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Paradoxical Effect of the Signal-to-Noise Ratio of GRAPPA Calibration Lines: A Quantitative Study

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Abstract

Purpose—Intuitively, GRAPPA auto-calibration signal (ACS) lines with higher signal-to-noise ratio (SNR) may be expected to boost the accuracy of kernel estimation and increase the SNR of GRAPPA reconstructed images. Paradoxically, Sodickson and his colleagues pointed out that using ACS lines with high SNR may actually lead to lower SNR in the GRAPPA reconstructed images. A quantitative study of how the noise in the ACS lines affects the SNR of the GRAPPA reconstructed images is presented.

Methods—In a phantom, the singular values of the GRAPPA encoding matrix and the root-mean-square error of GRAPPA reconstruction were evaluated using multiple sets of ACS lines with variant SNR. In volunteers, ACS lines with high and low SNR were estimated, and the SNR of corresponding TGRAPPA reconstructed images was evaluated.

Results—We show that the condition number of the GRAPPA kernel estimation equations is proportional to the SNR of the ACS lines. In dynamic image series reconstructed with TGRAPPA, high SNR ACS lines result in reduced SNR if appropriate regularization is not applied.

Conclusion—Noise has a similar effect to Tikhonov regularization. Without appropriate regularization, a GRAPPA kernel estimated from ACS lines with higher SNR amplifies random noise in the GRAPPA reconstruction.

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Additional Supporting Information may be found in the online version of this article.

Keywords

GRAPPA; condition number; signal to noise ratio; real-time MRI

INTRODUCTION

GRAPPA, a k-space based parallel MRI (pMRI) technique (1), is widely used clinically for dynamic cardiac imaging because of its robustness (1,2). In GRAPPA, missing k-space data in each channel are estimated by convolving the acquired k-space points with GRAPPA convolution kernels, which are estimated from fully sampled k-space auto-calibration signal (ACS) lines using linear regression.

There are two widely used methods to acquire the ACS lines needed for GRAPPA reconstruction. The first method uses a fully sampled center of k-space block as the ACS lines in each image or temporal frame (1); the second method uses a single, separately acquired, fully sampled k-space as the ACS lines for multiple temporal image frames. The first method requires sampling of additional lines for every image frame, and therefore lowers the effective acceleration and efficiency; the second method is prone to motion artifacts and requires longer scan time. TGRAPPA (3), an important variant of the GRAPPA technique, is commonly used in dynamic imaging applications. Multiple frames of uniformly down-sampled, temporally interleaved k-space data are acquired, and the ACS lines are estimated from temporally low-pass filtered frames. In practice, the temporal average of *all* acquired k-space frames is frequently used as the ACS lines; this approach results in ACS lines with potentially very high SNR.

Intuitively, one would expect ACS lines with higher SNR to boost the accuracy of the kernel estimation, and increase the SNR of GRAPPA reconstructed images. Paradoxically, Sodickson and his colleagues pointed out that higher SNR ACS lines may lead to lower SNR in GRAPPA reconstructed images (4,5). They hypothesized that a higher SNR in the ACS lines increases the condition number of the GRAPPA kernel estimation equations (4,5). If the condition number is too high, the GRAPPA kernel estimation equations become ill-conditioned, the estimated GRAPPA kernel becomes corrupted by amplified random noise, and the SNR in the corresponding GRAPPA reconstructed images is reduced. Tikhonov regularization is a commonly used technique to regularize the ill-conditioning problem (6), and has been used to stabilize the GRAPPA kernel estimation (7). Here, we take one step further, and present a quantitative study of how the noise in the ACS lines has a regularizing effect. Similar beneficial effects of noise has been observed in the training of neural networks (8).

To illustrate the impact of ACS SNR on image quality, we compare the common approach of ACS line generation by averaging all frames (AAF) with one of tiling-all-frames (TAF) of locally averaged k-space; TAF results in lower SNR ACS lines than the common AAF approach. We demonstrate that the TAF method has a higher inherent regularizing effect due to the lower SNR of the ACS lines; this in turn enhances the SNR of the reconstructed images even when Tikhonov regularization is not used in GRAPPA kernel estimation.

In the remainder of this article, we briefly review the GRAPPA technique, we test the relation between SNR and condition number in a phantom study; then demonstrate how changing the strategy to generate ACS lines affects the SNR of TGRAPPA; at last, we discuss the relation between ACS lines SNR and Tikhonov regularization and summarize the study.

METHODS

Theory

The GRAPPA method reconstructs missing k-space points using a linear combination of neighborhood acquired k-space points from all channels. The linear combination coefficients comprise a k-space interpolation kernel, i.e., (9)

$$\sum_{h=-H_l}^{H_r} \sum_{b=-N_b}^{N_a} \sum_{l=0}^{n_c-1} S_1(k_x + h\Delta k_x, k_y + bR\Delta k_y) G_{j,r}(l, b, h) = S_j(k_x, k_y + r\Delta k_y) \quad [1]$$

where R is the acceleration rate, $[-H_l, H_r]$ and $[-N_b, N_a]$ represent the k-space spans of the GRAPPA kernel, n_c is the number of channels; j represents j -th channel; S_j is the k-space of j -th channel; $G_{j,r}$ is the GRAPPA kernel for j -th channel and r -th missing k-space line, $r = 1, \dots, R-1$; and $n_c(R-1)$ is the total number of GRAPPA kernels.

The GRAPPA method uses the channel sensitivity implicitly, i.e., the GRAPPA kernel exploits k-space local correlation across multiple channels. In addition, the GRAPPA kernel is k-space translationally invariant. The GRAPPA kernel can be estimated using fully sampled k-space as ACS lines, and Eq. [1] can be written as a linear regression equation:

$$AG = b \quad [2]$$

where $m \times n$ matrix A is the GRAPPA encoding matrix. Each row of matrix A represents a GRAPPA sliding window at a particular location in k-space, which has the same number of elements as the GRAPPA kernel size, e.g., for a 12-channel coil and a 4×5 kernel ($n_c = 12$, $n_y = 4$, and $n_x = 5$), the GRAPPA kernel size is $12 \times 4 \times 5 = 240$, i.e., $n = 240$. b represents the GRAPPA reconstructed k-space points. Both entries of A and b are k-space points of the ACS lines. m is the number of reconstructed k-space points/sliding window/number of equations, which is always less than or equal to the total number of k-space points in the ACS lines. For a typical parallel imaging acquisition, it is required to have sufficient calibration data to ensure $m \gg n$; thus Eq. [2] is over-determined, i.e., the number of rows of matrix A is always larger than the number of columns.

Equation [1] can be solved by inverting matrix A , and all $n_c(R-1)$ GRAPPA kernels can be estimated simultaneously. The singular value decomposition (SVD) method:

$$G = V(\Sigma)^+ U^H b \quad [3]$$

and the Moore–Penrose pseudoinverse method:

$$G = (A^H A)^{-1} A^H b \quad [4]$$

are two widely used solution methods. According to SVD, $A = U \Sigma V^H$, where matrices V and U are unitary matrices, and Σ is a diagonal matrix with the nonnegative singular values on the diagonal. The condition number of matrix A , $\kappa(A)$, is defined as $\kappa(A) = \sigma(A)_{\max}/\sigma(A)_{\min}$ where $\sigma(A)_{\max}$ and $\sigma(A)_{\min}$ are the largest and smallest singular values of matrix A . Usually, the condition number of Eq. [1] is high (>10), especially when a large kernel or more channels are used. Therefore, Eqs. [2], [3], and [4] are usually ill-conditioned. We also note that Eq. [3] is computationally more demanding, but more accurate than Eq. [4]. To save computational time, our GRAPPA implementation uses Eq. [4].

Tikhonov regularization is the most commonly used regularization method to improve the conditioning of the linear system. Tikhonov regularization has been proposed to stabilize the GRAPPA kernel estimation (7). With Tikhonov regularization, the solution in Eq. [4] become:

$$G = (A^H A + \lambda^2 I_{n \times n})^{-1} A^H b \quad [5]$$

where parameter λ is the regularization strength. Because the elements of matrix A are the k-space points of ACS lines, then matrix A can be regarded as the summation of a signal matrix A_S and a noise matrix A_N , i.e., $A = A_S + A_N$. When $m \gg n \gg 1$, under certain assumptions Eq. [4] asymptotically becomes (4,10):

$$G \approx (A_S^H A_S + \sigma_n^2 I_{n \times n})^{-1} A^H b. \quad [6]$$

See Appendix for more details.

If matrix A_S is rank deficient, then its smallest singular value is zero; the smallest singular value of matrix A is determined by the noise standard deviation, given A_N has independent and identically distributed (IID) entries. More details on the singular value distribution of random matrices with IID entries can be found in (11–13). Obviously, matrix A has some structure caused by the repeated entries, and thus the noise in matrix A_N is not IID. Our numerical studies included in the Appendix show that the singular value distribution of matrix A_N is observed to follow the prediction of the random matrix theory, and neglected terms in Eq. [6] asymptotically go to zero when $m \gg n \gg 1$. Therefore, even if this IID

assumption is violated, Eq. [6] still remains a reasonable asymptotic representation of Eq. [4].

The similarity of Equations [5] and [6] suggests that noise of the ACS lines may have an effect similar to Tikhonov regularization (4). We hypothesize that the largest singular value of A_S is proportional to the signal strength of the ACS lines. If this is true, then the condition number of A is proportional to the SNR of the ACS lines. That is, GRAPPA kernel estimation equations from ACS lines with high SNR are ill-conditioned, and the corresponding GRAPPA kernel is expected to result in images with elevated random noise. Conversely, due to the regularizing effects of noise, ACS lines with relatively low SNR are expected to result in images with higher SNR, but potentially also a higher level of artifacts. These are the same well-known trade-offs that come with adjusting regularization strength in Eq. [5].

In this study, we explore the qualitative relation between the SNR of ACS lines and the condition number of the GRAPPA estimation equations. We experimentally test the following three hypotheses: (i) The largest and smallest singular values of the GRAPPA encoding matrix are proportional to the signal and noise of the ACS lines, respectively, and the condition number of the GRAPPA encoding matrix is proportional to the SNR of the ACS lines. As long as $m \gg n \gg 1$, which is the case when moderate acceleration rates are used (this assumption is also required for hypothesis (ii) and (iii)), the largest and smallest singular values and the condition number of the GRAPPA encoding matrix are only weakly dependent on the acceleration rate. (ii) the smallest singular value of the GRAPPA encoding matrix is close to the noise standard deviation over a wide range of GRAPPA kernel sizes. (iii) The root-mean-square-error (RMSE) of the GRAPPA reconstruction versus SNR of ACS lines is approximately a U-shaped function with minima at moderate SNR. Finally, we use in vivo cardiac images to demonstrate how the SNR of images reconstructed using TGRAPPA is affected by strategies that impact the SNR of the ACS lines.

Phantom and Volunteer Study

Phantom and volunteer studies were performed on a 1.5 Tesla (T) MR scanner (MAGNETOM Avanto, Siemens Healthcare, Germany) using a balanced steady state free precession (SSFP) sequence. Both a Siemens standard 12-channel body matrix array coil and a 32-channel cardiac array coil (RapidMRI, Columbus, OH) were used for data acquisition in the phantom study to test the dependency on the number of channels. A different 32-channel cardiac array coil (Quality Electrodynamics, Mayfield Village, OH) was used for data acquisition in volunteer studies. The GRAPPA reconstruction method was implemented and all data was processed using Matlab[®] 2011a (MathWorks, Natick, Massachusetts) running on a personal computer with Intel[®] Core(TM)2 Quad 3.0 GHz CPU, 16 GB system RAM. The volunteer study was approved by our institution's Human Subjects Committee and all subjects gave written informed consent to participate.

We acquired multiple image series with variant signal level by adjusting the flip angle in the SSFP sequence from 1° to 70°. It has been shown that the signal of the SSFP sequence is proportional to the flip angle over a wide range (14). The noise level is dominated by Johnson noise and is determined by the receiver bandwidth and the resistance in the receiver

circuit (15), independent of the sequence parameters. Therefore, the noise level is independent of flip angle and constant in all acquired image series. Noise level was varied separately by varying the number of frames averaged.

Fourteen images series each with 86 frames were acquired with flip angles varying from 1° to 70° . All k-space was fully sampled. Twenty different sets of ACS lines with varying SNR were prepared. The first 14 sets of ACS lines were taken from the first frame of each image series acquired with different flip angles, thereby varying the SNR by varying the signal only. The last six sets of ACS lines were prepared by averaging 2, 4, 8, 16, 32, and 64 frames of a single series acquired using a 70° flip angle; thus in these four sets of ACS lines the SNR was varied by changing only the noise level. The RMSE of GRAPPA reconstructed k-space using kernels estimated from ACS lines with SNR varied in this manner was calculated. The aliasing artifact was assessed using a metric based on the cross-correlation between a template and the image in the phase encoding direction (16). The template was comprised of a region around the edge of the phantom, which contained approximately 5% of the total pixels. This was convolved with the reconstructed image, and the cross-correlation coefficient at FOV/R was defined as the aliasing score. Please refer to (16) for more details. The random noise variance in the final images, after applying the sum-of-squares coil combine, was assessed using the MP-law method (13).

All of the experimental results shown in this study used a 4×5 GRAPPA kernel, i.e., four k-space phase-encoding lines and five data points in each k-space line were included in the GRAPPA kernel. The GRAPPA kernel span in k-space is dependent on the acceleration rate, e.g., at acceleration rate = 2, the 4×5 GRAPPA kernel spans nine phase encoding lines, whereas at acceleration rate = 4, the 4×5 GRAPPA kernel spans 17 phase encoding lines. The results shown were evaluated at acceleration rate $R = 4$, unless otherwise stated.

To test the hypothesis that the smallest singular value of the GRAPPA encoding matrix is close to the noise standard deviation, the GRAPPA kernel encoding matrix was constructed using the following kernel size: 2×3 , 2×5 , 2×7 , 4×3 , 4×5 , and 4×7 for all 14 image series using the 12-channel coil. The smallest singular value was evaluated and compared with the noise standard deviation.

The effect of ACS line SNR was compared directly with the effect of very high SNR ACS lines plus Tikhonov regularization. The highest SNR ACS lines (acquired using 70° flip angle and averaged 64 frames), and a series of Tikhonov regularization parameters with variant strength were used to estimate multiple GRAPPA kernels. The condition number was defined as:

$$\sqrt{\text{cond}(\mathbf{A}^H \mathbf{A} + \lambda^2 \mathbf{I}_{n \times n})}, \quad [7]$$

where λ is the Tikhonov regularization parameter, $\text{cond}(\cdot)$ represents the condition number of the matrix. Note that $\text{cond}(\mathbf{A}^H \mathbf{A}) = \text{cond}(\mathbf{A})^2$. The RMSE, artifact level, and the random noise level of images reconstructed using the Tikhonov regularized GRAPPA kernel were plotted against the condition number, and compared with the ACS lines SNR effect.

We also acquired SSFP real-time cine images in vertical and horizontal long-axis views and one short-axis view in two healthy volunteers using TGRAPPA with parallel acceleration rate = 5. Imaging parameters were: 95 phase encoding lines (19 acquired), 192×144 reconstructed matrix using zero-padding interpolation, 8.0 mm thick slice, flip angle = 71° , TE/TR 1.05/2.38 ms, pixel bandwidth = 1488 Hz/pixel, FOV = $380 \times 285 \text{ mm}^2$. A total of 255 images were acquired per image series to provide sufficient data for statistical analysis. The tile-all-frames (TAF) kernel was estimated using the following steps: first, 255 frames were partitioned into 17 temporal windows with 15 frames each; second, all frames within each temporal window were averaged (i.e., three fully sampled k-space were averaged as suggested by Breuer et al) (3) to minimize the motion artifacts; third, all 17 fully sampled and averaged k-space datasets were tiled together to estimate the GRAPPA reconstruction kernel. In other words, the TAF ACS lines has 17-times the k-space points of the AAF ACS lines, and subsequently, more linear equations in Eq. [1]. Note that an asymmetric echo will cause a signal discontinuity that may affect the accuracy of the kernel estimation near the boundary. In this case, instead of tiling the ACS lines one can simply use these 17 sets of ACS lines to estimate the GRAPPA kernel. We found the difference between this method and the TAF method was not significant when a symmetric echo was acquired. Therefore, we only used the TAF implementation in this study. TGRAPPA kernels were estimated using both the AAF and TAF methods, and two separate image series were reconstructed respectively. A 4×5 GRAPPA kernel was used in all image reconstructions. Please refer to Figure 1 illustrating the two strategies used to estimate the ACS lines of the TGRAPPA reconstruction from the same raw data. The image SNR was defined as the ratio of the spatially averaged signal level divided by the spatially averaged noise level. The signal level was defined as the root-mean-square of the temporally averaged image, and the image noise level was measured using the MP-law method (13).

RESULTS

Phantom Study

The phantom image SNR of the first 14 image series varied from 5.8 (1° flip angle) to 174 (70° flip angle) using the 12-channel coil, and 8.3 (1° flip angle) to 160 (70° flip angle) using the 32-channel coil. The relation between SNR and flip angle is close to a simple linear relation in a balanced SSFP sequence (14).

Figure 2 shows the relation between the largest/smallest singular values of the GRAPPA encoding matrix and signal/noise of the ACS lines. When the signal increased with the increasing flip angle of the SSFP sequence, the largest singular value increased linearly, while the smallest singular values remained the same, as shown Figure 2a. When multiple sets of ACS lines were averaged to enhance the SNR, the signal remained the same, as did the largest singular values; as the noise level decreased proportional to the square root of the number of averages, so did the smallest singular values (Fig. 2b). The effects demonstrated in Figures 2a and b are summarized in Figure 2c as the linear relation between the SNR of the ACS lines and the condition number (ratio of the largest and the smallest singular values of the encoding matrix) of the GRAPPA encoding matrix. Please note that the SNR varied by over two orders of magnitude in the phantom experiment. Counterintuitively, the

condition number of the GRAPPA encoding matrix is not sensitive to acceleration rate. The condition number only changed minimally from acceleration rate 2 to rate 8 (Fig. 2d).

The smallest singular values of GRAPPA encoding matrix only show weak dependence on the GRAPPA kernel size. For GRAPPA kernels 2×3 , 2×5 , 2×7 , 4×3 , 4×5 , and 4×7 , the average smallest singular values for acceleration rate $R = 4$ were found to be 69%, 55%, 47%, 66%, 53%, and 45% of the noise standard deviation. The larger the kernel size, the smaller the smallest singular value. The relative variation across the image series is minimal (around 1%), and the relative variation from $R = 2$ to $R = 8$ is less than 10%.

Figure 3 shows that the RMSE of the GRAPPA reconstruction reached its minimum when the ACS lines had moderate SNR, while the random noise variance increased and the aliasing score decreased almost monotonically with increasing SNR of the ACS lines. The flip-angle of the down-sampled raw data was 50° , and the RMSE minimum or “sweet spot” seen in Figure 3a corresponds to a separately acquired ACS lines with flip angle = 30° . The RMSE includes both random deviation (random noise) and structured deviation (aliasing artifacts). Figure 3b demonstrates the trade-off between random noise and artifacts. When the SNR of ACS lines is high, the RMSE of the GRAPPA reconstruction is dominated by random noise; conversely when the SNR of ACS lines is low, the RMSE of GRAPPA reconstruction is dominated by aliasing artifacts. Figure 3c are the example GRAPPA reconstructed images using variant SNR ACS lines. Obviously, the higher the ACS lines SNR, the lower the artifacts and the higher the random noise level.

Figure 4 shows the comparison of the condition number and the RMSE of GRAPPA reconstructed images using ACS lines generated by the average-all-frames or tile-all-frames methods. When the TAF method was used to estimate the ACS lines, both the largest and the smallest singular values of the GRAPPA encoding matrix increased linearly with the number of tiled frames, as shown in Figure 4a. Therefore, the condition number varied only minimally with increasing the number of tiled frames, as shown in Figure 4b. The RMSE of the GRAPPA reconstructed images was also lower when the TAF ACS lines were used (Fig. 4c).

Figure 5 shows the quantitative comparison between GRAPPA reconstruction using ACS lines with variable SNR, and the GRAPPA reconstruction using the highest SNR ACS lines while varying the strength of Tikhonov regularization. The reconstruction RMSE of both GRAPPA kernels follows a U-shape curve. The difference between the two curves is minimal when the condition number is high; see Figure 5a. The random noise variance and the aliasing artifact curves of both GRAPPA kernels show similar trends; see Figures 5b and c. Therefore, the SNR of ACS lines and the Tikhonov regularization show qualitatively the same effect in the GRAPPA reconstruction. Tikhonov regularized GRAPPA reconstruction has lower artifacts at the same condition number.

Volunteer Study

Supporting Figure S1, which is available online, shows the same short axis view cardiac image reconstructed using the AAF-TGRAPPA method, the TAF-TGRAPPA method, and the Tikhonov regularized AAF-TGRAPPA method. It is important to note that the Tikhonov

regularization parameter was selected to match the condition number of the TAF-TGRAPPA, and not necessarily to generate optimal results from an SNR or image quality perspective. Both images reconstructed by the TAF-TGRAPPA method and the Tikhonov regularized AAF-TGRAPPA method demonstrated significantly higher SNR than AAF-TGRAPPA without regularization.

Quantitative measurements performed on the volunteer image data indicate that the SNR of the TAF-TGRAPPA method was more than 40% higher (mean: 53%, max: 68%, min: 44%) than the AAF-TGRAPPA reconstruction method in all three cardiac views. The Tikhonov regularized AAF-TGRAPPA method also showed significant SNR gain (mean: 29%, max: 37%, min: 24%). The condition number of the GRAPPA kernel estimation equation using the TAF ACS lines was reduced by 4.22 ± 0.077 times compared with AAF ACS lines. The condition number change is close to $17 \approx 1.12$ (17 is the number of tiled frames, or the number of sets of ACS lines used in the GRAPPA kernel estimate, and the 17 is the SNR gain in the AAF ACS lines), which is in agreement with the quantitative relation found in the phantom experiment.

DISCUSSION

In this study, phantom experiments revealed a simple proportional relation between the SNR of ACS lines and the condition number of the GRAPPA kernel regression equation. As a result, ACS lines with high SNR can lead to ill-conditioning of the GRAPPA estimation equations, which in turn can amplify random noise in the GRAPPA reconstructed images. In addition, the largest singular value of the GRAPPA kernel encoding matrix was found to be proportional to the signal level of the ACS lines, while the smallest singular value was proportional to the noise level of the ACS lines. Counterintuitively, high SNR ACS lines, without appropriate regularization, can actually degrade the SNR of the GRAPPA reconstructed images because of the high condition number in the kernel estimation. We have shown in both phantom and volunteer studies that slightly lower SNR ACS lines may increase SNR and decrease RMS error in GRAPPA reconstructed images. At the same time, however, low SNR ACS lines may increase image artifacts. Effectively, the noise in the ACS lines has a regularization effect, similar to the noise in training neural networks (8). When the SNR of ACS lines is high, a stronger Tikhonov regularization should be used in GRAPPA kernel estimation to maintain sufficient image SNR.

Our results also confirmed that the smallest singular value decreases with increasing GRAPPA kernel size. The small singular values in the GRAPPA encoding matrix become large singular values after matrix inversion and cause the noise amplification in GRAPPA reconstruction. If the noise in the GRAPPA encoding matrix is IID, the smallest eigenvalue follows the singular value distribution of a random matrix (11–13). The random matrix theory predicts that the smallest singular value should decrease with increasing kernel size. However, the noise in the GRAPPA encoding matrix has complex correlation (4th order tensor) because the k-space sliding window is used to construct it. The quantitative result of the random matrix theory is not completely accurate under these conditions; however, our results show that the qualitative prediction of the random matrix theory is still valid.

The results of our experimental study show, in agreement with the qualitative observations of Sodickson et al (4,5), that noise in the ACS lines has the same qualitative effect as Tikhonov regularization; higher SNR in the ACS lines is similar to applying a weaker Tikhonov regularization and may lead to under-regularization of the GRAPPA encoding matrix. The converse is also true; i.e., lower SNR ACS lines have an effect similar to stronger regularization. Therefore, the SNR of the ACS lines can be considered as a form of inherent regularization and should be factored in when choosing a Tikhonov regularization parameter for the GRAPPA kernel encoding matrix.

It is well-known that Tikhonov regularization leads to a generic trade-off between reconstruction artifacts and random noise (7). When stronger regularization is applied, random noise is suppressed but image artifacts and blurring become more pronounced; weaker regularization has the opposite effect. Because ACS line noise exhibits effects similar to Tikhonov regularization, lower noise in the ACS lines, without regularization, may amplify random noise in GRAPPA reconstructed images, but preserves the image sharpness; conversely, higher noise in the ACS lines may boost image SNR but excessively smooth the image and increase artifact level.

The in vivo volunteer study shows that the regularizing effect of the noise in ACS lines can be significant in a clinically relevant imaging protocol. The GRAPPA encoding matrix condition number in the volunteer study also follows the quantitative relation revealed in the phantom study. In real-time cardiac cine acquired using the TGRAPPA technique, the SNR of ACS lines can become very high if the AAF method is used. Without proper Tikhonov regularization, the intrinsic regularizing effect of noise in ACS lines may be too weak; the GRAPPA kernel estimation equation can be under-regularized, and the TGRAPPA reconstruction may have sub-optimal SNR. It was also demonstrated that lower SNR ACS lines are generated using an alternative TAF method, and the regularization effect from noise in the ACS lines is high enough to significantly improve the SNR of the TGRAPPA reconstruction. These results should make it clear that increasing the SNR of the ACS lines makes the problem more ill-conditioned, and without proper regularization, e.g., Tikhonov regularization, it may lead to noise amplification in the reconstructed image.

We also observed that the Tikhonov regularized AAF-TGRAPPA and TAF-TGRAPPA reconstruction do not have identical SNR gain even when the condition number is matched. The Tikhonov regularization parameter could be further increased to boost the SNR gain of the AAF-TGRAPPA method to match that of the TAF-TGRAPPA method. The SNR gain in the TAF-TGRAPPA reconstruction cannot be fully explained by the regularization effect of the lower SNR ACS lines. Because more equations are included in the TAF-TGRAPPA kernel estimate, and the respiratory motion information can be better preserved in multiple sets of ACS lines, the GRAPPA kernel may benefit from TAF ACS lines. A thorough comparison between TAF-TGRAPPA and AAF-TGRAPPA is beyond the scope of this work; however, the reported comparison between TAF and AAF expands our understanding of GRAPPA reconstruction and explains how noise in ACS impacts image quality. The results do not suggest the superiority of one technique over the other, but merely demonstrate the impact of ACS noise on the reconstructed images.

In this work, we have used RMSE as an image quality metric; however, we recognize that artifacts and noise, which both contribute to RMSE, may affect the diagnostic quality and confidence of interpretation differently (17). The minimum RMSE condition does not necessarily correspond to the optimal image quality because the diagnostic accuracy of MRI depends on metrics other than RMSE. The “optimal” GRAPPA reconstruction may require higher SNR ACS lines than the minimal RMSE may indicate, i.e., it may be beneficial to trade some degree of random noise for lower artifact level or reduced blurring. However, because RMSE is representative of the level of variation from the gold standard, and there is no other well-established method to assess image quality differences, we have chosen it as our metric. Consequently, the result shown in Figure 3 is a rather qualitative measure of image quality. Results in Figure 5 confirm similarity of the effect of the Tikhonov regularization and the ACS lines SNR.

Our analysis relies on an important assumption: the signal matrix A_S is a low-rank matrix. In other words, the GRAPPA convolution kernel has more k-space support than it requires, i.e., the kernel shape is not optimal. Our implementation follows the classical GRAPPA algorithm, i.e., it uses an identical kernel size in each channel, regardless of the specific channel sensitivity map. If a small enough kernel is used, and the signal matrix A_S has full-rank, the noise-amplification effect described would be minimal, although severe artifacts may appear. In other words, there is a generic trade-off between artifact and noise in GRAPPA kernel size selection. In clinical practice, a relatively large GRAPPA kernel is preferred for robustness. Alternatively, recently proposed methods to optimally determine the size and shape of the GRAPPA kernel selection may be able to improve SNR and reduce artifacts simultaneously (18,19). We expect that optimizing the shape of the GRAPPA kernel improves the condition number of the GRAPPA kernel encoding matrix, and hence the quality of image reconstruction. The regularization effect provided by the noise in ACS lines can be minimized because the smallest singular values in the GRAPPA encoding matrix may not be dominated by noise.

CONCLUSIONS

In conclusion, ill-conditioning of the GRAPPA kernel estimation equations leads to noise amplification in the reconstructed images. This study revealed a simple proportional relation between the SNR of ACS lines and the condition number of the GRAPPA kernel regression equation. Perhaps counterintuitively, increasing the SNR of ACS lines degrades the SNR of GRAPPA reconstructed images. The noise in the ACS lines has an effect similar to applying Tikhonov regularization. Adopting a set of ACS lines with high SNR may require a higher Tikhonov regularization parameter to avoid under-regularization in GRAPPA kernel estimation and noise amplification of GRAPPA reconstructed images at the cost of image artifacts.

Supplementary Material

Refer to Web version on PubMed Central for supplementary material.

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APPENDIX

Matrix $\mathbf{A}$ can be regarded as the summation of a noise-free signal matrix $\mathbf{A}_S$ and a noise matrix $\mathbf{A}_N$; $\mathbf{A} = \mathbf{A}_S + \mathbf{A}_N$ then:

$$\mathbf{A}^H \mathbf{A} = \mathbf{A}_S^H \mathbf{A}_S + \mathbf{A}_S^H \mathbf{A}_N + \mathbf{A}_N^H \mathbf{A}_S + \mathbf{A}_N^H \mathbf{A}_N. \quad [\text{A.1}]$$

The dimension of matrices $\mathbf{A}$, $\mathbf{A}_S$, and $\mathbf{A}_N$ are $m \times n$. To further simplify the analysis, we make the following three assumptions: 1) entries of matrix $\mathbf{A}_N$ are zero-mean independent and identically distributed Gaussian noise with standard deviation σ/m ; 2) the mean signal of each entries of $\mathbf{A}_S$ is s_0/m , and the SNR of the $\mathbf{A}$ is s_0/σ ; 3) entries of $\mathbf{A}_S$ and $\mathbf{A}_N$ are completely uncorrelated. Please note that parameter m in assumption 1) and 2) is a common scaling factor to simplify the derivation.

Asymptotically, as the number of rows $m \rightarrow \infty$, entries of matrix $\mathbf{A}_S^H \mathbf{A}_S$ and diagonal entries of matrix $\mathbf{A}_N^H \mathbf{A}_N$ approach limited values, but entries of matrices $\mathbf{A}_S^H \mathbf{A}_N$ and $\mathbf{A}_N^H \mathbf{A}_S$, and off-diagonal entries of matrix $\mathbf{A}_N^H \mathbf{A}_N$ approach zero, i.e.:

$$\mathbf{A}^H \mathbf{A} = \mathbf{A}_S^H \mathbf{A}_S + \sigma^2 \mathbf{I} + O\left(m^{-\frac{1}{2}}\right) \approx \mathbf{A}_S^H \mathbf{A}_S + \sigma^2 \mathbf{I}_{n \times n}. \quad [\text{A.2}]$$

In practice, when parameter $m \gg n \gg 1$ (usually $m > 10^3$, $n > 10^2$), the Moore–Penrose pseudoinverse can be approximately written as:

$$(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \approx (\mathbf{A}_S^H \mathbf{A}_S + \sigma_n^2 \mathbf{I}_{n \times n})^{-1} \mathbf{A}^H. \quad [\text{A.3}]$$

The approximation in Eq. [A.2] has been reported before for $\mathbf{A}_N$ with IID entries (4,10). Because the GRAPPA encoding matrix has repeated entries from the overlap of k-space sliding windows, entries of $\mathbf{A}_N$ has structure. Hence, it is questionable whether the approximation in Eq. [A.2] is applicable. We use numerical simulations to demonstrate that the added structure in $\mathbf{A}_N$ does not alter its intended behavior; therefore, the approximation in Eq. [A.2] is still applicable. We will show the following two properties using two numerical simulations: first, the difference between the empirical eigenvalue distribution of

covariance matrix $A_N^H A_N$ and that of a random matrix is negligible; second, the entries of $A_S^H A_N$ in Eq. [A.2] are proportional to $m^{-\frac{1}{2}}$. In both simulations, the null hypothesis is that replacing matrix A_N by a IID random noise matrix A_N^0 with the same size does not cause a qualitative difference.

The first numerical simulation constructed a noise-only GRAPPA encoding matrix from simulated multiple-channel random noise k-space data. The empirical eigenvalue distribution functions of the corresponding covariance matrices $A_N^H A_N$ and $A_N^{0H} A_N^0$ were plotted. The difference between them was evaluated using the Kolmogorov-Smirnov test (KS-test) (13).

The k-space data in the second numerical simulation use high SNR (>50, achieved using multiple averages) MR phantom data as signal (to construct matrix A_S) and simulated noise. The k-space size varies from 64×64 to 128×256 to vary m , while the image SNR was fixed to 10 (measured in image space). The entries of cross-term $A_S^H A_N$ in Eq. [A.2] was plotted against m .

Supporting Figure S2A shows the empirical eigenvalue distribution functions of covariance matrices $A_N^H A_N$ and $A_N^{0H} A_N^0$ using 2×10^4 Monte Carlo simulations. The two distribution functions are almost identical. Supporting Figure S2B shows the P -value of the KS-test versus the number of Monte Carlo simulations. When the number of channels = 12, the difference becomes significant after more than 1.6×10^4 trials, i.e., the systematic difference between the eigenvalue spectrum of $A_N^H A_N$ and its null hypothesis is much smaller than the random fluctuation in the eigenvalue spectrum. In both Supporting Figures S2A and S2B, the kernel size is 4×5 , the k-space size is 128×256 . Varying acceleration rate and kernel size does not change the results qualitatively.

Supporting Figure S3 shows the rms of matrix $A_S^H A_S$ off-diagonal entries, rms and maximum of matrix $A_S^H A_N$ entries (cross-term in Eq. [A.2]), and the corresponding null hypothesis (entries of matrix $A_S^H A_N^0$). After applying proper scaling, the rms of matrix $A_S^H A_S$ off-diagonal entries does not change significantly with m . We believe correlations in matrix A_N entries cause the maximum entry of matrix $A_S^H A_N$ to be significantly higher than that of the null hypothesis. But the qualitative scaling relation, i.e., $\sim m^{-1/2}$, is still applicable. Therefore, when m is large, entries of $A_S^H A_N$ asymptotically goes to zero, as mentioned in Eq. [A.2].

In both Supporting Figures S2 and S3, the acceleration rate $R = 2$ was used. The simulation results did not show a qualitative difference when a higher acceleration rate was used.

In summary, the numerical simulation results using clinical relevant parameters demonstrate that the noise in the GRAPPA encoding matrix can be treated as an additive random matrix with IID noise entries, and the approximations in Eq. [A.2] are still applicable.

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FIG. 1.

The diagram of the AAF and TAF TGRAPPA ACS lines estimates. Averaging over all k-space generates the AAF ACS lines; tiling multiple temporal sliding window averaged k-space into a large k-space generates the TAF ACS lines (the TAF ACS lines are multiple sets of ACS lines). The SNR of AAF ACS lines is significantly higher than TAF, but also has significantly less number of k-space data points.

FIG. 2.

The relation between the condition number of the GRAPPA encoding matrix and the SNR of ACS lines. **a:** The largest singular values varied linearly with the mean signal of ACS lines, while the smallest singular values remained constant. **b:** Conversely, the smallest singular values varied with the number of averages of ACS lines (i.e., linearly with the noise level), while the largest singular values remained constant. **c:** Combining the results from a and b, the condition number increased linearly with the SNR of ACS lines. **d:** The condition number remained nearly constant (less than 25% variation for 12-channel and less than 33% variation for 32-channel) across acceleration rates varied from 2 to 8. **e:** The reconstructed images with acceleration rate = 2, 4, 6, and 8 (from left to right, raw data acquired using 12-channel coil). The raw data and ACS lines were acquired using flip angle = 60° in (d) and (e). When no parallel imaging was used, the image SNR of 60° flip angle image series was 162 for 12-channel coil, and 141 for 32-channel coil. A–C: The GRAPPA acceleration rate $R = 4$.

FIG. 3.

a: The relation between the RMSE of GRAPPA reconstruction and the SNR of ACS lines. **b:** The random noise variance (square) and the aliasing artifacts (circle) versus the SNR of ACS lines. **c:** (from left to right) Three example images corresponding to GRAPPA reconstruction using ACS lines with 5° flip angle, 30° flip angle, and 70° degree flip angle with 64 averages. When the flip angle of ACS lines was 30°, the GRAPPA reconstruction reached minimal RMSE. The random noise variance increased monotonically but the aliasing score decreased almost monotonically with increasing SNR of the ACS lines. The raw data was acquired using 12-channel coil, flip angle = 50°, acceleration rate $R = 4$.

FIG. 4.

The condition number and the RMSE of GRAPPA reconstruction versus the number of tiled frames in the ACS lines. **a:** Both the largest and the smallest singular values increased with the number of tiled frames. **b:** The condition number decreased slightly (<4% in 12-channel, <11% in 32-channel) with increasing number of tiled frames. **c:** RMSE of GRAPPA reconstruction increased with the number of averages when using AAF ACS lines, but decreased slightly with the number of tiled frames when using TAF ACS lines.

FIG. 5.

The quantitative comparison between GRAPPA reconstruction using ACS lines with variant SNR (dashed line), and the GRAPPA reconstruction using highest SNR ACS lines with variant Tikhonov regularization (solid line). The x-axis is the condition number of the GRAPPA encoding matrix. **a:** the RMSE versus condition number. **b:** The random noise variance versus the condition number. **c:** The aliasing artifacts versus the condition number. The SNR of ACS lines and the Tikhonov regularization show qualitatively the same effect in the GRAPPA reconstruction.