates using the diffeomorphism (relations about state variables transformation) provided by differential flatness theory. After estimating the disturbance terms in the neurons' model their compensation becomes possible. The performance of the synchronization control loop is tested through simulation experiments.

Keywords Coupled neural oscillators · FitzHugh–Nagumo neuron model · Differential geometric methods · Differential flatness theory · Derivative-free nonlinear Kalman Filter · Disturbances compensation

Introduction

The voltage of the neuron's membrane exhibits oscillatory variations after receiving suitable external excitation either when the neuron is independent from neighboring neural cells or when the neuron is coupled to neighboring neural cells through synapses or gap junctions (Baird-Emertrout and Terman 2009). In the latter case it is significant to analyze conditions under which synchronization between coupled neural oscillators takes place, which means that the neurons generate the same voltage variation pattern possibly subject to a phase difference. The loss of synchronism between neurons can cause several neurodegenerative disorders. Moreover, it can affect several basic functions of the body such as gait, respiration and heart's rhythm. For this reason synchronization of coupled neural oscillators has become a topic of significant research during the last years. The associated results have been also used in several engineering applications, such as biomedical engineering and robotics. For example, synchronization between neural cells can result in a rhythm generator that controls joints motion in quadruped, multi-legged and biped robots.

Dynamical models of coupled neural oscillators can serve as central pattern generators (Grillner 2006; Wu et al. 2009). This means that they stand for higher level control elements in a multi-layered control scheme which provide the activation frequency and rhythm for controllers operating at the lower level, e.g. controllers that provide motion to robot's joints. Central pattern generator (CPG) methods have been used to control various kinds of robots, such as crawling robots and legged robots and various modes of locomotion such as basic gait control, gait transitions control, dynamic adaptive locomotion control, etc. (Ijspeert 2008). CPG models have been used with hexapod and

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octopod robots inspired by insect locomotion (Arena et al. 2004). CPGs have been also used for controlling swimming robots, such as lamprey robots (Crespi and Ijspeert 2008). Quadruped walking robots controlled with the use of CPGs have been studied in Fukuoka et al. (2003). Models of CPGs are also increasingly used for the control of biped locomotion in humanoid robots (Héliot and Espiau 2008).

The problem of synchronization of coupled neural oscillators becomes more difficult when there is uncertainty about the parameters of the dynamical model of the neurons. Thus, it is rather unlikely that coupled neurons will have identical dynamical models and that these models will be free of parametric variations and external disturbances. To synchronize neurons subject to model uncertainties and external disturbances several approaches have been proposed. In Nguyen and Hong (2011) by using the Lyapunov function method and calculating Lyapunov exponents, respectively, necessary and sufficient conditions for achieving synchronization between two coupled FitzHugh–Nagumo neurons are established. In Aqil et al. (2012) matrix inequalities on the basis of Lyapunov stability theory are used to design a robust synchronizing controller for the model of the coupled FitzHugh–Nagumo neurons. In Yu et al. (2012) robust control system combining backstepping and sliding mode control techniques is used to realize the synchronization of two gap junction coupled chaotic FitzHugh–Nagumo neurons under external electrical stimulation. In Wei et al. (2010) a Lyapunov function-based control law is introduced, which transforms the FitzHugh–Nagumo neurons into an equivalent passive system. It is proved that the equivalent system can be asymptotically stabilized at any desired fixed state, which means that, synchronization can be succeeded. In Rehan et al. (2011) synchronizing control for coupled FitzHugh–Nagumo neurons is succeeded using a Lyapunov function approach. The control signal is based on feedback of the synchronization error between the master and the slave neurons. Finally, the use of differential geometric methods in the modeling and control of FitzHugh–Nagumo neuron dynamics has been studied in (Zhang et al. 2007; Denham 2005). Other indicative results on synchronizing control of coupled neural oscillators can be found in Cao and Wan (2014), Cao et al. (2013), Li and Cao (2014), Liu and Cao (2011), Shen and Cao (2011) and Yang et al. (2013).

In this paper, differential flatness theory has been proposed for the synchronization of coupled nonlinear oscillators of the FitzHugh–Nagumo type. Differential flatness theory is a main direction in nonlinear science and enables linearization for a wide class of systems (Sira-Ramirez and Agrawal 2004; Rigatos 2011; Rudolph 2003). To find out if a dynamical system is differentially flat, the following should be examined: (1) the existence of the so-called flat

output, i.e. a new variable which is expressed as a function of the system's state variables. The flat output and its derivatives should not be coupled in the form of an ordinary differential equation, (2) the components of the system (i.e. state variables and control input) should be expressed as functions of the flat output and its derivatives (Bououden et al. 2011; Laroche et al. 2007; Lévine 2011; Martin and Rouchon 1999; Sira-Ramirez and Agrawal 2004; Rigatos 2011; Rouchon 2005). In certain cases the differential flatness theory enables transformation to a linearized form (canonical Brunovsky form) for which the design of the controller becomes easier. In other cases by showing that a system is differentially flat one can easily design a reference trajectory as a function of the so-called flat output and can find a control law that assures tracking of this desirable trajectory (Fliess and Mounier 1999; Rouchon 2005).

This paper is concerned with proving differential flatness of the model of the coupled FitzHugh–Nagumo neural oscillators and its resulting description in the Brunovsky (canonical) form (Martin and Rouchon 1999). By defining specific state variables as flat outputs an equivalent description of the coupled FitzHugh–Nagumo neurons in the Brunovsky (linear canonical) form is obtained. It is shown that for the linearized model of the coupled neural oscillators it is possible to design a feedback controller that succeeds their synchronization. At a second stage, a novel Kalman Filtering method, the derivative-free nonlinear Kalman Filter, is used as a disturbance observer, thus making possible to estimate disturbance terms affecting the model of the coupled FitzHugh–Nagumo neurons and to use these terms in the feedback controller. By avoiding linearization approximations, the proposed filtering method, improves the accuracy of estimation, and results in smooth control signal variations and in minimization of the synchronization error (Rigatos 2012a, b; Rigatos and Rigatou 2013).

Comparing to other approaches for synchronizing control of coupled neural oscillators the approach presented in this paper exhibits the following advantages: (1) the differential flatness theory-based synchronizing control is based on an exact linearization and decoupling transformation of the neurons' model which facilitates the design of a state-feedback controller, (2) the derivative-free nonlinear Kalman enables the estimation of the non-measurable state vector variables in the neurons' model as well as of terms representing modelling uncertainty and the effects of external perturbations. Once these uncertainty and disturbance terms are identified it becomes also possible to compensate for them. The latter is important for the control of biological networks, such as coupled neural oscillators, because in such systems the dynamic model parameters cannot be precisely known, cannot be identical for all

nodes of the biological network and finally it varies in time under different environmental conditions.

The structure of the paper is as follows: In “**Models of neural oscillators**” section basic models of neural oscillators are analyzed. These include the Hodgkin–Huxley model and the FitzHugh–Nagumo model. The use of coupled neural oscillator models in the generation of synchronized motion patterns and its possible use in robotics is explained. In “**Differential flatness theory**” section differential flatness theory is introduced. The basic properties of differentially flat systems are analyzed. The use of differential flatness theory for transforming MIMO systems into the linear canonical form is explained. In “**Linearization of the FitzHugh–Nagumo neuron**” section it is shown how linearization of the independent FitzHugh–Nagumo neurons can be succeeded using differential geometric methods (Lie algebra) and differential flatness theory. In “**Linearization of coupled FitzHugh–Nagumo neuron using differential geometry**” section linearization of the model of the coupled FitzHugh–Nagumo neurons is performed using differential geometric methods. In “**Linearization of coupled FitzHugh–Nagumo neuron using differential flatness theory**” section linearization of the model of the coupled FitzHugh–Nagumo neurons is performed using differential flatness theory. In “**State and disturbances estimation with the derivative-free nonlinear Kalman Filter**” section the derivative-free nonlinear Kalman Filter is proposed for disturbances estimation in the model of the coupled FitzHugh–Nagumo neurons. It is shown that once disturbances estimation is performed a modified control signal enables their compensation. In “**Simulation tests**” section simulation tests are presented to evaluate the performance of the proposed synchronization control loop for the model of the coupled neural oscillators. Finally, in “**Conclusions**” section concluding remarks are provided.

Models of neural oscillators

The FitzHugh–Nagumo model of neurons

The FitzHugh–Nagumo model is a two-dimensional reduced model of the Hodgkin–Huxley dynamics. The model captures all the qualitative properties of the of the Hodgkin–Huxley dynamics, in the sense that it possesses the same bifurcation diagram. This diagram describes how the trajectories in the phase-plane change according to variations in the values of specific parameters. The advantage of the FitzHugh–Nagumo model is that it permits to visualize and understand complex physiological behaviors of the neuron which remain obscure in the four-dimensional Hodgkin–Huxley model (Baird-Emertrout and Terman 2009).

Fig. 1 Biological mechanism of activation of coupled neural oscillators (Central Patterns Generators)

The FitzHugh–Nagumo model comprises two differential equations

$$\begin{aligned} \frac{dV}{dt} &= V(V - a)(1 - V) - w + I \\ \frac{dw}{dt} &= \epsilon(V - \gamma w) \end{aligned} \quad (1)$$

where $0 < a < 1$, $\epsilon > 0$ and $\gamma \geq 0$. Parameter V stands for the membrane's voltage, parameter w known as recovery variable is associated with ionic currents and parameter I is an external input current. The FitzHugh–Nagumo model is equivalent to Van Der Pol oscillator, the latter being described by

$$\begin{aligned} C \frac{dV}{dt} &= -F(V) - J \\ L \frac{dJ}{dt} &= V \end{aligned} \quad (2)$$

where $F(V)$ is a nonlinear function of the voltage, e.g. $F(V) = V(V - a)(1 - V)$.

Coupled neural oscillators as coordinators of motion

Coupled neural oscillators can provide as output signals that maintain a specific phase difference and which can be used for the coordination of joints' motion, either in robotic or in biomedical applications. These coupled neural oscillators are also known as central pattern generators and can synchronize the motion performed by the legs of quadruped and biped robots. The concept of central pattern generators comes from biological neurons, located at the spinal level, and which are able of generating rhythmic

Fig. 2 Models of coupled neural oscillators in engineering: Fitz-Hugh–Nagumo neurons for controlling robot’s motion

commands for the muscles. CPGs receive commands from higher levels of the central nervous system. Thus their functioning is the result of the interaction between central commands and local reflexes.

Neuronal mechanisms in the brain select which central pattern generators will be activated at every time instant. The basal ganglia play an important role in this (see Fig. 1). Under resting conditions the output layer of the basal ganglia (the pallidum) maintains different CPG neurons under inhibition. To achieve CPG activation, striatal neurons, the input layer of the basal ganglia, inhibits cells in the pallidum which keep the CPG neurons inactive. The striatal neurons can, in turn, be activated from either neocortex or directly from the thalamus. The responsiveness of striatal neurons to activation can be facilitated by dopaminergic inputs. On the other hand, deficiencies or enhanced levels of dopaminergic inputs results in movements distortion and malfunction of the locomotion system (Grillner 2006; Wu et al. 2009).

Coupled neuron models which have been used as central pattern generators are the Hodgkin–Huxley neurons, the Stein’s neural oscillator, Matsuoka’s neuron oscillator, Kimura’s neuron oscillator, Kuramoto’s neural oscillator, the Hopf model, Van der Pol oscillator and Rayleigh’s model. Central pattern generator (CPG) methods have been used to control various kinds of robots, such as crawling robots and legged robots (see Fig. 2) and various modes of locomotion such as basic gait control, gait transitions control, dynamic adaptive locomotion control, etc. (Ijspeert 2008).

Remark 1 The reason for using central pattern generators in the design of humanoid robots is to replicate the biological mechanisms of gait and locomotion. Moreover, such neuronal oscillators can be used in biomedical engineering, e.g in the rehabilitation of human motor functions and in the treatment of neurological diseases. According to the diagram depicted in Fig. 2, a master CPG neuron provides the reference motion pattern and the slave neuron is synchronized to this setpoint. Thus finally, in usual gait, the neurons track the same, in shape, reference setpoint but possibly with a phase lag. This solves the problem of setpoint generation for the joints of biped or multi-legged robots (e.g. when performing a specific motion such as walking).

Differential flatness theory

Definition of differentially flat systems

Differential flatness theory will be used for implementing feedback control and synchronization of the coupled neurons. The main principles of the differential flatness theory are as follows (Fliess and Mounier 1999; Rouchon 2005): A finite dimensional system is considered. This can be written in the general form of an ordinary differential equation (ODE), i.e. $S_i(w, \dot{w}, \ddot{w}, \dots, w^{(i)})$, $i = 1, 2, \dots, q$. The term w denotes the system variables (these variables are for instance the elements of the system’s state vector and the control input) while $w^{(i)}$, $i = 1, 2, \dots, q$ are the associated derivatives. Such a system is said to be differentially flat if there is a collection of m functions $y = (y_1, \dots, y_m)$ of the system variables and of their time-derivatives, i.e. $y_i = \phi(w, \dot{w}, \ddot{w}, \dots, w^{(z_i)})$, $i = 1, \dots, m$ satisfying the following two conditions (Fliess and Mounier 1999; Rouchon 2005; Rigatos 2014): (1) There does not exist any differential relation of the form $R(y, \dot{y}, \dots, y^{(\beta)}) = 0$ which implies that the derivatives of the flat output are not coupled in the sense of an ODE, or equivalently it can be said that the flat output is differentially independent, (2) All system variables (i.e. the elements of the system’s state vector w and the control input) can be expressed using only the flat output y and its time derivatives $w_i = \psi_i(y, \dot{y}, \dots, y^{(r_i)})$, $i = 1, \dots, s$.

Conditions for applying differential flatness theory

The following generic class of nonlinear systems is considered

$$\dot{x} = f(x, u) \quad (3)$$

Such an equation can be transformed to the form of an affine in the input system by adding an integrator to each input (Bououden et al. 2011)

Fig. 3 Equivalent circuit diagram of coupled FitzHugh–Nagumo neurons

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i \quad (4)$$

If the system of Eq. (4) can be linearized by a diffeomorphism $z = \phi(x)$ and a static state feedback $u = \alpha(x) + \beta(x)v$ into the following form

$$\begin{aligned} \dot{z}_{i,j} &= z_{i+1,j} \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq v_j - 1 \\ \dot{z}_{v_i,j} &= v_j \end{aligned} \quad (5)$$

with $\sum_{j=1}^m v_j = n$, then $y_j = z_{1,j}$ for $1 \leq j \leq m$ are the 0-flat outputs which can be written as functions of only the elements of the state vector x (when the flat output of a system is only function of its states x , then this is called 0-flat). To define conditions for transforming the system of Eq. (4) into the canonical form described in Eq. (5) the following theorem holds (Bououden et al. 2011).

Theorem 1 For the nonlinear systems described by Eq. (4) the following variables are defined: (i) $G_0 = \text{span}[g_1, \dots, g_m]$, (ii) $G_1 = \text{span}[g_1, \dots, g_m, \text{ad}_f g_1, \dots, \text{ad}_f g_m, \dots]$ (k) $G_k = \text{span}\{\text{ad}_f^j g_i \text{ for } 0 \leq j \leq k, 1 \leq i \leq m\}$ (where $\text{ad}_f^j g$ stands for a Lie Bracket). Then, the linearization problem for the system of Eq. (4) can be solved if and only if: (1). The dimension of G_i , $i = 1, \dots, k$ is constant for $x \in X \subseteq \mathbb{R}^n$ and for $1 \leq i \leq n-1$, (2). The dimension of G_{n-1} is of order n , (3). The distribution G_k is involutive for each $1 \leq k \leq n-2$.

Transformation of MIMO nonlinear systems into the Brunovsky form

It is assumed now that after defining the flat outputs of the initial MIMO nonlinear system (this approach will be also

shown to hold for the model of the coupled FitzHugh–Nagumo neurons), and after expressing the system state variables and control inputs as functions of the flat output and of the associated derivatives, the system can be transformed in the Brunovsky canonical form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\dots \\ \dot{x}_{r_1-1} &= x_{r_1} & y_1 &= x_1 \\ \dot{x}_{r_1} &= f_1(x) + \sum_{j=1}^p g_{1j}(x)u_j + d_1 & y_2 &= x_2 \\ \dot{x}_{r_1+1} &= x_{r_1+2} & &\dots \\ &\dots & y_p &= x_{n-r_p+1} \\ \dot{x}_{p-1} &= x_p \\ \dot{x}_p &= f_p(x) + \sum_{j=1}^p g_{pj}(x)u_j + d_p \end{aligned} \quad (6)$$

where $x = [x_1, \dots, x_n]^T$ is the state vector of the transformed system (according to the differential flatness formulation), $u = [u_1, \dots, u_p]^T$ is the set of control inputs, $y = [y_1, \dots, y_p]^T$ is the output vector, f_i are the drift functions and g_{ij} , $i, j = 1, 2, \dots, p$ are smooth functions corresponding to the control input gains, while d_j is a variable associated to external disturbances. It holds that $r_1 + r_2 + \dots + r_p = n$. Having written the initial nonlinear system into the canonical (Brunovsky) form it holds

$$y_i^{(r_i)} = f_i(x) + \sum_{j=1}^p g_{ij}(x)u_j + d_j \quad (7)$$

Next the following vectors and matrices can be defined $f(x) = [f_1(x), \dots, f_n(x)]^T$, $g(x) = [g_1(x), \dots, g_n(x)]^T$ with $g_i(x) = [g_{1i}(x), \dots, g_{pi}(x)]^T$, $A = \text{diag}[A_1, \dots, A_p]$, $B = \text{diag}[B_1, \dots, B_p]$, $C^T = \text{diag}[C_1, \dots, C_p]$, $d = [d_1, \dots, d_p]^T$,

where matrix A has the MIMO canonical form, i.e. with block-diagonal elements

$$A_i = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{r_i \times r_i} \quad (8)$$

$$B_i^T = (0 \quad 0 \quad \cdots \quad 0 \quad 1)_{1 \times r_i}$$

$$C_i = (1 \quad 0 \quad \cdots \quad 0 \quad 0)_{1 \times r_i}$$

Thus, Eq. (7) can be written in state-space form

$$\dot{x} = Ax + Bv + B\tilde{d}$$

$$y = Cx \quad (9)$$

where the control input is written as $v = f(x) + g(x)u$.

Linearization of the FitzHugh–Nagumo neuron

Linearization of the FitzHugh–Nagumo model using a differential geometric approach

It will be shown that the model of the coupled neural oscillators can be linearized using a differential geometric method and the computation of Lie derivatives. The model of the dynamics of the FitzHugh–Nagumo neuron is considered

$$\frac{dV}{dt} = V(V - a)(1 - V) - w + I$$

$$\frac{dw}{dt} = \epsilon(V - \gamma w) \quad (10)$$

By defining the state variables $x_1 = w, x_2 = V$ and setting current I as the external control input one has

$$\frac{dx_1}{dt} = -\epsilon\gamma x_1 + \epsilon x_2$$

$$\frac{dx_2}{dt} = x_2(x_2 - a)(1 - x_2) - x_1 + u$$

$$y = h(x) = x_1 \quad (11)$$

Equivalently one has

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\epsilon\gamma x_1 + \epsilon x_2 \\ x_2(x_2 - a)(1 - x_2) - x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = h(x) = x_1 \quad (12)$$

The state variable $z_1 = h_1(x) = x_1$ is defined. It holds that

$$z_2 = L_f h_1(x) = \frac{\partial h_1}{\partial x_1} f_1 + \frac{\partial h_1}{\partial x_2} f_2 = 1 \cdot f_1 + 0 \cdot f_2 = f_1 \quad (13)$$

and

$$L_f^2 h_1(x) = \frac{\partial z_2}{\partial x_1} f_1 + \frac{\partial z_2}{\partial x_2} f_2 = \frac{\partial}{\partial x_1} (-\epsilon\gamma x_1 + \epsilon x_2) f_1 + \frac{\partial}{\partial x_2} (-\epsilon\gamma x_1 + \epsilon x_2) f_2 \Rightarrow L_f^2 h_1(x) = (-\epsilon\gamma) f_1 + \epsilon f_2 \Rightarrow L_f^2 h_1(x) = -\epsilon\gamma(-\epsilon\gamma x_1 + \epsilon x_2) + \epsilon[x_2(x_2 - a)(1 - x_2) - x_1].$$

Moreover, it holds

$$L_g h_1(x) = \frac{\partial h_1}{\partial x_1} g_1 + \frac{\partial h_1}{\partial x_2} g_2 = 1 \cdot g_1 + 0 \cdot g_2 = g_1 = 0 \quad (14)$$

and

$$L_g L_f h_1(x) = \frac{\partial z_2}{\partial x_1} g_1 + \frac{\partial z_2}{\partial x_2} g_2 = \frac{\partial}{\partial x_1} (-\epsilon\gamma x_1 + \epsilon x_2) g_1 + \frac{\partial}{\partial x_2} (-\epsilon\gamma x_1 + \epsilon x_2) g_2 \Rightarrow L_g L_f h_1(x) = -\epsilon\gamma g_1 + \epsilon g_2 \Rightarrow L_g L_f h_1(x) = (-\epsilon\gamma)0 + \epsilon 1 \Rightarrow L_g L_f h_1(x) = \epsilon.$$

It holds that

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = L_f^2 h_1(x) + L_g L_f h_1(x)u \quad (15)$$

or equivalently

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \{-\epsilon\gamma x_1 + \epsilon x_2 + \epsilon[x_2(x_2 - a)(1 - x_2) - x_1]\} + \epsilon u \quad (16)$$

The model of Eq. (16) can be also written in the matrix canonical form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \quad (17)$$

where $v = \{-\epsilon\gamma x_1 + \epsilon x_2 + \epsilon[x_2(x_2 - a)(1 - x_2) - x_1]\} + \epsilon u$. Finally, the relative degree of the system is computed. It holds that $L_g h_1(x) = 0$, whereas

$$L_g L_f h_1(x) = \frac{\partial f_1}{\partial x_1} g_1 + \frac{\partial f_1}{\partial x_2} g_2 \Rightarrow$$

$$L_g L_f h_1(x) = -\epsilon\gamma 0 + \epsilon 1 = \epsilon \neq 0 \quad (18)$$

and $L_g L_f^{n-1} h_1(x) \neq 0$ for $n = 2$. Therefore, the relative degree of the system is $n = 2$.

Linearization of the FitzHugh–Nagumo model using differential flatness theory

Next, the FitzHugh–Nagumo model is linearized with the use of the differential flatness theory. The flat output of the system is taken to be $y = h(x) = x_1$. It holds that $\dot{y} = \dot{x}_1$. From the first row of the state space equation of the neuron given in Eq. (12) one gets

$$x_2 = \frac{\dot{x}_1 + \epsilon\gamma x_1}{\epsilon} \Rightarrow x_2 = \frac{\dot{y} + \epsilon\gamma y}{\epsilon} \quad (19)$$

Moreover, from the second row of Eq. (12) one gets

$$u = \dot{x}_2 - x_2(x_2 - a)(1 - x_2) + x_1 \quad (20)$$

and since x_1, x_2 are functions of the flat output and its derivatives one has that the control input u is also a function of the flat output and its derivatives. Therefore, the

considered neuron model stands for a differentially flat dynamical system.

From the computation of the derivatives of the flat output one obtains

$$\dot{y}_1 = \dot{x}_1 = f_1 \quad (21)$$

and $\ddot{y}_1 = \dot{f}_1 = \frac{\partial}{\partial x_1}(-\epsilon\gamma x_1 + \epsilon x_2)\dot{x}_1 + \frac{\partial}{\partial x_1}(-\epsilon\gamma x_1 + \epsilon x_2)\dot{x}_2 \Rightarrow \ddot{y}_1 = -\epsilon\gamma\dot{x}_1 + \epsilon\dot{x}_2 \Rightarrow \ddot{y}_1 = -\epsilon\gamma(-\epsilon\gamma x_1 + \epsilon x_2) + \epsilon[x_2(x_2 - a)(1 - x_2) - x_1] + u$.

By denoting as $v = -\epsilon\gamma\dot{x}_1 + \epsilon\dot{x}_2 \Rightarrow \ddot{y}_1 = -\epsilon\gamma(-\epsilon\gamma x_1 + \epsilon x_2) + \epsilon[x_2(x_2 - a)(1 - x_2) - x_1] + u$, one arrives again at a state space description of the system in the linear canonical form. Thus setting $z_1 = y_1$ and $z_2 = \dot{y}_1 = y_2$ one gets Eq. (17)

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad (22)$$

For the linearized neuron model one can define a feedback control signal using that $\ddot{y} = v$.

$$v = \ddot{y}^d - k_d(\dot{y} - \dot{y}_d) - k_p(y - y_d) \Rightarrow \quad (23)$$

and using that $L_f^2 h(x) + L_g L_f h(x)u = v$ one has

$$u = (v - L_f^2 h(x))/L_g L_f h(x) \quad (24)$$

Linearization of coupled FitzHugh–Nagumo neurons using differential geometry

The following system of coupled neurons is considered

$$\begin{aligned} \frac{dV_1}{dt} &= V_1(V_1 - \alpha)(1 - V_1) - w_1 + g(V_1 - V_2) + I_1 \\ \frac{dw_1}{dt} &= \epsilon(V_1 - \gamma w_1) \\ \frac{dV_2}{dt} &= V_2(V_2 - \alpha)(1 - V_2) - w_2 + g(V_2 - V_1) + I_2 \\ \frac{dw_2}{dt} &= \epsilon(V_2 - \gamma w_2) \end{aligned} \quad (25)$$

The equivalent circuit of the model is shown in Fig. 3. The following state variables are defined $x_1 = w_1, x_2 = V_1, x_3 = w_2, x_4 = V_2, u_1 = I_1$ and $u_2 = I_2$. Thus, one obtains the following state-space description

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} -\epsilon\gamma x_1 + \epsilon x_2 \\ x_2(x_2 - a)(1 - x_2) - x_1 + g(x_2 - x_4) \\ -\epsilon\gamma x_3 + \epsilon x_4 \\ x_4(x_4 - \alpha)(1 - x_4) - x_3 + g(x_4 - x_2) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (26)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad (27)$$

It holds that $z_1^1 = h_1(x) = x_1, z_2^1 = L_f h_1(x) \Rightarrow z_2^1 = f_1 \Rightarrow z_2^1 = -\epsilon\gamma x_1 + \epsilon x_2$. Moreover, it holds that $\dot{z}_2^1 = L_f^2 h_1(x) \Rightarrow \dot{z}_2^1 = -\epsilon\gamma f_1 + \epsilon f_2 \Rightarrow \dot{z}_2^1 = -(\epsilon\gamma)(-\epsilon\gamma x_1 + \epsilon x_2) + \epsilon[x_2(x_2 - a)(1 - x_2) - x_1 + g(x_2 - x_4)]$

In a similar manner one computes $L_{g_1} h_1(x) = 1g_{11} + 0g_{12} + 0g_{13} + 0g_{14} \Rightarrow L_{g_1} h_1(x) = 0$ and $L_{g_2} h_1(x) = 1g_{21} + 0g_{22} + 0g_{23} + 0g_{24} \Rightarrow L_{g_2} h_1(x) = 0$.

Similarly one gets $L_{g_1} L_f h_1(x) = (-\epsilon\gamma)g_{11} + \epsilon g_{12} + 0g_{13} + 0g_{14} \Rightarrow L_{g_1} L_f h_1(x) = \epsilon$ and $L_{g_2} L_f h_1(x) = (-\epsilon\gamma)g_{21} + \epsilon g_{22} + 0g_{23} + 0g_{24} \Rightarrow L_{g_2} L_f h_1(x) = 0$.

Moreover, it holds $z_1^2 = h_2(x) = x_3$ and $z_2^2 = f_3 \Rightarrow z_2^2 = -\epsilon\gamma x_3 + \epsilon x_4$. Additionally, one has $\dot{z}_2^2 = L_f^2 h_2(x) \Rightarrow \dot{z}_2^2 = -\epsilon\gamma(-\epsilon\gamma x_3 + \epsilon x_4) + \epsilon[x_4(x_4 - a)(1 - x_4) - x_3 + g(x_4 - x_2)]$.

Additionally $L_{g_1} h_2(x) = 0g_{11} + 0g_{12} + 1g_{13} + 0g_{14} \Rightarrow L_{g_1} h_2(x) = 0$ and $L_{g_2} h_2(x) = 1g_{21} + 0g_{22} + 0g_{23} + 0g_{24} \Rightarrow L_{g_2} h_2(x) = 0$.

Equivalently $L_{g_1} L_f h_2(x) = L_{g_1} L_f h_2(x) = 0 \cdot 0 + 0 \cdot 1 - \epsilon\gamma 0 + \epsilon 0 \Rightarrow L_{g_1} L_f h_2(x) = 0$ and $L_{g_2} L_f h_2(x) = 0 \cdot 0 + 0 \cdot 0 - \epsilon\gamma 0 + \epsilon 1 \Rightarrow L_{g_2} L_f h_2(x) = \epsilon$. Therefore, it holds

$$\begin{aligned} z_1^1 &= z_2^1 \\ z_2^1 &= L_f^2 h_1(x) + L_{g_1} L_f h_1(x)u_1 + L_{g_2} L_f h_1(x)u_2 \\ z_1^2 &= z_2^2 \\ z_2^2 &= L_f^2 h_2(x) + L_{g_1} L_f h_2(x)u_1 + L_{g_2} L_f h_2(x)u_2 \end{aligned} \quad (28)$$

or equivalently,

$$\begin{aligned} z_2^1 &= L_f^2 h_1(x) + L_{g_1} L_f h_1(x)u_1 + L_{g_2} L_f h_1(x)u_2 \\ z_2^2 &= L_f^2 h_2(x) + L_{g_1} L_f h_2(x)u_1 + L_{g_2} L_f h_2(x)u_2 \end{aligned} \quad (29)$$

which can be also written in matrix form

$$\begin{pmatrix} z_2^1 \\ z_2^2 \end{pmatrix} = \begin{pmatrix} L_f^2 h_1(x) \\ L_f^2 h_2(x) \end{pmatrix} + \begin{pmatrix} L_{g_1} L_f h_1(x) & L_{g_2} L_f h_1(x) \\ L_{g_1} L_f h_2(x) & L_{g_2} L_f h_2(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (30)$$

or $\ddot{z} = D_m + G_m u$. Denoting $v_1 = L_f^2 h_1(x) + L_{g_1} L_f h_1(x)u_1 + L_{g_2} L_f h_1(x)u_2$ and $v_2 = L_f^2 h_2(x) + L_{g_1} L_f h_2(x)u_1 + L_{g_2} L_f h_2(x)u_2$ one also has $\dot{z}_1^1 = v_1$ and $\dot{z}_1^2 = v_2$.

The control inputs v_1 and v_2 are chosen as $v_1 = \ddot{z}_{1d}^1 - k_d^1(\dot{z}_{1d}^1 - \dot{z}_1^1) - k_p^1(z_{1d}^1 - z_1^1)$ and $v_2 = \ddot{z}_{1d}^2 - k_d^2(\dot{z}_{1d}^2 - \dot{z}_1^2) - k_p^2(z_{1d}^2 - z_1^2)$. Therefore, the associated control signal is $u = G_m^{-1}(v - D_m(x))$.

Linearization of coupled FitzHugh–Nagumo neurons using differential flatness theory

Differential flatness of the model of the coupled neurons

It can be proven that the model of the coupled FitzHugh–Nagumo neurons described in Eq. (26) is a differentially flat one. The following flat outputs are defined $y_1 = h_1(x) = x_1$ and $y_2 = h_2(x) = x_3$. From the first row of the dynamical model of the coupled neurons one has

$$\dot{y}_1 = -\epsilon\gamma y_1 + \epsilon x_2 \Rightarrow x_2 = \frac{\dot{y}_1 + \epsilon\gamma y_1}{\epsilon} \quad (31)$$

Similarly, from the third row of the dynamical model of the coupled neurons one has

$$\dot{y}_2 = -\epsilon\gamma y_2 + \epsilon x_4 \Rightarrow x_4 = \frac{\dot{y}_2 + \epsilon\gamma y_2}{\epsilon} \quad (32)$$

Thus all state variables x_i , $i = 1, \dots, 4$ can be written as functions of the flat outputs and their derivatives. From the second row of the dynamical model of the coupled neurons one has

$$\begin{aligned} u_1 &= \dot{x}_2 - x_2(x_2 - a)(1 - x_2) + x_1 - g(x_2 - x_1) \Rightarrow \\ u_1 &= f_1(y_1, \dot{y}_1, y_2, \dot{y}_2) \end{aligned} \quad (33)$$

Similarly, from the second row of the dynamical model of the coupled neurons one has

$$\begin{aligned} u_2 &= \dot{x}_4 - x_4(x_4 - a)(1 - x_4) + x_3 - g(x_4 - x_2) \Rightarrow \\ u_2 &= f_2(y_1, \dot{y}_1, y_2, \dot{y}_2) \end{aligned} \quad (34)$$

Consequently, all state variables x_i , $i = 1, \dots, 4$ and all control inputs u_i $i = 1, 2$ of the model of the coupled FitzHugh–Nagumo neurons can be written as functions of the flat outputs y_i $i = 1, 2$ and their derivatives. Therefore, the model of the coupled neurons is a differentially flat one.

Linearization of the coupled neurons using differential flatness theory

The application of differential flatness theory enables linearization and decoupling of the neurons model in accordance to the results of “Differential flatness theory” section. For the model of the coupled FitzHugh–Nagumo neurons described in Eq. (26) the flat outputs $y_1 = h_1(x) = x_1$ and $y_2 = h_2(x) = x_3$ have been defined. It holds that

$$\dot{y}_1 = \dot{x}_1 \Rightarrow \dot{y}_1 = f_1 \Rightarrow \dot{y}_1 = -\epsilon\gamma x_1 + \epsilon x_2 \quad (35)$$

and $\ddot{y}_1 = (-\epsilon\gamma)(-\epsilon\gamma x_1 + \epsilon x_2) + \epsilon[x_2(x_2 - a)(1 - x_2) - x_1 + g(x_2 - x_4)] + \epsilon u_1 \Rightarrow$, or equivalently $\ddot{y}_1 = L_f^2 h_1(x) + L_{g_1} L_f h_1(x) u_1 + L_{g_2} L_f h_1(x) u_2$, where $L_f^2 h_1(x) = (-\epsilon\gamma)(-\epsilon\gamma x_1 + \epsilon x_2) + \epsilon[x_2(x_2 - a)(1 - x_2) - x_1 + g(x_2 - x_4)]$ and

$L_{g_1} L_f h_1(x) = \epsilon$ $L_{g_2} L_f h_1(x) = 0$. Equivalently, one computes

$$\dot{y}_2 = \dot{x}_3 \Rightarrow \dot{y}_2 = f_3 \Rightarrow \dot{y}_2 = -\epsilon\gamma x_3 + \epsilon x_4 \quad (36)$$

and $\ddot{y}_2 = L_f^2 h_2(x) + L_{g_1} L_f h_2(x) u_1 + L_{g_2} L_f h_2(x) u_2$, where $L_f^2 h_2(x) = (-\epsilon\gamma)(-\epsilon\gamma x_3 + \epsilon x_4) + \epsilon[x_4(x_4 - a)(1 - x_4) - x_3 + g(x_4 - x_2)]$ and $L_{g_1} L_f h_2(x) = 0$ $L_{g_2} L_f h_2(x) = \epsilon$.

By defining the state variables $z_1^1 = y_1$, $\dot{z}_1^1 = \dot{y}_1$, $z_1^2 = y_2$ and $\dot{z}_1^2 = \dot{y}_2$ one obtains a description of the coupled neurons in the state-space form of Eq. (30). The control signal is obtained as in the case of the control derived with the use of differential geometric methods.

State and disturbances estimation with the derivative-free nonlinear Kalman Filter

Design of a disturbance observer for the model of the coupled neurons

Using Eq. (30), the nonlinear model of the coupled neurons can be written in the MIMO canonical form that was described in “Differential flatness theory” section, that is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (37)$$

where $z_1 = y_1$, $z_2 = \dot{y}_1$, $z_3 = y_2$ and $z_4 = \dot{y}_2$, while $v_1 = L_f^2 h_1(x) + L_{g_1} L_f h_1(x) u_1 + L_{g_2} L_f h_1(x) u_2$ and $v_2 = L_f^2 h_2(x) + L_{g_1} L_f h_2(x) u_1 + L_{g_2} L_f h_2(x) u_2$. Thus one has a MIMO linear model of the form

$$\begin{aligned} \dot{z} &= Az + Bv \\ z_m &= Cz \end{aligned} \quad (38)$$

where $z = [z_1, z_2, z_3, z_4]^T$ and matrices A, B, C are in the MIMO canonical form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad C^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (39)$$

Assuming now the existence of additive input disturbances in the linearized model of the coupled neurons described in Eq. (30) one gets

$$\begin{pmatrix} \dot{z}_1^1 \\ \dot{z}_2^1 \\ \dot{z}_2^2 \end{pmatrix} = \begin{pmatrix} L_f^2 h_1(x) \\ L_f^2 h_2(x) \end{pmatrix} + \begin{pmatrix} L_{g_1} L_f h_1(x) & L_{g_2} L_f h_1(x) \\ L_{g_1} L_f h_2(x) & L_{g_2} L_f h_2(x) \end{pmatrix} \begin{pmatrix} u_1 + d_1 \\ u_2 + d_2 \end{pmatrix} \quad (40)$$

These disturbances can be due to external perturbations affecting the coupled neurons and can also represent parametric uncertainty for the neuron's model. It can be assumed that the additive disturbances d_i , $i = 1, 2$ are described by the n -th order derivatives of d_i and the associated initial conditions. Without loss of generality it will be considered that $n = 2$. This means

$$\begin{aligned}\ddot{d}_1 &= f_a(y_1, \dot{y}_1, y_2, \dot{y}_2) \\ \ddot{d}_2 &= f_b(y_1, \dot{y}_1, y_2, \dot{y}_2)\end{aligned}\quad (41)$$

By defining the additional state variables $z_5 = d_1, z_6 = \dot{d}_1, z_7 = d_2, z_8 = \dot{d}_2$ one has $\dot{z}_1 = z_2, \dot{z}_2 = v_1 + d_1, \dot{z}_3 = z_4, \dot{z}_4 = v_2 + d_2, \dot{z}_5 = z_6, \dot{z}_6 = f_a, \dot{z}_7 = z_8$ and $\dot{z}_8 = f_b$. Thus, in the case of additive input disturbances the state-space equations of the model of the coupled neurons can be written in the following matrix form

$$\begin{aligned}\dot{z} &= Az + Bv \\ z_m &= Cz\end{aligned}\quad (42)$$

where the state vector is $z = [z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8]^T$, the control input vector is $u = [v_1, v_2, f_a, f_b]^T$ and the measured output is $z_m = [z_1, z_3]^T$, while matrices A, B and C are defined as

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\quad (43)$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\quad (44)$$

The associated disturbance observer is

$$\begin{aligned}\dot{\hat{z}} &= A_o \hat{z} + B_o u + K(z_m - \hat{z}_m) \\ \hat{z}_m &= C_o \hat{z}\end{aligned}\quad (45)$$

where $A_o = A, C_o = C$ and

$$B_o^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}\quad (46)$$

where now $u = [v_1, v_2]^T$

Disturbances compensation for the model of the coupled neurons

For the aforementioned model, and after carrying out discretization of matrices A_o, B_o and C_o with common discretization methods one can perform Kalman Filtering (Rigatos and Zhang 2009; Rigatos 2011). This is *Derivative-free nonlinear Kalman filtering* which, unlike EKF, is performed without the need to compute Jacobian matrices and does not introduce numerical errors due to approximative linearization with Taylor series expansion.

In the design of the associated disturbances' estimator one has the dynamics defined in Eq. (45), where $K \in \mathbb{R}^{8 \times 2}$ is the state estimator's gain and matrices A_o, B_o and C_o have been defined in Eqs. (43), (44) and (46). The discrete-time equivalents of matrices A_o, B_o and C_o are denoted as $\tilde{A}_d, \tilde{B}_d$ and $\tilde{C}_d$ respectively, and are computed with the use of common discretization methods. Next, a derivative-free nonlinear Kalman Filter can be designed for the aforementioned representation of the system dynamics (Rigatos 2011, 2012a). The associated Kalman Filter-based disturbance estimator is given by the recursion (Rigatos and Tzafestas 2007; Rigatos and Zhang 2009)

measurement update:

$$\begin{aligned}K(k) &= P^-(k) \tilde{C}_d^T [\tilde{C}_d P^-(k) \tilde{C}_d^T + R]^{-1} \\ \hat{z}(k) &= \hat{z}^-(k) + K(k) [\tilde{C}_d z(k) - \tilde{C}_d \hat{z}^-(k)] \\ P(k) &= P^-(k) - K(k) \tilde{C}_d P^-(k)\end{aligned}\quad (47)$$

time update:

$$\begin{aligned}P^-(k+1) &= \tilde{A}_d(k) P(k) \tilde{A}_d^T(k) + Q(k) \\ \hat{z}^-(k+1) &= \tilde{A}_d(k) \hat{z}(k) + \tilde{B}_d(k) \hat{v}(k)\end{aligned}\quad (48)$$

Moreover, by using the inverse transformations designated by Eqs. (31) and (32) one can obtain also estimates of the state variables of the initial nonlinear system. Finally, to compensate for the effects of the disturbance inputs, it suffices to use in the control loop the estimates of the disturbance terms that is $\hat{z}_5 = d_1$ and $\hat{z}_7 = d_2$ and the modified control input vector

$$v = \begin{pmatrix} v_1 - \hat{f}_a \\ v_2 - \hat{f}_b \end{pmatrix} \text{ or } v = \begin{pmatrix} v_1 - \hat{z}_5 \\ v_2 - \hat{z}_7 \end{pmatrix}\quad (49)$$

Fig. 4 Test case 1: Synchronization in the model of the two coupled FitzHugh–Nagumo neurons **a** between state variables x_1 (master neuron) and x_3 (slave neuron), **b** between state variables x_2 (master neuron) and x_4 (slave neuron)

Fig. 5 Test case 1: Phase diagrams of the synchronized FitzHugh–Nagumo neurons **a** between state variables x_1 – x_2 (master neuron) and between x_3 – x_4 (slave neuron), **b** estimation of disturbances affecting

the master neuron (state variables z_5 and z_6) and the slave neuron (state variables z_7 and z_8)

Simulation tests

Results on the synchronization between the master and the slave FitzHugh–Nagumo neuron are provided for three different cases, each one associated with different disturbance and uncertainty terms affecting the neuron model, as well as with different setpoints for the state vector elements of the coupled neurons' model. The results obtained in the first test case are shown in Figs. 4 and 5. The results obtained in the second case are shown in Figs. 6 and 7.

Finally, the results obtained in the third case are shown in Figs. 8 and 9.

In Figs. 4a, 6a and 8a it is shown how the proposed Kalman Filter-based feedback control scheme succeeded the convergence of state variable x_3 of the second neuron to state variable x_1 of the master neuron. Similarly, in Figs. 4b, 6b and Fig. 8b it is shown how the proposed Kalman Filter-based feedback control scheme succeeded the convergence of state variable x_4 of the second neuron to state variable x_2 of the master neuron. In Figs. 5a, 7a and 9a the

Fig. 6 Test case 2: Synchronization in the model of the two coupled FitzHugh–Nagumo neurons **a** between state variables x_1 (master neuron) and x_3 (slave neuron), **b** between state variables x_2 (master neuron) and x_4 (slave neuron)

Fig. 7 Test case 2: Phase diagrams of the synchronized FitzHugh–Nagumo neurons **a** between state variables x_1 – x_2 (master neuron) and between x_3 – x_4 (slave neuron), **b** estimation of disturbances affecting

the master neuron (state variables z_5 and z_6) and the slave neuron (state variables z_7 and z_8)

phase diagrams between the state variables x_1, x_2 of the master neuron and the phase diagrams between the state variables x_3, x_4 of the slave neuron are presented. The phase diagrams for the master and slave neuron become identical and this is an indication about the succeeded synchronization. Finally, in Figs. 5b, 7b and 9b the estimates of the disturbance terms affecting the neurons' model are provided. It can be noticed that the Kalman Filter-based observer provides accurate estimates about the non-measurable disturbances and this information enables the efficient compensation of the perturbations' effects.

Remark 2 The proposed control is possible both for engineering and for biomedical applications. Results on CPGs use for coordination of gait and locomotion in biped, multi-legged and humanoid robots can be found in Aoi and Tsuchiya (2006); Aoi et al. (2012); Hong et al. (2014); Liu et al. (2013); Morimoto et al. (2008); Nakada et al. (2003); Niu et al. (2014); Or (2009); Still et al. (2006), and Yu et al. (2014). The method analyzed in the previous sections on synchronizing control of coupled neuronal models can find similar application in the design of a Central Pattern Generation for coordinating the motion of robot's joints. In

Fig. 8 Test case 3: Synchronization in the model of the two coupled FitzHugh–Nagumo neurons **a** between state variables x_1 (master neuron) and x_3 (slave neuron), **b** between state variables x_2 (master neuron) and x_4 (slave neuron)

Fig. 9 Test case 3: Phase diagrams of the synchronized FitzHugh–Nagumo neurons **a** between state variables x_1 – x_2 (master neuron) and between x_3 – x_4 (slave neuron), **b** estimation of disturbances affecting

the master neuron (state variables z_5 and z_6) and the slave neuron (state variables z_7 and z_8)

an equivalent manner, the results of the paper can be used for the assistance and rehabilitation of the human motion. Indicative results on the use of Central Patterns Generators for coordination of the motion of human joints in limbs or in controlling spinal cord locomotion can be found in Amini et al. (2005); Guevremont et al. (2006); Noble et al. (2011); Simoni and DeWeerth (2006) and Vogelstein et al. (2006). The results presented in the previous sections can also find similar application in the assistance and correction of the motor functions of the human body and in the treatment of neurological diseases (e.g. motion desynchronization symptoms appearing in Parkinson's disease).

Conclusions

A new method for robust synchronization of coupled neural oscillators has been developed. The model of the FitzHugh–Nagumo neural oscillators has been considered which represents efficiently voltage variations on the neuron's membrane, due to ionic currents and external activation currents. It was pointed out that synchronism between the coupled neurons is responsible for rhythm generation and control of several functions in living species such as gait, breathing, heart's pulsing etc. Moreover, models of synchronized neurons can be used in several engineering applications e.g. in robotics for

controlling the locomotion of multilegged, quadruped and biped robots.

With the application of differential geometric methods and differential flatness theory it was shown that the nonlinear model of the FitzHugh–Nagumo neuron can be written in the linear canonical form. It was also shown that by applying differential geometric methods and by computing Lie derivatives, exact linearization of the model of the coupled FitzHugh–Nagumo neurons (connected through a gap junction) can be succeeded. Moreover, it was proven that the model of the coupled FitzHugh–Nagumo neuron is a differentially flat one and by defining appropriate flat outputs it can be transformed to the MIMO linear canonical form. For the linearized representation of the coupled neuron's model the design of a feedback control is possible and synchronization between the two neurons can be attained.

Next, the problem of synchronization of the coupled neurons under external perturbations and parametric uncertainties was examined. To obtain simultaneous state and disturbances estimation a disturbance observer based on the derivative-free nonlinear Kalman Filter has been used. The derivative-free nonlinear Kalman Filter consists of the standard Kalman Filter recursion on the linearized equivalent model of the coupled neurons and on computation of state and disturbance estimates using the diffeomorphism (relations about state variables transformation) provided by differential flatness theory. After estimating the disturbance terms in the neurons' model their compensation has become possible.

The performance of the synchronizing control loop has been tested through simulation experiments. It was shown that the proposed method assures that the neurons will remain synchronized, despite parametric variations and uncertainties in the associated dynamical model and despite the existence of external perturbations. The method can be extended to multiple interconnected neurons, thus enabling to succeed more complicated motion patterns either in robotic or in biomedical applications.

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