Segregated and synchronized vector solutions to linearly coupled systems of Schrödinger equations

Wei Long^{1,a)} and Qingfang Wang^{2,b)}

¹College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, People's Republic of China

²School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, People's Republic of China

(Received 18 November 2014; accepted 24 August 2015; published online 11 September 2015)

In this paper, we study the following linearly coupled system $-\varepsilon^2 \Delta u_i + P_i(x)u_i = u_i^3 + \sum_{j \neq i}^N \lambda_{ij} u_j$, $u_i \in H^1(\mathbb{R}^3)$, $i = 1, \dots, N$, where $\varepsilon > 0$ is a small parameter, $P_i(x)$ are positive potentials, and $\lambda_{ij} = \lambda_{ji} > 0$ ($i \neq j$) are coupling constants for $i, j = 1, \dots, N$. We investigate the effect of potentials to the structure of the solutions. More precisely, we construct multi-spikes solutions concentrating near the local maximum point x_0^i of $P_i(x)$. When $x_0^i = x_0^j$, $P_i(x_0^i) = P_j(x_0^j) = a$, $i \neq j$, $i, j = 1, \dots, N$, the components have spikes clustering at the same point as $\varepsilon \rightarrow 0^+$. When $x_0^i \neq x_0^j$, $i \neq j$, the components have spikes clustering at the different points as $\varepsilon \rightarrow 0^+$. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4930189>]

I. INTRODUCTION

This paper is concerned with the following nonlinear Schrödinger (NLS) system in $\mathbb{R}^3$:

$$\begin{cases} -\varepsilon^2 \Delta u_i + P_i(x)u_i = u_i^3 + \sum_{j \neq i}^N \lambda_{ij} u_j, & x \in \mathbb{R}^3, \\ u_i \in H^1(\mathbb{R}^3), & i = 1, \dots, N, \end{cases} \quad (1.1)$$

where $\varepsilon > 0$ is a small parameter, $P_i(x)$ are positive potentials, and λ_{ij} are coupling constants satisfying $\lambda_{ij} = \lambda_{ji} > 0$ for $i, j = 1, \dots, N$. This type of system arises when one considers stationary pulselike (standing wave) solutions of the time-dependent N -coupled Schrödinger system of the form

$$\begin{cases} -i\hbar \frac{\partial \psi_j}{\partial t} - \frac{\hbar^2}{2} \Delta \psi_j + P_j(x)\psi_j = |\psi_j|^2 \psi_j + \sum_{k \neq j}^N \lambda_{jk} \psi_k, & x \in \mathbb{R}^3, t > 0, \\ \psi_j = \psi_j(x, t) \in \mathbb{C}, & j = 1, \dots, N, \end{cases} \quad (1.2)$$

where i denotes the imaginary unit and $\hbar$ is the Plank constant. In fact, system (1.2) appears in nonlinear optics (cf. Ref. 1). For example, the propagation of optical pulses in nonlinear N -core directional coupler can be described by N linearly coupled nonlinear Schrödinger system (1.2). Here, ψ_j ($j = 1, 2, \dots, N$) are envelope functions and λ_{jk} , which is the normalized coupling coefficient between the cores, is equal to the linear coupling coefficient times the dispersion length. Generally speaking, for system (1.2) with constant potentials, the sign of λ_{jk} determines whether the interactions of fiber couplers are repulsive or attractive. In the attractive case, the components of a vector solution tend to go along with each other leading to synchronization, and in the repulsive case, the components tend to segregate with each other leading to phase separations. These phenomena have been documented in numeric simulations (e.g., Ref. 1 and references therein) and

^{a)}E-mail address: hopelw@126.com

^{b)}E-mail address: hbwangqingfang@163.com

in mathematical proofs (e.g., Ref. 18 and references therein). However, if the potentials are not constants, what will happen? That is the main motivation of this paper. In the rest of this paper, we will investigate whether the interactions of fiber couplers are repulsive or attractive in the case of the potentials being not constants.

In particular for $N = 1$, problem (1.1) is the following classic Schrödinger equation

$$-\varepsilon^2 \Delta u + P(x)u = u^p, \quad x \in \mathbb{R}^n, \quad u > 0, \quad (1.3)$$

where $1 < p < \frac{n+2}{n-2}$. The Equation (1.3) arises in the study of solitary waves in nonlinear equations of the Klein-Gordon or Schrödinger type and has been under extensive studies in recent years (see, e.g., Refs. 8–10, 16, 17, 25, 27, 33, and 34 and references therein).

Recently, the system of Schrödinger equations has been broadly investigated in many aspects, such as the existence and qualitative properties of solutions, asymptotic behavior, concentration, and multi-bump phenomena for semiclassical states. For results in this direction, the readers can refer Refs. 3, 4, 14, 15, 17, 20–22, 26, and 30 and the references therein. Mathematical work on systems with the nonlinearly coupling terms (e.g., the term $\sum_{i \neq j} u_i$ in (1.1) being replaced by $\sum_{i \neq j}^N u_i^2 u_j$) has been studied extensively in recent years, for example, Refs. 6, 7, 12, 15, 19, 23, 24, 28, 29, 31, 32, and 35 and references therein, where phase separation or synchronization vector solutions have been proved in several cases. From these papers, we know that the existence of multi-bump solutions requires the presence of a suitable potential depending upon x which breaks the symmetry invariance of the autonomous NLS.

However, as far as we know, it seems that there are few results for linearly coupled system (1.1). In Ref. 18, for ε fixed and sufficiently small $\lambda_{ij} = \lambda(i \neq j) > 0$, Lin and Peng construct non-radial vector solutions of segregated type with two components having exactly l positive bumps, where l is a positive integer. In Ref. 13, Chen and Zou get positive solutions of the coupled system for sufficiently small $\varepsilon > 0$, which have concentration phenomenon as $\varepsilon \rightarrow 0$. In Ref. 2, by using concentration compactness principle, Ambrosetti, Cerami, and Ruiz studied the existence of both positive ground and bound states under some decaying assumptions of the potentials at infinity for $N = 2$. In Ref. 5, Ambrosetti *et al.* showed that there exists a solution with one multi-bump component having bumps located near some points, while the other component has one negative peak when $N = 2$.

The main purpose of this paper is to investigate how potentials $P_i(x)$ and coupling constants λ_{ij} influence the structure of solutions to Equation (1.1). Under some assumptions on λ_{ij} and $P_i(x)$, we construct multi-peak positive vector solutions for both the segregate case and the synchronize case for problem (1.1).

More precisely, we construct multi-spike solutions concentrating near the local maximum point x_0^i of $P_i(x)$. When $x_0^i = x_0^j$, $P_i(x_0^i) = P_j(x_0^j) = a$, $i \neq j$, $i, j = 1, \dots, N$, the components have spikes clustering at the same point as $\varepsilon \rightarrow 0^+$. When $x_0^i \neq x_0^j$, $i \neq j$, the components have spikes clustering at the different points as $\varepsilon \rightarrow 0^+$.

As for $i = 1, \dots, N$, we assume that $P_i(x)$ satisfy the following hypotheses.

(H₀): $P_i(x) \in C^1(\mathbb{R}^3, \mathbb{R})$ and there exist positive constants b_i such that $\inf_{\mathbb{R}^N} P_i(x) \geq b_i$ for all $x \in \mathbb{R}^3$.

(H₁): Suppose that $A_1, \dots, A_h \subset \{1, \dots, N\}$ disjoint mutually such that $\bigcup_{s=1}^h A_s = \{1, \dots, N\}$. For each A_s ($s = 1, \dots, h$), there exist $\delta_s > 0$ and $x_0^s \in \mathbb{R}^3$ such that $P_j(x) < P_j(x_0^s)$ for $x \in B_{\delta_s}(x_0^s) \setminus \{x_0^s\}$ and $j \in A_s$. For every s , we denote $P_{k_1}(x_0^s) = P_{k_2}(x_0^s) = a_s$ for $k_1, k_2 \in A_s$. Moreover, for every $k_1 \in A_{s_1}$, $k_2 \in A_{s_2}$ ($s_1 \neq s_2$), we assume $x_0^{s_1} \neq x_0^{s_2}$.

(H₂): There exist positive constants L_i and θ_i such that $|P_i(x) - P_i(y)| \leq L_i |x - y|^{\theta_i}$ for all $x, y \in \mathbb{R}^3$, $\theta_i > 0$.

Our main result is as follows.

Theorem 1.1. *Suppose that (H₀), (H₁), and (H₂) hold. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, problem (1.1) has a positive solution $(u_{1\varepsilon}, \dots, u_{N\varepsilon})$, with $u_{j\varepsilon}$ (for every $j \in \{1, \dots, N\}$) having multiple spikes clustering near x_0^s ($s = 1, \dots, h$) provided that*

- (1) $\lambda_{|A_s|-1}^* < \lambda_{ij} := \lambda < \frac{\min_{i \in A_s} b_i}{|A_s|-1}$, if i, j belong to the same A_s ,
 (2) $\lambda_{ij} \sim \varepsilon^\alpha$, if i, j do not belong to the same A_s ,

where $\lambda_{|A_s|-1}^*$, α are constants, $\alpha \geq \frac{\min\{\theta_i, 1\}}{2}$, and $|A_s|$ denotes the number of the set A_s 's elements.

In particular, when $h = 1$, we rewrite Theorem 1.1 in the following form.

Theorem 1.2. Suppose that (H_0) , (H_1) , and (H_2) hold and $h = 1$. Then there is $\lambda^* > 0$ such that $\lambda_{N-1}^* < \lambda_{ij} := \lambda < \frac{\min_{i \in \{1, \dots, N\}} b_i}{N-1}$ ($i \neq j$, $i, j = 1, \dots, N$). For any positive integer $k \in \mathbb{Z}^+$, problem (1.1) has a positive solution $(u_{1\varepsilon}, \dots, u_{N\varepsilon})$, with k spikes concentrating near x_0 and $(u_{1\varepsilon}, \dots, u_{N\varepsilon})$ is a synchronized vector solution for ε sufficient small.

Let us outline the main idea in the proof of Theorem 1.1. Without loss of generality, we only consider the case $N = 3$, i.e., problem (1.1) can be written as

$$\begin{cases} -\varepsilon^2 \Delta u + P_1(x)u = u^3 + \lambda_1 v + \lambda_2 w, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta v + P_2(x)v = v^3 + \lambda_1 u + \lambda_3 w, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta w + P_3(x)w = w^3 + \lambda_2 u + \lambda_3 v, & x \in \mathbb{R}^3, \end{cases} \quad (1.4)$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$ are coupling constants. Suppose that

$$\max_{B_\delta(x_0^1) \setminus \{x_0^1\}} P_i(x) = P_i(x_0^1) = a_1, \quad i = 1, 2$$

and

$$\max_{B_\delta(x_0^2) \setminus \{x_0^2\}} P_3(x) = P_3(x_0^2) = a_2, \quad x_0^1 \neq x_0^2.$$

We denote $\theta_1 = \min\{\theta_1, \theta_2, \theta_3\}$.

Hereafter,

$$H = \left\{ (u, v, w) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} P_1(x)u^2 < \infty, \int_{\mathbb{R}^3} P_2(x)v^2 < \infty, \int_{\mathbb{R}^3} P_3(x)w^2 < \infty \right\}$$

endowed with following norm:

$$\|(u, v, w)\|^2 = \langle (u, v, w), (u, v, w) \rangle = \|u\|_{\varepsilon, P_1}^2 + \|v\|_{\varepsilon, P_2}^2 + \|w\|_{\varepsilon, P_3}^2, \quad \forall (u, v, w) \in H,$$

where

$$\|u\|_{\varepsilon, P_i}^2 = \langle u, u \rangle_{\varepsilon, P_i} = \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u|^2 + P_i(x)u^2), \quad i = 1, 2, 3.$$

Let W_{a_i} denote the solution of the equation

$$\begin{cases} -\Delta w + a_i w = w^3, & w > 0, x \in \mathbb{R}^3, \\ w(0) = \max_{x \in \mathbb{R}^3} w(x), & w(x) \in H^1(\mathbb{R}^3). \end{cases} \quad (1.5)$$

Particularly, we set $W = W_1$. It is well-known that $W(x) = W(|x|)$ is nondegeneracy and satisfies

$$W'(r) < 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{2}} e^r W(r) = C > 0, \quad \lim_{r \rightarrow \infty} \frac{W'(r)}{W(r)} = -1.$$

We will use $(U, V, W_{a_2}) = (W_{a_1-\lambda_1}, W_{a_1-\lambda_1}, W_{a_2})$ to build up the approximate solutions for (1.4), where $(U, V) = (W_{a_1-\lambda_1}, W_{a_1-\lambda_1})$ is the solution of following equation:

$$\begin{cases} -\Delta u + a_1 u = u^3 + \lambda_1 v, & x \in \mathbb{R}^3, \\ -\Delta v + a_1 v = v^3 + \lambda_1 u, & x \in \mathbb{R}^3. \end{cases} \quad (1.6)$$

We define

$$D_{\varepsilon,\delta}^{k,y} = \left\{ x = (y^1, \dots, y^k) \in (\mathbb{R}^3)^k : y^j \in B_\delta(x_0^1), \frac{|y^i - y^j|}{\varepsilon} \geq |\ln \varepsilon|^{\frac{1}{2}}, i \neq j, i, j = 1, \dots, k \right\}$$

and

$$D_{\varepsilon,\delta}^{m,z} = \left\{ z = (z^1, \dots, z^m) \in (\mathbb{R}^3)^m : z^j \in B_\delta(x_0^2), \frac{|z^i - z^j|}{\varepsilon} \geq |\ln \varepsilon|^{\frac{1}{2}}, i \neq j, i, j = 1, \dots, m \right\}.$$

Fix $y = (y^1, \dots, y^k) \in D_{\varepsilon,\delta}^{k,y}$, $z = (z^1, \dots, z^m) \in D_{\varepsilon,\delta}^{m,z}$, we set

$$U_{\varepsilon,y} = \sum_{i=1}^k U_{\varepsilon,y^i}(x), \quad V_{\varepsilon,y} = \sum_{i=1}^k V_{\varepsilon,y^i}(x), \quad W_{\varepsilon,z} = \sum_{i=1}^m W_{\varepsilon,z^i}(x),$$

where

$$U_{\varepsilon,y^i} = W_{a_1-\lambda_1}\left(\frac{x-y^i}{\varepsilon}\right), \quad V_{\varepsilon,y^i} = W_{a_1-\lambda_1}\left(\frac{x-y^i}{\varepsilon}\right), \quad W_{\varepsilon,z^i} = W_{a_2}\left(\frac{x-z^i}{\varepsilon}\right).$$

We write

$$E_{k,m}^{y,z} = \left\{ (\varphi, \psi, \phi) \in H : \left\langle \left(\frac{\partial U_{\varepsilon,y^i}}{\partial y_l^i}, \frac{\partial V_{\varepsilon,y_l^i}}{\partial y_l^i} \right), (\varphi, \psi) \right\rangle_{P_1 P_2} = 0, \left\langle \frac{\partial W_{\varepsilon,z^j}}{\partial z_l^j}, \phi \right\rangle_{\varepsilon, P_3} = 0, \right. \\ \left. i = 1, \dots, k, j = 1, \dots, m, l = 1, 2, 3 \right\}$$

and

$$M_{k,m}^{y,z} = \left\{ (y^1, \dots, y^k, z^1, \dots, z^m, \varphi, \psi, \phi) : (y^1, \dots, y^k) \in D_{\varepsilon,\delta}^{k,y}, (z^1, \dots, z^m) \in D_{\varepsilon,\delta}^{m,z}, \right. \\ \left. (\varphi, \psi, \phi) \in E_{k,m}^{y,z} \right\}.$$

Here,

$$\left\langle \left(\frac{\partial U_{\varepsilon,y^i}}{\partial y_l^i}, \frac{\partial V_{\varepsilon,y_l^i}}{\partial y_l^i} \right), (\varphi, \psi) \right\rangle_{P_1 P_2} = \left\langle \frac{\partial U_{\varepsilon,y^i}}{\partial y_l^i}, \varphi \right\rangle_{\varepsilon, P_1} + \left\langle \frac{\partial V_{\varepsilon,y_l^i}}{\partial y_l^i}, \psi \right\rangle_{\varepsilon, P_2}.$$

It is easy to see that Theorem 1.1 is a direct consequence of the following result in the case of $N = 3$.

Theorem 1.3. *Under the assumptions of Theorem 1.1, $\lambda^* < \lambda_1 < a_1$, $\lambda_2, \lambda_3 \sim \varepsilon^\alpha$, $\alpha \geq \frac{\min\{\theta_1, 1\}}{2}$, for ε sufficiently small, problem (1.4) has a solution of the form*

$$(u_\varepsilon, v_\varepsilon, w_\varepsilon) = (U_{\varepsilon,y}(x) + \varphi_\varepsilon, V_{\varepsilon,y}(x) + \psi_\varepsilon, W_{\varepsilon,z}(x) + \phi_\varepsilon),$$

where $(\varphi_\varepsilon, \psi_\varepsilon, \phi_\varepsilon) \in E_{k,m}^{y,z}$.

Remark 1.4. In Theorem (1.3), taking $w_\varepsilon = 0$. In a similar way, we can obtain that $(u_\varepsilon, v_\varepsilon) = (U_{\varepsilon,y}(x) + \varphi_\varepsilon, V_{\varepsilon,y}(x) + \psi_\varepsilon)$ is a synchronized vector solution of Equation (1.1) for $N = 2$. Similarly, Theorem (1.2) is proved.

To prove Theorem 1.3, we will use the well-known Lyapunov-Schmidt reduction in dealing with the singularly perturbed elliptic problems. For Lyapunov-Schmidt reduction, one needs to use the nondegeneracy of solution of (2.1). So, we will adopt the idea of Ambrosetti, Cerami, and Ruiz in Ref. 2 to research the nondegeneracy of the solutions for the l -linearly coupled Schrödinger equations.

The main difficulty in constructing solutions to the linearly coupled system is to get a better control of the error terms. Compared with non-linearly coupled system, the linear term $\sum u_j$'s attenuation speed is not enough quick. To overcome this difficulty, we need to adjust the parameter λ_{ij} to increase the speed of attenuation.

Our paper is organized as follows. In Section II, we will give nondegeneracy of the solution. In Sec. III, we will establish some preliminary estimate. We will carry out a reduction procedure and study the reduced one dimensional problem to prove Theorem 1.3 in Sec. IV. In the Appendix, an energy expansion for the functional corresponding to problem (1.1) will be established.

II. NONDEGENERACY OF THE SOLUTION

First, we will introduce the nondegeneracy of the solutions $(u_1, \dots, u_l)$ for the $l \geq 2$ coupled Schrödinger equations,

$$\begin{cases} -\Delta u_i + au_i = u_i^3 + \lambda \sum_{i \neq j}^l u_j, & x \in \mathbb{R}^3, \\ u_i \in H^1(\mathbb{R}^3), & i = 1, \dots, l, \end{cases} \quad (2.1)$$

where $u_1(x) = \dots = u_l(x) = W_{a-(l-1)\lambda}(x)$.

Proposition 2.1. If $\lambda \in (\lambda_{l-1}^*, \frac{a}{l-1})$, where λ_{l-1}^* is a positive constant, $(u_1, \dots, u_l)$ is nondegeneracy for system (2.1) in $(H^1(\mathbb{R}^3))^l$ in the sense that the kernel is given by $\text{span} \left\{ \left(\frac{\partial W}{\partial x_i}, \dots, \frac{\partial W}{\partial x_i} \right) | i = 1, 2, 3 \right\}$.

Proof. Following from Ref. 2, let us denote

$$Z_\lambda =: \left\{ \left(W_{a-(l-1)\lambda}(x - \xi), \dots, W_{a-(l-1)\lambda}(x - \xi) \right) : \xi \in \mathbb{R}^3 \right\}$$

and

$$\begin{aligned} F_\lambda(u_1, \dots, u_l) =: & \left(-\Delta u_1 + au_1 - u_1^3 - \lambda \sum_{j \neq 1} u_j, \dots, -\Delta u_i + au_i - u_i^3 - \lambda \sum_{j \neq i} u_j, \dots, \right. \\ & \left. -\Delta u_l + au_l - u_l^3 - \lambda \sum_{j \neq l} u_j \right). \end{aligned} \quad (2.2)$$

We may take $\xi = 0$. It is easy to compute

$$T_{(W_{a-(l-1)\lambda}, \dots, W_{a-(l-1)\lambda})} Z = \text{span} \left\{ \left(\frac{\partial W}{\partial x_i}, \dots, \frac{\partial W}{\partial x_i} \right), i = 1, 2, 3 \right\}.$$

On the other hand,

$$\begin{aligned} F'_\lambda(W_{a-(l-1)\lambda}, \dots, W_{a-(l-1)\lambda})(\phi_1, \dots, \phi_l) = & \left(-\Delta \phi_1 + a\phi_1 - 3W_{a-(l-1)\lambda}^2 \phi_1 - \lambda \sum_{j \neq 1} \phi_j \right. \\ & \dots \\ & -\Delta \phi_i + a\phi_i - 3W_{a-(l-1)\lambda}^2 \phi_i - \lambda \sum_{j \neq i} \phi_j \\ & \dots \\ & \left. -\Delta \phi_l + a\phi_l - 3W_{a-(l-1)\lambda}^2 \phi_l - \lambda \sum_{j \neq l} \phi_j \right). \end{aligned} \quad (2.3)$$

Letting $(\phi_1, \dots, \phi_l)$ belong to its kernel, by the subtraction of the l equations with each other, we get

$$-\Delta(\phi_i - \phi_j) + a(\phi_i - \phi_j) + (\lambda - 3W_{a-(l-1)\lambda}^2)(\phi_i - \phi_j) = 0, \quad (i \neq j, i, j = 1, \dots, l).$$

Taking into account the choice of λ , we have $\lambda - 3W_{a-(l-1)\lambda}^2 \geq 0$. Hence, $\phi_i - \phi_j = 0$. Moreover, ϕ_j satisfies the equation

$$-\Delta \phi_j + (a - (l-1)\lambda)\phi_j - 3W_{a-(l-1)\lambda}^2 \phi_j = 0 \quad (j = 1, \dots, l).$$

We conclude that $\phi_j \in \text{span} \left\{ \frac{\partial W}{\partial x_i}, i = 1, 2, 3 \right\}$. □

III. SOME PRELIMINARIES

Let

$$\begin{aligned} I_\varepsilon(u, v, w) := & \frac{1}{2} \int_{\mathbb{R}^3} \left(\varepsilon^2 |\nabla u|^2 + P_1(x)u^2 + \varepsilon^2 |\nabla v|^2 + P_2(x)v^2 + \varepsilon^2 |\nabla w|^2 + P_3(x)w^2 \right) \\ & - \frac{1}{4} \int_{\mathbb{R}^3} (u^4 + v^4 + w^4) - \lambda_1 \int_{\mathbb{R}^3} uv - \lambda_2 \int_{\mathbb{R}^3} uw - \lambda_3 \int_{\mathbb{R}^3} vw \end{aligned}$$

and

$$J_\varepsilon(\varphi, \psi, \phi) = I_\varepsilon(U_{\varepsilon,y} + \varphi, V_{\varepsilon,y} + \psi, W_{\varepsilon,z} + \phi), \quad \forall (\varphi, \psi, \phi) \in E_{k,m}^{y,z}.$$

Expand $J_\varepsilon(\varphi, \psi, \phi)$ as follows:

$$J_\varepsilon(\varphi, \psi, \phi) = J_\varepsilon(0, 0, 0) + \ell_\varepsilon(\varphi, \psi, \phi) + \frac{1}{2} \langle L_\varepsilon(\varphi, \psi, \phi), (\varphi, \psi, \phi) \rangle + R_\varepsilon(\varphi, \psi, \phi), \quad (3.1)$$

where

$$\begin{aligned} \ell_\varepsilon(\varphi, \psi, \phi) &= \int_{\mathbb{R}^3} (P_1(x) - a_1) U_{\varepsilon,y} \varphi + (P_2(x) - a_1) V_{\varepsilon,y} \psi + (P_3(x) - a_2) W_{\varepsilon,z} \phi \\ &\quad + \int_{\mathbb{R}^3} \left(\sum_{i=1}^k U_{\varepsilon,y}^3 - U_{\varepsilon,y}^3 \right) \varphi + \left(\sum_{i=1}^k V_{\varepsilon,y}^3 - V_{\varepsilon,y}^3 \right) \psi + \left(\sum_{i=1}^m W_{\varepsilon,z}^3 - W_{\varepsilon,z}^3 \right) \phi \\ &\quad - \lambda_2 \int_{\mathbb{R}^3} (U_{\varepsilon,y} \phi + W_{\varepsilon,z} \varphi) - \lambda_3 \int_{\mathbb{R}^3} (V_{\varepsilon,y} \phi + W_{\varepsilon,z} \psi) \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} L_\varepsilon(\varphi, \psi, \phi) &= \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla \varphi|^2 + P_1(x) \varphi^2 + \varepsilon^2 |\nabla \psi|^2 + P_2(x) \psi^2 + \varepsilon^2 |\nabla \phi|^2 + P_3(x) \phi^2) \\ &\quad - \int_{\mathbb{R}^3} (3U_{\varepsilon,y}^2 \varphi^2 + 3V_{\varepsilon,y}^2 \psi^2 + 3W_{\varepsilon,z}^2 \phi^2) - 2\lambda_1 \int_{\mathbb{R}^3} \varphi \psi \\ &\quad - 2\lambda_2 \int_{\mathbb{R}^3} \varphi \phi - 2\lambda_3 \int_{\mathbb{R}^3} \psi \phi. \end{aligned} \quad (3.3)$$

In order to find a critical point for $J_\varepsilon(\varphi, \psi, \phi)$, we need to discuss each term in expansion (3.1).

Lemma 3.1. *There exist a constant $C > 0$ independent of ε such that*

$$R_\varepsilon^{(i)}(\varphi, \psi, \phi) \leq C(\varepsilon^{-\frac{3}{2}} \|(\varphi, \psi, \phi)\|^{3-i} + \varepsilon^{-3} \|(\varphi, \psi, \phi)\|^{4-i}), \quad i = 0, 1, 2.$$

Proof. By (3.1)–(3.3), we know that

$$|R(\varphi, \psi, \phi)| \leq \left| \int_{\mathbb{R}^3} \sum_{i=1}^k (U_{\varepsilon,y} \varphi^3 + V_{\varepsilon,y} \psi^3) + \sum_{i=1}^m W_{\varepsilon,z} \phi^3 + \frac{1}{4} \varphi^4 + \frac{1}{4} \psi^4 + \frac{1}{4} \phi^4 \right|.$$

Let $\tilde{\varphi}(x) = \varphi(\varepsilon x)$. Since $\inf_{\mathbb{R}^3} P_i(x) \geq b_i > 0$, we see

$$\begin{aligned} \int_{\mathbb{R}^3} |\varphi|^4 dx &= \varepsilon^3 \int_{\mathbb{R}^3} |\tilde{\varphi}|^4 dx \leq C\varepsilon^3 \left(\int_{\mathbb{R}^3} |\nabla \tilde{\varphi}|^2 + |\tilde{\varphi}|^2 \right)^2 \\ &= C\varepsilon^3 \left(\varepsilon^{-3} \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla \varphi|^2 + P_1(x) |\varphi|^2) dx \right)^2 \\ &\leq C\varepsilon^{-3} \|\varphi\|_{\varepsilon, P_1}^4. \end{aligned} \quad (3.4)$$

Similar to (3.4), we obtain

$$\int_{\mathbb{R}^3} |\psi|^4 dx \leq C\varepsilon^{-3} \|\psi\|_{\varepsilon, P_2}^4, \quad \int_{\mathbb{R}^3} |\phi|^4 dx \leq C\varepsilon^{-3} \|\phi\|_{\varepsilon, P_3}^4. \quad (3.5)$$

Using the Hölder inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^3} \sum_{i=1}^k U_{\varepsilon,y} \varphi^3 &\leq \sum_{i=1}^k \left(\int_{\mathbb{R}^3} U_{\varepsilon,y}^4 \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}^3} \varphi^4 \right)^{\frac{3}{4}} \\ &\leq C \sum_{i=1}^k (\varepsilon^3 \int_{\mathbb{R}^3} W^4)^{\frac{1}{4}} (\varepsilon^{-3} \|\varphi\|_{\varepsilon, P_1}^4)^{\frac{3}{4}} \\ &\leq C\varepsilon^{-\frac{3}{2}} \|\varphi\|_{\varepsilon, P_1}^3, \end{aligned} \quad (3.6)$$

and analogously,

$$\int_{\mathbb{R}^3} \sum_{i=1}^k V_{\varepsilon, y^i} \psi^3 \leq C \varepsilon^{-\frac{3}{2}} \|\psi\|_{\varepsilon, P_2}^3, \quad \int_{\mathbb{R}^3} \sum_{i=1}^m W_{\varepsilon, y^i} \phi^3 \leq C \varepsilon^{-\frac{3}{2}} \|\phi\|_{\varepsilon, P_3}^3. \quad (3.7)$$

Combining (3.4)–(3.7), we find

$$\|R_{\varepsilon}(\varphi, \psi, \phi)\| \leq C(\varepsilon^{-\frac{3}{2}} \|(\varphi, \psi, \phi)\|^3 + \varepsilon^{-3} \|(\varphi, \psi, \phi)\|^4).$$

Similarly, we deduce that

$$|\langle R'_{\varepsilon}(\varphi, \psi, \phi), (g, h, f) \rangle| \leq C(\varepsilon^{-\frac{3}{2}} \|(\varphi, \psi, \phi)\|^2 + \varepsilon^{-3} \|(\varphi, \psi, \phi)\|^3) \|(g, h, f)\|$$

and

$$\begin{aligned} & |\langle R''_{\varepsilon}(\varphi, \psi, \phi)(g, h, f), (w_1, w_2, w_3) \rangle| \\ & \leq C(\varepsilon^{-\frac{3}{2}} \|(\varphi, \psi, \phi)\| + \varepsilon^{-3} \|(\varphi, \psi, \phi)\|^2) \|(g, h, f)\| \|(w_1, w_2, w_3)\|. \end{aligned}$$

□

Lemma 3.2. There exist a constant $C > 0$ independent of ε such that

$$\begin{aligned} \|\ell_{\varepsilon}\| & \leq C \left((\lambda_2 + \lambda_3) \varepsilon^{\frac{3}{2}} + \varepsilon^{\frac{3}{2} + \theta_1} + \varepsilon^{\frac{3}{2}} \sum_{i=1}^k |(P_1(y^i) - P_1(x_0^1))| + \varepsilon^{\frac{3}{2}} \sum_{i=1}^k |(P_2(y^i) - P_2(x_0^1))| \right. \\ & \quad \left. + \varepsilon^{\frac{3}{2}} \sum_{i=1}^m |(P_3(z^i) - P_3(x_0^2))| + \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} + \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2} |z^i - z^j|}{\varepsilon}} \right). \end{aligned}$$

Proof. Directly calculating, we have

$$\begin{aligned} & \left| \sum_{i=1}^k \int_{\mathbb{R}^3} (P_1(x) - P_1(x_0^1)) U_{\varepsilon, y^i} \varphi \right| \\ & = \left| \sum_{i=1}^k \int_{\mathbb{R}^3} (P_1(x) - P_1(y^i) + P_1(y^i) - P_1(x_0^1)) U_{\varepsilon, y^i} \varphi \right| \\ & \leq C \left(\sum_{i=1}^k \varepsilon^{\frac{3}{2}} |P_1(x_0^1) - P_1(y^i)| + \varepsilon^{\frac{3}{2} + \theta_1} \right) \|\varphi\|_{\varepsilon, P_1}. \end{aligned} \quad (3.8)$$

Analogously,

$$\left| \sum_{i=1}^k \int_{\mathbb{R}^3} (P_2(x) - P_2(x_0^1)) V_{\varepsilon, y^i} \psi \right| \leq C \left(\sum_{i=1}^k \varepsilon^{\frac{3}{2}} |P_2(x_0^1) - P_2(y^i)| + \varepsilon^{\frac{3}{2} + \theta_2} \right) \|\psi\|_{\varepsilon, P_2} \quad (3.9)$$

and

$$\left| \sum_{i=1}^m \int_{\mathbb{R}^3} (P_3(x) - P_3(x_0^2)) W_{\varepsilon, z^i} \phi \right| \leq C \left(\sum_{i=1}^m \varepsilon^{\frac{3}{2}} |P_3(x_0^2) - P_3(z^i)| + \varepsilon^{\frac{3}{2} + \theta_3} \right) \|\phi\|_{\varepsilon, P_3}. \quad (3.10)$$

By direct computation, we obtain

$$\left| \int_{\mathbb{R}^3} \left(\sum_{j=1}^k U_{\varepsilon, y^j}^3 - U_{\varepsilon, y}^3 \right) \varphi \right| \leq C \left| \int_{\mathbb{R}^3} \sum_{i \neq j}^k U_{\varepsilon, y^i}^2 U_{\varepsilon, y^j} \varphi \right| \leq C \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} \|\varphi\|_{\varepsilon, P_1}. \quad (3.11)$$

Similar to (3.12), we get the following result:

$$\left| \int_{\mathbb{R}^3} \left(\sum_{j=1}^k V_{\varepsilon, y^j}^3 - V_{\varepsilon, y}^3 \right) \psi \right| \leq C \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} \|\psi\|_{\varepsilon, P_2} \quad (3.12)$$

and

$$\left| \int_{\mathbb{R}^3} \left(\sum_{j=1}^m W_{\varepsilon, z^j}^3 - W_{\varepsilon, z}^3 \right) \phi \right| \leq C \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2}|z^i - z^j|}{\varepsilon}} \|\phi\|_{\varepsilon, P_3}. \quad (3.13)$$

We know

$$\begin{aligned} \int_{\mathbb{R}^3} U_{\varepsilon, y} \phi &\leq C \varepsilon^{\frac{3}{2}} \|\phi\|_{\varepsilon, P_3}^2, \quad \int_{\mathbb{R}^3} V_{\varepsilon, y} \phi \leq C \varepsilon^{\frac{3}{2}} \|\phi\|_{\varepsilon, P_3}^2, \\ \int_{\mathbb{R}^3} W_{\varepsilon, z} \varphi &\leq C \varepsilon^{\frac{3}{2}} \|\varphi\|_{\varepsilon, P_1}^2, \quad \int_{\mathbb{R}^3} W_{\varepsilon, z} \psi \leq C \varepsilon^{\frac{3}{2}} \|\psi\|_{\varepsilon, P_2}^2. \end{aligned} \quad (3.14)$$

From (3.8)–(3.14) and (3.2), we obtain

$$\begin{aligned} \|\ell_\varepsilon\| &\leq C \left(\varepsilon^{\frac{3}{2} + \theta_1} + \varepsilon^{\frac{3}{2}} \sum_{i=1}^k (|P_1(y^i) - P_1(x_0^1)| + |P_2(y^i) - P_2(x_0^1)|) \right. \\ &\quad + \varepsilon^{\frac{3}{2}} \sum_{i=1}^m |P_3(z^i) - P_3(x_0^2)| + (\lambda_2 + \lambda_3) \varepsilon^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1}|y^i - y^j|}{\varepsilon}} \\ &\quad \left. + \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2}|z^i - z^j|}{\varepsilon}} \right). \end{aligned}$$

□

IV. THE FINITE-DIMENSIONAL REDUCTION AND PROOF OF THE MAIN RESULTS

In this section, we intend to prove the main theorem by the Lyapunov-Schmidt reduction.

First, we discuss the invertibility of L_ε .

Lemma 4.1. For $\lambda_1 \in (\lambda_1^*, a_1)$, $\lambda_2, \lambda_3 \sim \varepsilon^\alpha$ and ε sufficiently small, there is a constant $\varrho^* > 0$ independent of ε such that

$$L_\varepsilon(\varphi, \psi, \phi) \geq \varrho^* \|(\varphi, \psi, \phi)\|, \quad (\varphi, \psi, \phi) \in E_{k,m}^{y,z},$$

where λ_1^* is defined in Proposition 2.1.

Proof. Arguing by contradiction, we suppose there are $(y^n, z^n) = (y^{1,n}, \dots, y^{k,n}, z^{1,n}, \dots, z^{m,n}) \in D_{\varepsilon_n, \delta_n}^{k, y^n} \times D_{\varepsilon_n, \delta_n}^{m, z^n}$ and $(\varphi_n, \psi_n, \phi_n) \in E_{k,m}^{y^n, z^n}$, with

$$\langle L_\varepsilon(\varphi_n, \psi_n, \phi_n), (g, h, f) \rangle = o_n(1) \|(\varphi_n, \psi_n, \phi_n)\| \|(g, h, f)\|, \quad \forall (g, h, f) \in E_{k,m}^{y^n, z^n}. \quad (4.1)$$

We may assume that $\|(\varphi_n, \psi_n, \phi_n)\|^2 = \varepsilon_n^3$. Fix $i(i \in \{1, \dots, k\})$, $j(j \in \{1, \dots, m\})$ and let

$$\tilde{\varphi}_{n,i} = \varphi_n(\varepsilon_n x + y^{i,n}), \quad \tilde{\psi}_{n,i} = \psi_n(\varepsilon_n x + y^{i,n}), \quad \tilde{\phi}_{n,j} = \phi_n(\varepsilon_n x + z^{j,n}).$$

We have

$$\int_{\mathbb{R}^3} (|\nabla \tilde{\varphi}_{n,i}|^2 + P_1(\varepsilon_n x + y^{i,n}) \tilde{\varphi}_{n,i}^2 + |\nabla \tilde{\psi}_{n,i}|^2 + P_2(\varepsilon_n x + y^{i,n}) \tilde{\psi}_{n,i}^2) \leq C$$

and

$$\int_{\mathbb{R}^3} (|\nabla \tilde{\phi}_{n,j}|^2 + P_3(\varepsilon_n x + z^{j,n}) \tilde{\phi}_{n,j}^2) \leq C.$$

So, we suppose the existence of $(\varphi, \psi, \phi) \in (H^1(\mathbb{R}^3))^3$ such that

$$\tilde{\varphi}_{n,i} \rightarrow \varphi, \text{ weakly in } H_{loc}^1(\mathbb{R}^3), \quad \tilde{\varphi}_{n,i} \rightarrow \varphi, \text{ strongly in } L_{loc}^2(\mathbb{R}^3),$$

$$\tilde{\psi}_{n,i} \rightarrow \psi, \text{ weakly in } H_{loc}^1(\mathbb{R}^3), \quad \tilde{\psi}_{n,i} \rightarrow \psi, \text{ strongly in } L_{loc}^2(\mathbb{R}^3),$$

$$\tilde{\phi}_{n,j} \rightarrow \phi, \text{ weakly in } H_{loc}^1(\mathbb{R}^3), \quad \tilde{\phi}_{n,j} \rightarrow \phi, \text{ strongly in } L_{loc}^2(\mathbb{R}^3).$$

Moreover, (φ, ψ, ϕ) satisfy

$$\int_{\mathbb{R}^3} \left(\nabla \frac{\partial U}{\partial y_l} \nabla \varphi + \frac{\partial U}{\partial y_l} \varphi + \nabla \frac{\partial V}{\partial y_l} \nabla \psi + \frac{\partial V}{\partial y_l} \psi \right) = 0,$$

$$\int_{\mathbb{R}^3} \left(\nabla \frac{\partial W}{\partial y_l} \nabla \phi + \frac{\partial W}{\partial y_l} \phi \right) = 0, \quad l = 1, 2, 3.$$

We claim that

$$\varphi = 0, \quad \psi = 0, \quad \phi = 0.$$

For any $(g, h) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$, we decompose

$$(g_n, h_n) = (g, h) - \sum_{i=1}^k \sum_{l=1}^3 a_{n,i,l} \left(\frac{\partial \tilde{U}_{\varepsilon_n, y_l^{t,n}}}{\partial y_l^{t,n}}, \frac{\partial \tilde{V}_{\varepsilon_n, y_l^{t,n}}}{\partial y_l^{t,n}} \right),$$

where

$$\tilde{U}_{\varepsilon_n, y_l^{t,n}} = U_{\varepsilon_n, y_l^{t,n}}(\varepsilon_n x + y^{i,n}), \quad \tilde{V}_{\varepsilon_n, y_l^{t,n}} = V_{\varepsilon_n, y_l^{t,n}}(\varepsilon_n x + y^{i,n}).$$

Then, with the same argument as in Ref. 11, it is not difficult to check that $a_{n,i,l} \rightarrow a_{t,i,l}$, $a_{n,i,l} \rightarrow 0$, $i \neq t$, $l = 1, 2, 3$ as $n \rightarrow \infty$.

Taking $(g_n(\frac{x-y^{t,n}}{\varepsilon}), h_n(\frac{x-y^{t,n}}{\varepsilon}), 0)$ into (4.1), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla \varphi \nabla g + \varphi g - 3U^2 \varphi g) + \int_{\mathbb{R}^3} (\nabla \psi \nabla h + \psi h - 3V^2 \psi h) \\ & + \sum_{l=1}^3 a_{t,l} \left(\int_{\mathbb{R}^3} \left(\nabla \frac{\partial U}{\partial x_l} \nabla \varphi + \frac{\partial U}{\partial x_l} \varphi - 3U^2 \frac{\partial U}{\partial x_l} \varphi \right) + \left(\nabla \frac{\partial V}{\partial x_l} \nabla \psi + \frac{\partial V}{\partial x_l} \psi - 3V^2 \frac{\partial V}{\partial x_l} \psi \right) \right. \\ & \left. - \int_{\mathbb{R}^3} \lambda_1 \left(\frac{\partial U}{\partial x_l} \varphi + \frac{\partial V}{\partial x_l} \psi \right) \right) - \lambda_1 \int_{\mathbb{R}^3} (\varphi h + \psi g) = 0, \quad l = 1, 2, 3. \end{aligned} \quad (4.2)$$

Note that

$$\begin{aligned} & \int_{\mathbb{R}^3} \left(\nabla \frac{\partial U}{\partial x_l} \nabla \varphi + \frac{\partial U}{\partial x_l} \varphi + \nabla \frac{\partial V}{\partial x_l} \nabla \psi + \frac{\partial V}{\partial x_l} \psi \right. \\ & \left. - 3U^2 \frac{\partial U}{\partial x_l} \varphi - 3V^2 \frac{\partial V}{\partial x_l} \psi \right) - \lambda_1 \left(\frac{\partial U}{\partial x_l} \varphi + \frac{\partial V}{\partial x_l} \psi \right) = 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla \varphi \nabla g + \varphi g - 3U^2 \varphi g) + \int_{\mathbb{R}^3} (\nabla \psi \nabla h + \psi h - 3V^2 \psi h) \\ & - \lambda_1 \int_{\mathbb{R}^3} (\varphi h + \psi g) = 0. \end{aligned}$$

Then, we get (φ, ψ) that satisfies

$$\begin{cases} -\Delta \varphi + a_1 \varphi - 3U^2 \varphi - \lambda_1 \psi = 0, & x \in \mathbb{R}^3, \\ -\Delta \psi + a_1 \psi - 3V^2 \psi - \lambda_1 \varphi = 0, & x \in \mathbb{R}^3. \end{cases}$$

From Proposition 2.1 for $l = 2$, we have $(U, V) = (W_{a_1-\lambda_1}, W_{a_1-\lambda_1})$ which is nondegeneracy, with $\lambda^* < \lambda_1 < a_1$, which implies that $(\varphi, \psi) = (0, 0)$.

Next, we denote

$$f_n = f - \sum_{j=1}^m \sum_{l=1}^3 a_{n,j,l} \frac{\partial \tilde{W}_{\varepsilon_n, z_l^{r,n}}}{\partial z_l^{r,n}},$$

where $\tilde{W}_{\varepsilon_n, z_l^{r,n}} = W_{\varepsilon_n, z_l^{r,n}}(\varepsilon_n x + z^{j,n})$ and taking $(0, 0, f_n(\frac{x-z^{r,n}}{\varepsilon}))$ into (4.1), we get

$$\int_{\mathbb{R}^3} (\nabla \phi \nabla f + \phi f - 3W^2 \phi f) = 0.$$

We get that $\phi = 0$.

Hence, by $\lambda_2, \lambda_3 \sim \varepsilon^\alpha$, $\alpha > \frac{\min\{\theta_1, 1\}}{2}$, we obtain

$$\begin{aligned} o_n(1)\varepsilon_n^3 &= \int_{\mathbb{R}^3} \left(\varepsilon_n^2 |\nabla \varphi_n|^2 + P_1(x) \varphi_n^2 + \varepsilon_n^2 |\nabla \psi_n|^2 + P_2(x) \psi_n^2 + \varepsilon_n^2 |\nabla \phi_n|^2 + P_3(x) \phi_n^2 \right) \\ &\quad - \int_{\mathbb{R}^3} \left(3U_{\varepsilon_n, y}^2 \varphi_n^2 + 3V_{\varepsilon_n, y}^2 \psi_n^2 + 3W_{\varepsilon_n, z}^2 \phi_n^2 \right) - 2\lambda_1 \int_{\mathbb{R}^3} \varphi_n \psi_n \\ &\quad - 2\lambda_2 \int_{\mathbb{R}^3} \varphi_n \phi_n - 2\lambda_3 \int_{\mathbb{R}^3} \psi_n \phi_n \\ &= \left(1 - \frac{\lambda_1}{b} \right) \|(\varphi_n, \psi_n, \phi_n)\|^2 - \int_{\mathbb{R}^3} \left(3U_{\varepsilon_n, y}^2 \varphi_n^2 + 3V_{\varepsilon_n, y}^2 \psi_n^2 + 3W_{\varepsilon_n, z}^2 \phi_n^2 \right) \\ &\quad + O(\varepsilon_n^\alpha) \|(\varphi_n, \psi_n, \phi_n)\|^2 \\ &\geq \left(1 - \frac{\lambda_1}{b} \right) \|(\varphi_n, \psi_n, \phi_n)\|^2 + o(e^{-R}) \varepsilon_n^3 + O(\varepsilon_n^\alpha) \varepsilon_n^3. \end{aligned}$$

Since $b > \lambda_1$, we get contradiction. Then, we complete the result. $\square$

Proposition 4.2. For $\varepsilon > 0$ small enough, there is a C^1 map from $D_{\varepsilon, \delta}^{k, y} \times D_{\varepsilon, \delta}^{k, y} \times D_{\varepsilon, \delta}^{m, z}$ to H such that $(\varphi, \psi, \phi) \in E_{k, m}^{y, z}$ satisfying

$$\left\langle \frac{\partial J_\varepsilon(y, z, \varphi, \psi, \phi)}{\partial(\varphi, \psi, \phi)}, (f_1, f_2, f_3) \right\rangle = 0, \quad \forall (f_1, f_2, f_3) \in E_{k, m}^{y, z}.$$

Moreover,

$$\begin{aligned} \|(\varphi, \psi, \phi)\| &\leq C \left(\varepsilon^{\frac{3}{2} + \theta_1} + (\lambda_2 + \lambda_3) \varepsilon^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}} \left(\sum_{i=1}^k (|P_1(y^i) - P_1(x_0^1)|) + \sum_{i=1}^k (|P_2(y^i) - P_2(x_0^1)|) \right) \right. \\ &\quad \left. + \sum_{i=1}^m (|P_3(z^i) - P_3(x_0^2)|) + \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} + \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2} |z^i - z^j|}{\varepsilon}} \right). \end{aligned}$$

Proof. We know that $\ell_\varepsilon(\varphi, \psi, \phi)$ is a bounded linear function in $E_{k, m}^{y, z}$ by Lemma 3.2. Thus, there is $\bar{\ell}_\varepsilon$ such that

$$\ell_\varepsilon(\varphi, \psi, \phi) = \langle \bar{\ell}_\varepsilon, (\varphi, \psi, \phi) \rangle.$$

So, finding a critical point for $J_\varepsilon(\varphi, \psi, \phi)$ in $E_{k, m}^{y, z}$ is equivalent to solving

$$\bar{\ell}_\varepsilon - L_\varepsilon(\varphi, \psi, \phi) + R'_\varepsilon(\varphi, \psi, \phi) = 0.$$

Since L_ε is invertible by Lemma 4.1, we get

$$A(\varphi, \psi, \phi) = -L_\varepsilon^{-1} \bar{\ell}_\varepsilon - L_\varepsilon^{-1} R'_\varepsilon(\varphi, \psi, \phi).$$

Set

$$\begin{aligned} D = \left\{ (\varphi, \psi, \phi) \in E_{k, m}^{y, z} : \|(\varphi, \psi, \phi)\| \leq \varepsilon^{\frac{3}{2} + \theta_1 - \tau} + (\lambda_2 + \lambda_3) \varepsilon^{\frac{3}{2} - \tau} + \varepsilon^{\frac{3}{2}} \left(\sum_{i=1}^k (|P_1(y^i) - P_1(x_0^1)|^{1-\tau}) \right. \right. \\ \left. \left. + \sum_{i=1}^k (|P_2(z^i) - P_2(x_0^1)|^{1-\tau}) + \sum_{i=1}^m (|P_3(z^i) - P_3(x_0^2)|^{1-\tau}) \right) \right. \\ \left. + \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} (1-\tau) |y^i - y^j|}{\varepsilon}} + \varepsilon^{\frac{3}{2}} \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2} (1-\tau) |z^i - z^j|}{\varepsilon}} \right\}. \end{aligned}$$

Now, we verify that A is a contraction from D to D . Indeed, for any $(\varphi, \psi, \phi) \in D$, by Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}
\|A(\varphi, \psi, \phi)\| &\leq C(\|\bar{\ell}_\varepsilon\| + \|R'_\varepsilon\|) \leq C(\|\bar{\ell}_\varepsilon\| + \varepsilon^{-\frac{3}{2}}\|(\varphi, \psi, \phi)\|^2) \\
&\leq C\left(\varepsilon^{\frac{3}{2}+\theta_1} + (\lambda_2 + \lambda_3)\varepsilon^{\frac{3}{2}-\tau} + \varepsilon^{\frac{3}{2}}\left(\sum_{i=1}^k |(P_1(y^i) - P_1(x_0^1))|^{2(1-\tau)}\right.\right. \\
&\quad \left.+\sum_{i=1}^k |(P_2(y^i) - P_2(x_0^1))|^{2(1-\tau)} + \sum_{i=1}^m |(P_3(z^i) - P_3(x_0^2))|^{2(1-\tau)}\right. \\
&\quad \left.+\varepsilon^{\frac{3}{2}}\sum_{i\neq j}^k e^{-\frac{2\sqrt{\lambda_1-\lambda_1}(1-\tau)|y^i-y^j|}{\varepsilon}} + \varepsilon^{\frac{3}{2}}\sum_{i\neq j}^m e^{-\frac{2\sqrt{\lambda_2}(1-\tau)|z^i-z^j|}{\varepsilon}}\right) \\
&\leq \varepsilon^{\frac{3}{2}+\theta_1-\tau} + (\lambda_2 + \lambda_3)\varepsilon^{\frac{3}{2}-\tau} + \varepsilon^{\frac{3}{2}}\left(\sum_{i=1}^k |(P_1(y^i) - P_1(x_0^1))|^{1-\tau}\right. \\
&\quad \left.+\sum_{i=1}^k |(P_2(y^i) - P_2(x_0^1))|^{1-\tau} + \sum_{i=1}^m |(P_3(z^i) - P_3(x_0^2))|^{1-\tau}\right) \\
&\quad +\varepsilon^{\frac{3}{2}}\sum_{i\neq j}^k e^{-\frac{\sqrt{\lambda_1-\lambda_1}(1-\tau)|y^i-y^j|}{\varepsilon}} + \varepsilon^{\frac{3}{2}}\sum_{i\neq j}^m e^{-\frac{\sqrt{\lambda_2}(1-\tau)|z^i-z^j|}{\varepsilon}},
\end{aligned}$$

and we get A maps from D to D . On the other hand, for $(\varphi_1, \psi_1, \phi_1), (\varphi_2, \psi_2, \phi_2) \in D$ and ε sufficiently small, we have

$$\begin{aligned}
\|A(\varphi_1, \psi_1, \phi_1) - A(\varphi_2, \psi_2, \phi_2)\| &= \|L_\varepsilon^{-1}R'_\varepsilon(\varphi_1, \psi_1, \phi_1) - L_\varepsilon^{-1}R'_\varepsilon(\varphi_2, \psi_2, \phi_2)\| \\
&\leq C\|R''_\varepsilon((\varphi_1, \psi_1, \phi_1) + s(\varphi_2, \psi_2, \phi_2))\| \|(\varphi_1, \psi_1, \phi_1) - (\varphi_2, \psi_2, \phi_2)\| \\
&\leq C\left(\varepsilon^{-\frac{3}{2}}\|(\varphi_1, \psi_1, \phi_1)\| + \varepsilon^{-3}\|(\varphi_1, \psi_1, \phi_1)\|^2\right) \|(\varphi_1, \psi_1, \phi_1) - (\varphi_2, \psi_2, \phi_2)\| \\
&\leq \frac{1}{2}\|(\varphi_1, \psi_1, \phi_1) - (\varphi_2, \psi_2, \phi_2)\|.
\end{aligned}$$

Therefore, A is a contraction map from D to D . By the contraction mapping theorem, there exist $(\varphi, \psi, \phi) \in E_{k,m}^{y,z}$ such that $A(\varphi, \psi, \phi) = (\varphi, \psi, \phi)$.

Moreover,

$$\begin{aligned}
\|(\varphi, \psi, \phi)\| &\leq C\left(\varepsilon^{\frac{3}{2}+\theta_1} + (\lambda_2 + \lambda_3)\varepsilon^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}}\left(\sum_{i=1}^k |P_1(y^i) - P_1(x_0^1)| + \sum_{i=1}^k |P_2(y^i) - P_2(x_0^1)|\right.\right. \\
&\quad \left.+\sum_{i=1}^m |P_3(z^i) - P_3(x_0^2)|\right) + \varepsilon^{\frac{3}{2}}\sum_{i\neq j}^k e^{-\frac{\sqrt{\lambda_1-\lambda_1}|y^i-y^j|}{\varepsilon}} + \varepsilon^{\frac{3}{2}}\sum_{i\neq j}^m e^{-\frac{\sqrt{\lambda_2}|z^i-z^j|}{\varepsilon}}.
\end{aligned}$$

□

Proof of Theorem 1.3. It follows from Lemma 4.1 and Propositions 4.2 and A.2 that

$$\begin{aligned}
F(y, z) &= J_\varepsilon(\varphi(y, z), \psi(y, z), \phi(y, z)) \\
&= I_\varepsilon(U_{\varepsilon, y} + \varphi, V_{\varepsilon, y} + \psi, W_{\varepsilon, z} + \phi) \\
&= J_\varepsilon(0, 0, 0) + \ell_\varepsilon(\varphi, \psi, \phi) + \frac{1}{2}\langle L_\varepsilon(\varphi, \psi, \phi), (\varphi, \psi, \phi) \rangle + R_\varepsilon(\varphi, \psi, \phi) \\
&= I_\varepsilon(U_{\varepsilon, y}, V_{\varepsilon, y}, W_{\varepsilon, z}) + O(\|\ell_\varepsilon\| \|(\varphi, \psi, \phi)\| + \|(\varphi, \psi, \phi)\|^2) \\
&= \frac{1}{4}A\varepsilon^3 - \varepsilon^3\left(C_1\sum_{i=1}^k (P_1(x_0^1) - P_1(y^i)) + C_1\sum_{i=1}^k (P_2(x_0^1) - P_2(y^i))\right. \\
&\quad \left.+ C_2\sum_{i=1}^m (P_3(x_0^2) - P_3(z^i))\right) - \varepsilon^3C_3\sum_{i\neq j}^k e^{-\frac{\sqrt{\lambda_1-\lambda_1}|y^i-y^j|}{\varepsilon}} \\
&\quad - \varepsilon^3C_4\sum_{i\neq j}^m e^{-\frac{\sqrt{\lambda_2}|z^i-z^j|}{\varepsilon}} + O(\varepsilon^{3+\theta_1}),
\end{aligned}$$

since $\lambda_2, \lambda_3 \sim \varepsilon^\alpha, \alpha \geq \frac{\min\{\theta_1, 1\}}{2}$.

Consider the following maximization problem:

$$\max_{(y,z) \in D_{\varepsilon,\delta}^{k,y} \times D_{\varepsilon,\delta}^{m,z}} F(y,z). \quad (4.3)$$

Assume (4.3) is achieved by some $(y_\varepsilon, z_\varepsilon) \in D_{\varepsilon,\delta}^{k,y} \times D_{\varepsilon,\delta}^{m,z}$, next we claim that $(y_\varepsilon, z_\varepsilon)$ is an interior point of $D_{\varepsilon,\delta}^{k,y} \times D_{\varepsilon,\delta}^{m,z}$.

Let us choose $\bar{y}^j = x_0^1 + L_1 \varepsilon |\ln \varepsilon| e_j$, $\bar{z}^j = x_0^2 + L_2 \varepsilon |\ln \varepsilon| e_j$ for $L_1, L_2 > 0$, and some vectors $e_1, e_2, \dots, e_n$, with $|e_i - e_j| = 1$ for $i \neq j$. Then, $\frac{|\bar{y}^i - \bar{y}^j|}{\varepsilon} = L_1 |\ln \varepsilon|$, $\frac{|\bar{z}^i - \bar{z}^j|}{\varepsilon} = L_2 |\ln \varepsilon|$, which means that $\bar{y} := (\bar{y}^1, \dots, \bar{y}^k) \in D_{\varepsilon,\delta}^{k,y}$, $\bar{z} := (\bar{z}^1, \dots, \bar{z}^m) \in D_{\varepsilon,\delta}^{m,z}$ for ε sufficiently small.

Thus, for L_1, L_2 large enough, we have

$$\begin{aligned} \frac{1}{4} A \varepsilon^3 - C \varepsilon^3 (\varepsilon |\ln \varepsilon|)^{\min\{\theta_1, 1\}} &\leq F(\bar{y}, \bar{z}) \leq F(y_\varepsilon, z_\varepsilon) \\ &\leq \frac{1}{4} A \varepsilon^3 - \varepsilon^3 \sum_{i=1}^k ((P_1(x_0^1) - P_1(y^i)) + (P_2(x_0^1) - P_2(y^i))) \\ &\quad - \varepsilon^3 \sum_{i=1}^m ((P_3(x_0^2) - P_3(z^i))) - \varepsilon^3 \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} \\ &\quad - \varepsilon^3 \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2} |z^i - z^j|}{\varepsilon}}, \end{aligned}$$

that is,

$$\begin{aligned} &\sum_{i=1}^k ((P_1(x_0^1) - P_1(y^i)) + (P_2(x_0^1) - P_2(y^i))) + \sum_{i=1}^m (P_3(x_0^2) - P_3(z^i)) \\ &+ \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} + \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2} |z^i - z^j|}{\varepsilon}} \leq C (\varepsilon |\ln \varepsilon|)^{\min\{\theta_1, 1\}}. \end{aligned}$$

Therefore, we can deduce

$$\begin{aligned} \sum_{i=1}^k (P_1(x_0^1) - P_1(y^i)) &\leq C (\varepsilon |\ln \varepsilon|)^{\min\{\theta_1, 1\}}, \quad \sum_{i=1}^k (P_2(x_0^1) - P_2(y^i)) \leq C (\varepsilon |\ln \varepsilon|)^{\min\{\theta_1, 1\}}, \\ \sum_{i=1}^m (P_3(x_0^2) - P_3(z^i)) &\leq C (\varepsilon |\ln \varepsilon|)^{\min\{\theta_1, 1\}}, \\ \sum_{i \neq j}^k \frac{|y^i - y^j|}{\varepsilon} &\geq C |\ln \varepsilon| > |\ln \varepsilon|^{\frac{1}{2}}, \quad \sum_{i \neq j}^m \frac{|z^i - z^j|}{\varepsilon} \geq C |\ln \varepsilon| > |\ln \varepsilon|^{\frac{1}{2}}, \end{aligned}$$

which imply that $(y_\varepsilon, z_\varepsilon)$ is an interior point of $D_{\varepsilon,\delta}^{k,y} \times D_{\varepsilon,\delta}^{m,z}$. Then, the corresponding vector function $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a critical point of I_ε , where $u_\varepsilon = \sum_{i=1}^k U_{\varepsilon, y^i} + \varphi_\varepsilon$, $v_\varepsilon = \sum_{i=1}^k V_{\varepsilon, y^i} + \psi_\varepsilon$, $w_\varepsilon = \sum_{i=1}^m W_{\varepsilon, z^i} + \phi_\varepsilon$.

That $u_\varepsilon, v_\varepsilon, w_\varepsilon$ are all positive and can be proved following the argument in Ref. 22. We get $(y_\varepsilon, z_\varepsilon)$ which is an interior point $D_{\varepsilon,\delta}^{k,y} \times D_{\varepsilon,\delta}^{m,z}$. Hence, $(U_{\varepsilon, y} + \varphi_\varepsilon, V_{\varepsilon, y} + \psi_\varepsilon, W_{\varepsilon, y} + \phi_\varepsilon)$ is the solution of problem (1.4). $\square$

ACKNOWLEDGMENTS

The authors are grateful to the editors and two reviewers for their suggestions and comments. W. Long was partially supported by NSFC (No. 11501264) and the NSF of Jiangxi Province.

Q. Wang acknowledges support from the excellent doctoral dissertation cultivation grant from Central China Normal University (2013YBZD16).

The authors sincerely thank Professor Shuangjie Peng for helpful discussions and suggestions.

APPENDIX: ENERGY EXPANSIONS

In this section, we will expand the energies $I_\varepsilon(U_{\varepsilon,y}, V_{\varepsilon,y}, W_{\varepsilon,z})$, where

$$I_\varepsilon(u, v, w) = \frac{1}{2} \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u|^2 + P_1(x)u^2 + \varepsilon^2 |\nabla v|^2 + P_2(x)v^2 + \varepsilon^2 |\nabla w|^2 + P_3(x)w^2) \\ - \frac{1}{4} \int_{\mathbb{R}^3} (u^4 + v^4 + w^4) - \lambda_1 \int_{\mathbb{R}^3} uv - \lambda_2 \int_{\mathbb{R}^3} uw - \lambda_3 \int_{\mathbb{R}^3} vw.$$

Proposition A.1 (see Ref. 5). Suppose that $u(x), v(x) \in H_r^1(\mathbb{R}^N)$ satisfy

$$u(r) \sim r^\alpha e^{-\beta r}, \quad v(r) \sim r^\gamma e^{-\eta r} \quad (r \rightarrow +\infty),$$

where $\alpha, \gamma \in \mathbb{R}$, $\beta > 0$, $\eta > 0$. Let $y \in \mathbb{R}^N$, with $|y| \rightarrow +\infty$. We have the following:

(i) if $\beta < \eta$, then

$$\int_{\mathbb{R}^3} u_y v \sim |y|^\alpha e^{-\beta|y|}.$$

(ii) If $\beta = \eta$, suppose for simplicity that $\alpha \geq \gamma$, then,

$$\int_{\mathbb{R}^3} u_y v \sim \begin{cases} e^{-\beta|y|} |y|^{\alpha+\gamma+\frac{1+N}{2}}, & \text{if } \gamma > -\frac{1+N}{2}, \\ e^{-\beta|y|} |y|^\alpha \ln |y|, & \text{if } \gamma = -\frac{N+1}{2}, \\ e^{-\beta|y|} |y|^\alpha, & \text{if } \gamma < -\frac{N+1}{2}. \end{cases} \quad (\text{A1})$$

Proposition A.2. There is a small constant $\sigma > 0$ and positive constants C_1, C_2, C_3, C_4 such that

$$I_\varepsilon(U_{\varepsilon,y}(x), V_{\varepsilon,y}(x), W_{\varepsilon,z}(x)) = \frac{1}{4} A \varepsilon^3 - \varepsilon^3 \left(C_1 \sum_{i=1}^k (P_1(x_0^1) - P_1(y^i)) \right. \\ + C_1 \sum_{i=1}^k (P_2(x_0^1) - P_2(y^i)) + C_2 \sum_{i=1}^m (P_3(x_0^2) - P_3(z^i)) \\ \left. - \varepsilon^3 C_3 \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} - \varepsilon^3 C_4 \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2} |z^i - z^j|}{\varepsilon}} \right) \\ + O(\varepsilon^{3+\theta_1}), \quad (\text{A2})$$

where $A = (2k + m) \int_{\mathbb{R}^3} W^4$.

Proof. By the definition of I_ε , we get

$$I_\varepsilon(U_{\varepsilon,y}, V_{\varepsilon,y}, W_{\varepsilon,z}) = \frac{1}{2} \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla U_{\varepsilon,y}|^2 + P_1(x)U_{\varepsilon,y}^2 + \varepsilon^2 |\nabla V_{\varepsilon,y}|^2 + P_2(x)V_{\varepsilon,y}^2 \\ + \varepsilon^2 |\nabla W_{\varepsilon,z}|^2 + P_3(x)W_{\varepsilon,z}^2) - \frac{1}{4} \int_{\mathbb{R}^3} (U_{\varepsilon,y}^4 + V_{\varepsilon,y}^4 + W_{\varepsilon,z}^4) \\ - \lambda_1 \int_{\mathbb{R}^3} U_{\varepsilon,y} V_{\varepsilon,y} - \lambda_2 \int_{\mathbb{R}^3} U_{\varepsilon,y} W_{\varepsilon,z} - \lambda_3 \int_{\mathbb{R}^3} V_{\varepsilon,y} W_{\varepsilon,z}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{\mathbb{R}^3} \left(\sum_{i=1}^k (P_1(x) - P_1(x_0^1)) U_{\varepsilon, y^i}^2 + \sum_{i \neq j}^k (P_1(x) - P_1(x_0^1)) U_{\varepsilon, y^i} U_{\varepsilon, y^j} \right. \\
&\quad + \sum_{i=1}^k (P_2(x) - P_2(x_0^1)) V_{\varepsilon, y^i}^2 + \sum_{i \neq j}^k (P_2(x) - P_2(x_0^1)) V_{\varepsilon, y^i} V_{\varepsilon, y^j} \\
&\quad + \sum_{i=1}^m (P_3(x) - P_3(x_0^2)) W_{\varepsilon, z^i}^2 + \sum_{i \neq j}^m (P_3(x) - P_3(x_0^2)) W_{\varepsilon, z^i} W_{\varepsilon, z^j} \Big) \\
&\quad - \frac{1}{4} \int_{\mathbb{R}^3} \left(\left(\sum_{i=1}^k U_{\varepsilon, y^i} \right)^4 - \sum_{i=1}^k U_{\varepsilon, y^i}^4 - 2 \sum_{i \neq j}^k U_{\varepsilon, y^i}^3 U_{\varepsilon, y^j} \right) \\
&\quad - \frac{1}{4} \int_{\mathbb{R}^3} \left(\left(\sum_{i=1}^k V_{\varepsilon, y^i} \right)^4 - \sum_{i=1}^k V_{\varepsilon, y^i}^4 - 2 \sum_{i \neq j}^k V_{\varepsilon, y^i}^3 V_{\varepsilon, y^j} \right) \\
&\quad - \frac{1}{4} \int_{\mathbb{R}^3} \left(\left(\sum_{i=1}^m W_{\varepsilon, z^i} \right)^4 - \sum_{i=1}^m W_{\varepsilon, z^i}^4 - 2 \sum_{i \neq j}^m W_{\varepsilon, z^i}^3 W_{\varepsilon, z^j} \right) \\
&\quad + \frac{1}{4} \int_{\mathbb{R}^3} \sum_{i=1}^k (U_{\varepsilon, y^i}^4 + \sum_{i=1}^k V_{\varepsilon, y^i}^4 + \sum_{i=1}^m W_{\varepsilon, z^i}^4) \\
&\quad - \lambda_2 \int_{\mathbb{R}^3} U_{\varepsilon, y} W_{\varepsilon, z} - \lambda_3 \int_{\mathbb{R}^3} V_{\varepsilon, y} W_{\varepsilon, z}. \tag{A3}
\end{aligned}$$

By direct computation, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \sum_{i=1}^k (P_1(x) - P_1(x_0^1)) U_{\varepsilon, y^i}^2 + \sum_{i \neq j}^k (P_1(x) - P_1(x_0^1)) U_{\varepsilon, y^i} U_{\varepsilon, y^j} \\
&= \int_{\mathbb{R}^3} \left(\sum_{i=1}^k (P_1(x) - P_1(y^i) + P_1(y^i) - P_1(x_0^1)) U_{\varepsilon, y^i}^2 \right. \\
&\quad \left. + \sum_{i \neq j}^k (P_1(x) - P_1(y^i) + P_1(y^i) - P_1(x_0^1)) U_{\varepsilon, y^i} U_{\varepsilon, y^j} \right) \\
&= -\varepsilon^3 C_1 \sum_{i=1}^k (P_1(x_0^1) - P_1(y^i)) + O(\varepsilon^{3+\theta_1}). \tag{A4}
\end{aligned}$$

Similar to (A4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\sum_{i=1}^k (P_2(x) - P_2(x_0^1)) V_{\varepsilon, y^i}^2 + \sum_{i \neq j}^k (P_2(x) - P_2(x_0^1)) V_{\varepsilon, y^i} V_{\varepsilon, y^j} \right) \\
&= -\varepsilon^3 C_1 \sum_{i=1}^k (P_2(x_0^1) - P_2(y^i)) + O(\varepsilon^{3+\theta_2}) \tag{A5}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\sum_{i=1}^m (P_3(x) - P_3(x_0^2)) W_{\varepsilon, z^i}^2 + \sum_{i \neq j}^m (P_3(x) - P_3(x_0^2)) W_{\varepsilon, z^i} W_{\varepsilon, z^j} \right) \\
&= -\varepsilon^3 C_1 \sum_{i=1}^m (P_3(x_0^2) - P_3(z^i)) + O(\varepsilon^{3+\theta_3}). \tag{A6}
\end{aligned}$$

With elementary computations, we have

$$\begin{aligned}
 & \int_{\mathbb{R}^3} \left(\sum_{i=1}^k U_{\varepsilon, y^i} \right)^4 - \sum_{i=1}^k U_{\varepsilon, y^i}^4 - 2 \sum_{i \neq j}^k U_{\varepsilon, y^i}^3 U_{\varepsilon, y^j} \\
 &= 2 \int_{\mathbb{R}^3} \sum_{i \neq j}^k U_{\varepsilon, y^i}^3 U_{\varepsilon, y^j} + O \left(\int_{\mathbb{R}^3} \sum_{i \neq j}^k U_{\varepsilon, y^i}^2 U_{\varepsilon, y^j}^2 \right) \\
 &= \varepsilon^3 C_2 \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} + O \left(\varepsilon^3 \sum_{i \neq j}^k e^{-\frac{2\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} \left| \frac{\varepsilon}{|y^i - y^j|} \right|^2 \ln |y^i - y^j| \right) \\
 &= \varepsilon^3 C_2 \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} + O(\varepsilon^{3+\theta_1}). \tag{A7}
 \end{aligned}$$

Similar to (A7), we obtain

$$\int_{\mathbb{R}^3} \left(\sum_{i=1}^k V_{\varepsilon, y^i} \right)^4 - \sum_{i=1}^k V_{\varepsilon, y^i}^4 - 2 \sum_{i \neq j}^k V_{\varepsilon, y^i}^3 V_{\varepsilon, y^j} = \varepsilon^3 C_2 \sum_{i \neq j}^k e^{-\frac{\sqrt{a_1 - \lambda_1} |y^i - y^j|}{\varepsilon}} + O(\varepsilon^{3+\theta_2}), \tag{A8}$$

$$\int_{\mathbb{R}^3} \left(\sum_{i=1}^m W_{\varepsilon, z^i} \right)^4 - \sum_{i=1}^m W_{\varepsilon, z^i}^4 - 2 \sum_{i \neq j}^m W_{\varepsilon, z^i}^3 W_{\varepsilon, z^j} = \varepsilon^3 C_2 \sum_{i \neq j}^m e^{-\frac{\sqrt{a_2} |z^i - z^j|}{\varepsilon}} + O(\varepsilon^{3+\theta_3}), \tag{A9}$$

$$\int_{\mathbb{R}^3} U_{\varepsilon, y^i} W_{\varepsilon, z^j} = C \varepsilon^3 e^{-\frac{\min\{\sqrt{a_1 - \lambda_1}, \sqrt{a_2}\} |y^i - z^j|}{\varepsilon}} = O(\varepsilon^{3+\theta_1}). \tag{A10}$$

Inserting (A4)–(A10) into (A3), we get (A2). $\square$

- ¹ Akhmediev, N. and Ankiewicz, A., “Novel soliton states and bifurcation phenomena in nonlinear fiber couplers,” *Phys. Rev. Lett.* **70**, 2395-2398 (1993).
- ² Ambrosetti, A., Cerami, G., and Ruiz, D., “Solitons of linearly coupled systems of semilinear non-autonomous equations on $\mathbb{R}^n$,” *J. Funct. Anal.* **254**, 2816-2845 (2008).
- ³ Ambrosetti, A. and Colorado, E., “Bound and ground states of coupled nonlinear Schrödinger equations,” *C. R. Math. Acad. Sci. Paris* **342**, 453-458 (2006).
- ⁴ Ambrosetti, A. and Colorado, E., “Standing waves of some coupled nonlinear Schrödinger equations,” *J. London Math. Soc.* **75**, 67-82 (2007).
- ⁵ Ambrosetti, A., Colorado, E., and Ruiz, D., “Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations,” *Calculus Var. Partial Differ. Equations* **30**, 85-112 (2007).
- ⁶ Bartsch, T., Dancer, N., and Wang, Z.-Q., “A Liouville theorem, a priori bounds and bifurcating branches of positive solutions for a nonlinear elliptic system,” *Calculus Var. Partial Differ. Equations* **37**, 345-361 (2010).
- ⁷ Bartsch, T., Wang, Z.-Q., and Wei, J., “Bound states for a coupled Schrödinger system,” *J. Fixed Point Theory Appl.* **2**, 353-367 (2007).
- ⁸ Byeon, J. and Jeanjean, L., “Standing waves for nonlinear Schrödinger equations with a general nonlinearity,” *Arch. Ration. Mech. Anal.* **185**, 185-200 (2007).
- ⁹ Byeon, J. and Wang, Z.-Q., “Standing waves with a critical frequency for nonlinear Schrödinger equations II,” *Calculus Var. Partial Differ. Equations* **18**, 207-219 (2003).
- ¹⁰ Cao, D., Noussair, E. S., and Yan, S., “Solutions with multiple peaks for nonlinear elliptic equations,” *Proc. -R. Soc. Edinburgh, Sect. A: Math.* **129**, 235-264 (1999).
- ¹¹ Cao, D. and Peng, S., “Semi-classical bound states for Schrödinger equations with potentials vanishing or unbounded at infinity,” *Commun. Partial Differ. Equations* **34**, 1566-1591 (2009).
- ¹² Chang, S., Lin, C. S., Lin, T. C., and Lin, W., “Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates,” *Physica D* **196**, 341-361 (2004).
- ¹³ Chen, Z. and Zou, W., “Standing waves for linearly coupled Schrödinger equations with critical exponent,” *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31**, 429-447 (2014).
- ¹⁴ Dancer, E. N., Wei, J., and Weth, T., “A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system,” *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27**, 953-969 (2010).
- ¹⁵ de Figueiredo, D. G. and Lopes, O., “Solitary waves for some nonlinear Schrödinger systems,” *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25**, 149-161 (2008).
- ¹⁶ Del Pino, M. and Felmer, P., “Local mountain passes for semilinear elliptic problems in unbounded domains,” *Calculus Var. Partial Differ. Equations* **4**, 121-137 (1996).
- ¹⁷ Del Pino, M. and Felmer, P., “Semi-classical states for nonlinear Schrödinger equations,” *J. Funct. Anal.* **149**, 245-265 (1997).

- ¹⁸ Lin, C. S. and Peng, S., "Segregated vector solutions for linearly coupled nonlinear Schrödinger systems," *Indiana Univ. Math. J.* **63**, 939-967 (2014).
- ¹⁹ Lin, T. C. and Wei, J., "Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$," *Commun. Math. Phys.* **255**, 629-653 (2005).
- ²⁰ Lin, T. C. and Wei, J., "Solitary and self-similar solutions of two-component system of nonlinear Schrödinger equations," *Physica D* **220**, 99-115 (2006).
- ²¹ Liu, Z. and Wang, Z.-Q., "Ground states and bound states of a nonlinear Schrödinger system," *Adv. Nonlinear Stud.* **10**, 175-193 (2010).
- ²² Long, W. and Peng, S., "Segregated vector solutions for a class of Bose-Einstein systems," *J. Differ. Equations* **257**, 207-230 (2014).
- ²³ Montefusco, E., Pellacci, B., and Squassina, M., "Semiclassical states for weakly coupled nonlinear Schrödinger systems," *J. Eur. Math. Soc.* **10**, 41-71 (2008).
- ²⁴ Noris, B., Tavares, H., Terracini, S., and Verzini, G., "Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition," *Commun. Pure Appl. Math.* **63**, 267-302 (2010).
- ²⁵ Oh, Y. G., "On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential," *Commun. Math. Phys.* **131**, 223-253 (1990).
- ²⁶ Peng, S. and Wang, Z.-Q., "Segregated and synchronized vector solutions for nonlinear Schrödinger systems," *Arch. Ration. Mech. Anal.* **208**, 305-339 (2013).
- ²⁷ Rabinowitz, P. H., "On a class of nonlinear Schrödinger equations," *Z. Angew. Math. Phys.* **43**, 270-291 (1992).
- ²⁸ Sirakov, B., "Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$," *Commun. Math. Phys.* **271**, 199-221 (2007).
- ²⁹ Terracini, S. and Verzini, G., "Multipulse Phase in k -mixtures of Bose-Einstein condensates," *Arch. Ration. Mech. Anal.* **194**, 717-741 (2009).
- ³⁰ Tian, R. and Wang, Z.-Q., "Multiple solitary wave solutions of nonlinear Schrödinger systems," *Topol. Methods Nonlinear Anal.* **37**, 203-223 (2011).
- ³¹ Wei, J. and Weth, T., "Nonradial symmetric bound states for a system of two coupled Schrödinger equations," *Rend. Lincei - Mat. Appl.* **18**, 279-293 (2007).
- ³² Wei, J. and Weth, T., "Radial solutions and phase separation in a system of two coupled Schrödinger equations," *Arch. Ration. Mech. Anal.* **190**, 83-106 (2008).
- ³³ Wei, J. and Yan, S., "Infinitely many solutions for the nonlinear Schrödinger equations in $\mathbb{R}^n$," *Calculus Var. Partial Differ. Equations* **37**, 423-439 (2010).
- ³⁴ Wei, J. and Yan, S., "Infinitely many solutions for the prescribed scalar curvature problem on S^n ," *J. Funct. Anal.* **258**, 3048-3081 (2010).
- ³⁵ Wei, J. and Yao, W., "Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations," *Commun. Pure Appl. Anal.* **11**, 1003-1011 (2012).