An analytic, Fourier domain description of shear wave propagation in a viscoelastic medium using asymmetric Gaussian sources

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Recent measurements of shear wave propagation in viscoelastic materials have been analyzed by constructing the two-dimensional Fourier transform (2D-FT) of the spatial-temporal shear wave signal and using an analysis procedure derived under the assumption the wave is described as a plane wave, or as the asymptotic form of a wave expanding radially from a cylindrically symmetric source. This study presents an exact, analytic expression for the 2D-FT description of shear wave propagation in viscoelastic materials following asymmetric Gaussian excitations and uses this expression to evaluate the bias in 2D-FT measurements obtained using the plane or cylindrical wave assumptions. A wide range of biases are observed depending on specific values of frequency, aspect ratio R of the source asymmetry, and material properties. These biases can be reduced significantly by weighting the shear wave signal in the spatial domain to correct for the geometric spreading of the shear wavefront using a factor of x^p . The optimal weighting power p is found to be near the theoretical value of 0.5 for the case of a cylindrical source with $R = 1$, and decreases for asymmetric sources with $R > 1$. © 2015 Acoustical Society of America.

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I. INTRODUCTION

Shear wave elasticity imaging¹ is a quantitative, non-invasive technique that can be used to measure the shear modulus of a material by observing shear wave propagation in the material. Under the assumption of a linear, isotropic, unbounded, elastic material, the shear modulus μ can be expressed in terms of the material density ρ and propagation speed c as^{2,3}

$$\mu = \rho c^2 = \frac{\rho \omega^2}{k^2} \quad (1)$$

where $\omega = 2\pi f$ is the angular frequency and $k = 2\pi/\lambda$ is the wavenumber. Often, focused acoustic radiation is used to generate the shear waves by applying a localized, impulsive force^{4,5} that is characterized by a range of spatial and temporal frequencies. However, for an elastic material, the shear modulus is independent of frequency, and the measured group velocity can be used in Eq. (1) to determine μ .

In contrast, when a stress is applied to a viscoelastic material, viscous forces in the material cause the strain to lag the applied stress and the shear modulus is a complex, frequency dependent quantity $\mu(\omega)$.⁶ Shear wave propagation in a viscoelastic material exhibits dispersion with a frequency dependent phase velocity $c(\omega)$ and shear attenuation with attenuation coefficient $\alpha(\omega)$. One method that can be used to measure the complex shear modulus $\mu(\omega)$ is to measure the shear wave phase velocity $c(\omega)$ over a range of frequencies^{7,8} and to use a viscoelastic model, such as a

Voigt model^{7,9} or linear dispersion model,^{10,11} to characterize the dispersion.

Alternatively, measurements of $c(\omega)$ and $\alpha(\omega)$ can be combined to give the real and imaginary parts of $\mu(\omega)$. Nenadic, *et al.*^{12,13} have described a method to measure $c(\omega)$ and $\alpha(\omega)$ from the two-dimensional Fourier transform (2D-FT) of the spatial-temporal shear wave signal using the location of the maximum 2D-FT signal at a given frequency to determine $c(\omega)$ and the full width at half-maximum (FWHM) of the 2D-FT signal at the given frequency to determine $\alpha(\omega)$. A summary of this method is included in Sec. II below. Nenadic, *et al.*^{12,13} refer to this approach as “model-free” because it does not rely on the assumption of a particular viscoelastic model.

Nightingale, *et al.*¹¹ have used the 2D-FT approach to measure the phase velocity $c(f)$ in simulation data sets that modeled the experimental excitation and shear wave propagation of measurements performed in human liver by Wang *et al.*¹⁴ and Palmeri *et al.*¹⁵ These measurements were analyzed by fitting $c(f)$ with a linear dispersion model of the form

$$c(f) = c_0 + \frac{dc}{df}f, \quad (2)$$

and the results of the fit were reported using the phase velocity at a frequency of 200 Hz, $c(200 \text{ Hz})$, and the dispersion slope dc/df . In this analysis, the shear attenuation $\alpha(\omega)$ was not measured from the 2D-FT data and, in addition, a weighting factor described by Nenadic *et al.*¹³ for the case of a cylindrical excitation source was not applied to the 2D-FT data. Comparing the $c(200 \text{ Hz})$ and dc/df values to the true

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values calculated from the model parameters used in the simulations showed that these results were systematically biased with errors on the order of 9% for $c(200 \text{ Hz})$ and 51% for dc/df . While application of the weighting factor described by Nenadic *et al.*¹³ would be expected to reduce these biases, particularly, for waves observed in the asymptotic range far from the source, additional factors that could contribute to these biases include (1) effects introduced by the finite temporal or lateral ranges of the shear wave propagation, (2) effects introduced by the spatial shape and temporal duration of the excitation force, and (3) effects introduced by the observation of a shear wave near the source.

In this paper, we investigate potential biases of measured phase velocity $c(\omega)$ and shear attenuation $\alpha(\omega)$ introduced by the 2D-FT analysis method by deriving an exact, analytic expression for 2D-FT data that describes shear wave propagation in a viscoelastic material following a localized, impulsive excitation. In particular, we solve the equation of motion that describes shear wave propagation generated by a spatially asymmetric Gaussian source in the Fourier domain. This solution is used to calculate the 2D-FT signal describing shear wave propagation in a viscoelastic material with the stress-strain relation described by a specific viscoelastic model so that the true values of phase velocity $c(\omega)$ and shear attenuation $\alpha(\omega)$ can be calculated from the model parameters. Thus, bias inherent in the 2D-FT analysis method can be evaluated without possible confounding factors introduced by finite spatial or lateral ranges or complicated excitation geometries such as those used in experimental measurements.

II. BACKGROUND AND MOTIVATION

In this section, we introduce the notation used in the paper and summarize some of the important relations for the description of shear wave propagation in a viscoelastic material.

The stress-strain relation in a viscoelastic material is characterized by a complex, frequency dependent shear modulus.⁶ For example, if a harmonic shear stress of the form $\sigma(t) = \sigma_0 e^{i\omega t}$ is applied to an infinitesimal volume, the resulting shear strain is also harmonic with the form $\varepsilon(t, \omega) = \varepsilon_0(\omega) e^{i(\omega t + \phi(\omega))}$ where the phase angle $\phi(\omega)$ is needed to account for the strain lag relative to the applied stress. Then the complex shear modulus $\mu(\omega)$ is given by the ratio of stress to strain,

$$\mu(\omega) = \frac{\sigma(t)}{\varepsilon(t, \omega)} = \frac{\sigma_0}{\varepsilon_0(\omega)} e^{-i\phi(\omega)}. \quad (3)$$

Often, $\mu(\omega)$ is expressed in terms of the real and imaginary parts as $\mu(\omega) = \mu'(\omega) + i\mu''(\omega)$. Wave propagation in a viscoelastic material is characterized by a complex wavenumber $k(\omega) - i\alpha(\omega)$, so that the shear modulus (1) is given by

$$\mu(\omega) = \mu'(\omega) + i\mu''(\omega) = \frac{\rho\omega^2}{[k(\omega) - i\alpha(\omega)]^2}. \quad (4)$$

Expressions for the phase velocity $c(\omega)$ and attenuation $\alpha(\omega)$ can be found by equating the real and imaginary parts of Eq. (4),

$$c(\omega) = \frac{\omega}{k(\omega)} = \sqrt{\frac{1}{\rho} \frac{2(\mu'^2 + \mu''^2)}{\mu' + \sqrt{\mu'^2 + \mu''^2}}} \quad (5)$$

and

$$\alpha(\omega) = \sqrt{\rho\omega^2 \frac{\sqrt{\mu'^2 + \mu''^2} - \mu'}{2(\mu'^2 + \mu''^2)}}. \quad (6)$$

These results agree with the specific expressions for the cases of Voigt and Maxwell models given by Catheline *et al.*⁹

Nenadic *et al.*^{12,13} have described a method to measure $c(\omega)$ and $\alpha(\omega)$, and thereby $\mu(\omega)$, by constructing the 2D-FT from the spatial-temporal description of a propagating shear wave. To illustrate this method, consider a plane wave propagating away from the origin in the $+x$ direction with amplitude A , speed $c = \omega_0/k_0$, and attenuation α ,

$$u(x, t) = AH(x)e^{-i(k_0x - \omega_0t)}e^{-\alpha x} \quad (7)$$

where $H(x)$ is the Heaviside step function. Note that the signs in Eqs. (4) and (7) are chosen so that the time dependence $e^{i\omega_0t}$ agrees with the time dependence in Eq. (9) below, and Eq. (7) describes a wave propagating in the $+x$ direction with a positive value for the attenuation α .

In this paper, we use the non-unitary, angular frequency form of the Fourier transform relations with the forward transform from the time to frequency domain given by

$$F(\omega) = \mathcal{F}_t[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \quad (8)$$

and the inverse transform from the frequency to time domain given by

$$f(t) = \mathcal{F}_t^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega. \quad (9)$$

Similar expressions hold for the transforms between the spatial coordinate and frequency domains with forward and inverse operators $\mathcal{F}_x$ and $\mathcal{F}_x^{-1}$, respectively. Calculations involving 2D-FT data are performed using sequential applications of one dimensional transforms.

Calculating the 2D-FT of Eq. (7) gives

$$U(k, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [AH(x)e^{-i(k_0x - \omega_0t)}e^{-\alpha x}]e^{-i\omega t}e^{-ikx} dx dt = 2\pi A \delta(\omega - \omega_0) F(k) \quad (10)$$

where $\delta(\cdot)$ is the delta function and the k -dependent factor $F(k)$ is given by

$$F(k) = \frac{1}{i(k + k_0) + \alpha}. \quad (11)$$

This expression for $U(k, \omega)$ agrees with Nenadic *et al.*¹³ after accounting for the different sign convention used for the plane wave description in Eq. (7) above.

In this paper, we present 2D-FT data by displaying the normalized spectral energy density using the magnitude squared $|U(k, \omega)|^2$ of the 2D-FT signal. Then for the plane wave case with 2D-FT given by Eq. (10), the k -dependence of the displayed signal is proportional to $|F(k)|^2$,

$$|F(k)|^2 = \frac{1}{(k + k_0)^2 + \alpha^2}. \quad (12)$$

Figure 1(a) shows a plot of $|F(k)|^2$. The phase velocity $c(\omega_0) = \omega_0/k_0$ can be measured from 2D-FT data by locating the position of the maximum signal at that frequency.^{12,13} For $|F(k)|^2$ given by Eq. (12), this maximum occurs at $k_{\max} = -k_0$ so that

$$c(\omega_0) = -\frac{\omega_0}{k_{\max}}. \quad (13)$$

This result is equivalent to the expression given by Nenadic *et al.*¹³ after accounting for the different sign convention used in Eq. (7) above.

In addition, the attenuation $\alpha(\omega_0)$ can be measured from the FWHM of the 2D-FT signal.^{12,13} As shown in Fig. 1(a) for the case of $|F(k)|^2$ given by Eq. (12), the half-maximum values occur at $k_{\pm} = -k_0 \pm \alpha$ so that $\alpha(\omega_0)$ is given by

$$\alpha(\omega_0) = \frac{1}{2} \text{FWHM}(k) = \pi \text{FWHM}(k/2\pi) \quad (14)$$

where $\text{FWHM}(k)$ and $\text{FWHM}(k/2\pi)$ are measured using the wavenumber or spatial frequency, respectively. If, alternatively, 2D-FT data were displayed using the magnitude $|U(k, \omega)|$ of the 2D-FT signal,^{12,13} the attenuation would be given in terms of the FWHM of $|F(k)|$ as

$$\alpha(\omega_0) = \frac{1}{2\sqrt{3}} \text{FWHM}(k) = \frac{\pi}{\sqrt{3}} \text{FWHM}(k/2\pi). \quad (15)$$

A key factor in the 2D-FT analysis just described is the observation that the phase velocity and attenuation at the frequency ω_0 were determined using only 2D-FT data at this specific frequency. Thus, this procedure can be extended to a

FIG. 1. Plots of the functions (a) $|F(k)|^2 = 1/[(k + k_0)^2 + \alpha^2]$ from Eq. (12) and (b) $|F(k)|^2 = \pi/\sqrt{(k + k_0)^2 + \alpha^2}$ from Eq. (19). For both cases, the maximum signal occurs at $k_{\max} = -k_0$ so that the phase velocity $c = \omega_0/k_0$ is given by $c = -\omega_0/k_{\max}$ as in Eq. (13). For (a), the half-maximum value of $1/2\alpha^2$ occurs at $k_{\pm} = -k_0 \pm \alpha$ so that $\text{FWHM} = 2\alpha$ as in Eq. (14). For (b), the half-maximum value of $\pi/2\alpha$ occurs at $k_{\pm} = -k_0 \pm \sqrt{3}\alpha$ so that $\text{FWHM} = 2\sqrt{3}\alpha$ as in Eq. (20).

superposition of plane waves with different frequencies to determine $c(\omega)$ and $\alpha(\omega)$ without the assumption of a specific viscoelastic material model.^{12,13}

As a second example, consider a harmonic excitation applied along the z axis that generates a cylindrically expanding shear wave that propagates radially outward from the source as described by Graff.² With the addition of an attenuation factor $e^{-\alpha x}$ (not considered by Graff), this shear wave can be described as

$$u(x, t) = AH(x)H_0^{(2)}(k_0x)e^{i\omega_0t}e^{-\alpha x} \quad (16)$$

where A is the amplitude and $H_0^{(2)}(k_0x)$ is the Hankel function of the second kind of order zero. [Note that the Hankel function and time dependence in Eq. (16) have been switched compared to the expressions in Graff² to agree with the time dependence in Eq. (9) used here.] Expressing the Hankel function in terms of its asymptotic form¹⁶ and absorbing constant factors into the amplitude A in Eq. (16) gives

$$u(x, t) \sim AH(x) \frac{1}{\sqrt{x}} e^{-i(k_0x - \omega_0t)} e^{-\alpha x} \quad (17)$$

where the factor of $1/\sqrt{x}$ is present because of the geometric spreading of the wavefront. The 2D-FT of Eq. (17) is evaluated using the same integral expression (10) as used for the plane wave example above. For this case, the spatial integral can be evaluated using Eq. (4.2.3) from Erdélyi¹⁷ to give the k -dependent factor

$$F(k) = \sqrt{\frac{\pi}{i(k + k_0) + \alpha}}. \quad (18)$$

For 2D-FT signals displayed using the spectral energy density, $|F(k)|^2$ is given by

$$|F(k)|^2 = \frac{\pi}{\sqrt{(k + k_0)^2 + \alpha^2}}. \quad (19)$$

Figure 1(b) shows a plot of $|F(k)|^2$ from Eq. (19). As with the case of the plane wave above, the phase velocity $c = \omega_0/k_0$ can be calculated from the location of the maximum 2D-FT signal using Eq. (13). However, as indicated in Fig. 1(b), the attenuation is not given by Eq. (14), but, instead, by the relation

$$\alpha(\omega_0) = \frac{1}{2\sqrt{3}} \text{FWHM}(k) = \frac{\pi}{\sqrt{3}} \text{FWHM}(k/2\pi). \quad (20)$$

If, alternatively, 2D-FT data were displayed using the magnitude $|U(k, \omega)|$ of the 2D-FT signal,^{12,13} the attenuation would be given in terms of the FWHM of $|F(k)|$ as

$$\alpha(\omega_0) = \frac{1}{2\sqrt{15}} \text{FWHM}(k) = \frac{\pi}{\sqrt{15}} \text{FWHM}(k/2\pi). \quad (21)$$

The two examples described above illustrate several issues related to the measurement of $c(\omega)$ and $\alpha(\omega)$ from the 2D-FT description of a propagating shear wave. In particular,

expressions (14) and (20) for the shear attenuation are different because they apply to plane and cylindrically expanding waves. Thus, the geometrical configuration of the excitation and wave propagation must be considered in the 2D-FT analysis. For the case of a cylindrical source, Nenadic *et al.*¹³ have proposed weighting the shear wave signal by a factor of $\sqrt{x}$ to compensate for the $1/\sqrt{x}$ factor in Eq. (17). Then the weighted signal would resemble a plane wave and the 2D-FT analysis for the case of a plane wave could be used to measure $c(\omega)$ and $\alpha(\omega)$. However, Eq. (17) only describes the asymptotic behavior of the wave and it is not clear if the $\sqrt{x}$ weighting factor would apply for shear wave motion observed near the source where the asymptotic description does not apply. It would also be expected that the $\sqrt{x}$ weighting factor would need to be modified for the case of shear wave propagation observed near an asymmetric excitation source as is commonly present in acoustic radiation force excitations.

Furthermore, even though Eqs. (12) and (19) indicate that $c(\omega)$ can be measured using Eq. (13) for both examples described above, it is also possible that different excitation configurations or an applied weighting factor could change this result. In particular, the biases observed by Nightingale *et al.*¹¹ in the linear dispersion analysis of phase velocity data could be caused in part by the omission of the $\sqrt{x}$ weighting factor described by Nenadic *et al.*,¹³ and also by the asymmetric excitation used in their simulations. Thus, a careful evaluation of the 2D-FT analysis method to measure $c(\omega)$ and $\alpha(\omega)$ is needed to understand its limitations and potential biases, including biases introduced by the use of an asymmetric source. In this paper, we present an exact, analytic expression for the 2D-FT description of shear wave propagation in a viscoelastic material following an asymmetric Gaussian excitation. This expression is used to evaluate biases in 2D-FT measurements derived by assuming the spatial-temporal shear wave signal is described as a plane wave, or as the asymptotic form of a wave expanding radially from a cylindrically symmetric source.

III. THEORY

A. Equation of motion

For a homogeneous isotropic linear elastic material, the derivation of the Navier equation of motion for the material displacement $\vec{u}(\vec{r}, t)$ requires three fundamental relations. First, conservation of linear momentum gives the acceleration of an infinitesimal volume of material with density ρ in terms of the applied force density $\vec{f}(\vec{r}, t)$ and the gradient of the stress tensor σ_{ij} across the volume,^{2,3}

$$\sigma_{ij,j}(\vec{r}, t) + f_i(\vec{r}, t) = \rho \ddot{u}_i(\vec{r}, t) \quad (22)$$

where summation over repeated indices is implied. Second, for infinitesimal displacements, the strain tensor ε_{ij} is a linear function of the displacements and can be written as^{2,3}

$$\varepsilon_{ij}(\vec{r}, t) = \frac{1}{2} [u_{i,j}(\vec{r}, t) + u_{j,i}(\vec{r}, t)]. \quad (23)$$

Third, for a homogeneous isotropic elastic material, the stress-strain relationship can be written in terms of the Lamé parameters λ and μ as^{2,3}

$$\sigma_{ij}(\vec{r}, t) = \lambda \varepsilon_{kk}(\vec{r}, t) \delta_{ij} + 2\mu \varepsilon_{ij}(\vec{r}, t). \quad (24)$$

As indicated by Eq. (3), a key difference between elastic and viscoelastic materials is that complex, frequency dependent moduli are required to characterize the stress-strain relationship in a viscoelastic material. Thus, it is convenient to express the equation of motion for a viscoelastic material in the frequency domain. Using a tilde to indicate the Fourier transform in time, Eqs. (22) and (23) can be written in the frequency domain as

$$\tilde{\sigma}_{ij,j}(\vec{r}, \omega) + \tilde{f}_i(\vec{r}, \omega) = -\rho \omega^2 \tilde{u}_i(\vec{r}, \omega) \quad (25)$$

and

$$\tilde{\varepsilon}_{ij}(\vec{r}, \omega) = \frac{1}{2} [\tilde{u}_{i,j}(\vec{r}, \omega) + \tilde{u}_{j,i}(\vec{r}, \omega)], \quad (26)$$

respectively. Similarly, as suggested by Eq. (3), the stress-strain relation (24) can be expressed in the frequency domain using the Fourier transform of the stress and strain tensors and complex moduli $\lambda(\omega)$ and $\mu(\omega)$,

$$\tilde{\sigma}_{ij}(\vec{r}, \omega) = \lambda(\omega) \tilde{\varepsilon}_{kk}(\vec{r}, \omega) \delta_{ij} + 2\mu(\omega) \tilde{\varepsilon}_{ij}(\vec{r}, \omega). \quad (27)$$

This relation is an example of the elastic-viscoelastic correspondence principle^{18,19} which states that the stress-strain relation for a viscoelastic material has the same form as for the corresponding elastic material when the stress and strain tensors are replaced by their Fourier (or Laplace) transforms in time and the moduli are interpreted as complex, frequency dependent quantities. In particular, Eq. (27) agrees with the stress-strain relations for a homogeneous isotropic linear viscoelastic material given by Read²⁰ and Borchardt,²¹ and also by Chao and Achenbach²² using Laplace transforms.

Combining Eqs. (25)–(27) gives the equation of motion for a viscoelastic material expressed in the frequency domain

$$[\lambda(\omega) + \mu(\omega)] \tilde{\varepsilon}_{kk,i}(\vec{r}, \omega) + \mu(\omega) \tilde{u}_{i,jj}(\vec{r}, \omega) + \tilde{f}_i(\vec{r}, \omega) = -\rho \omega^2 \tilde{u}_i(\vec{r}, \omega). \quad (28)$$

Finally, the equation of motion describing shear wave propagation in a viscoelastic material can be found by taking the curl of Eq. (28) or by assuming an incompressible material so that the dilation $\tilde{\varepsilon}_{kk}(\vec{r}, \omega)$ is zero,

$$\mu(\omega) \tilde{u}_{i,jj}(\vec{r}, \omega) + \tilde{f}_i(\vec{r}, \omega) = -\rho \omega^2 \tilde{u}_i(\vec{r}, \omega). \quad (29)$$

B. Cylindrically symmetric Gaussian excitation

For the case of a cylindrically symmetric Gaussian source, we choose the applied force to be directed in the $\hat{z}$ direction with infinite extent along the z axis so that it can be written as

$$\vec{f}(r, t) = F_0 e^{-r^2/\sigma^2} W(t) \hat{z}, \quad (30)$$

where F_0 is the amplitude, σ gives the width, and $W(t)$ gives the time dependence of the applied force. Because this force is cylindrically symmetric, the term $\tilde{u}_{i,jj} = \nabla^2 \tilde{u}_i$ in the equation of motion (29) can be expressed using cylindrical coordinates, and only terms involving the derivatives with respect to the coordinate r will contribute. Then the z component of displacement is given by

$$\mu(\omega) \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right] \tilde{u}(r, \omega) + F_0 e^{-r^2/\sigma^2} \tilde{W}(\omega) = -\rho\omega^2 \tilde{u}(r, \omega). \quad (31)$$

Calculating the Hankel transform^{2,23} of order zero using Eq. (B.4.9) of Graff² gives an expression for the transformed displacement,

$$\hat{\tilde{u}}(\xi, \omega) = \frac{F_0 \sigma^2 \tilde{W}}{2} \frac{e^{-\xi^2 \sigma^2/4}}{\mu \xi^2 - \rho\omega^2} \quad (32)$$

where the hat indicates the Hankel transform in r with transform variable ξ and the ω dependence of $\mu(\omega)$ and $\tilde{W}(\omega)$ is no longer explicitly indicated for brevity. For wave propagation along the x axis, the 2D-FT signal $U^{(x)}(k, \omega)$ can be found from Eq. (32) by applying the inverse Hankel transform to return to the spatial domain and calculating the Fourier transform in x . These operations can be performed by direct integration using Eqs. (1.12.1) and (4.2.26) from Erdélyi,¹⁷

$$\begin{aligned} U^{(x)}(k, \omega) &= \int_{-\infty}^{\infty} \int_0^{\infty} \hat{\tilde{u}}(\xi, \omega) J_0(\xi x) e^{-ikx} \xi d\xi dx \\ &= \frac{\pi F_0 \sigma^2 \tilde{W}}{2\mu} \frac{e^{-\sigma^2 \rho \omega^2/4\mu}}{\sqrt{k^2 - \rho\omega^2/\mu}} \\ &\quad \times \text{Erfc}\left(\frac{\sigma}{2} \sqrt{k^2 - \rho\omega^2/\mu}\right) \end{aligned} \quad (33)$$

where $\text{Erfc}(z)$ is the complementary error function. Alternatively, these operations can be performed by noting that the combination of Hankel and Fourier transforms is equivalent to an Abel transform.^{24,25}

Typically, 2D-FT analysis only considers shear wave propagation at coordinates located on one side of the source, for example, for wave propagation along the $+x$ axis. Then, Eq. (33) can be corrected by transforming to the spatial domain, selecting only positions $x \geq 0$ using the step function $H(x)$, and transforming back to the frequency domain,

$$U^{(+x)}(k, \omega) = \mathcal{F}_x\{H(x) \mathcal{F}_x^{-1}[U^{(x)}(k, \omega)]\}. \quad (34)$$

For the results described in Sec. V, this step is performed numerically using a discrete Fourier transform. In addition, weighting the shear wave signal in the spatial domain by the factor of $\sqrt{x}$ as described by Nenadic *et al.*¹³ to account for the cylindrical rather than plane wave source can be performed by including this factor in Eq. (34),

$$U^{(+x)}(k, \omega) = \mathcal{F}_x\{H(x) \sqrt{x} \mathcal{F}_x^{-1}[U^{(x)}(k, \omega)]\}. \quad (35)$$

Finally, experimental measurements typically detect the particle velocity $v = \partial u / \partial t$ of the medium instead of the particle displacement. Then the 2D-FT of the particle velocity signal is given by

$$V(k, \omega) = i\omega U(k, \omega). \quad (36)$$

C. Asymmetric Gaussian excitation

For the case of an asymmetric Gaussian excitation, the applied force can be written as

$$\vec{f}(x, y, t) = F_0 e^{-x^2/\sigma_x^2} e^{-y^2/\sigma_y^2} W(t) \hat{z} \quad (37)$$

where σ_x and σ_y give the widths of the Gaussian shape along the x and y axes, respectively. Inserting this expression into the equation of motion (29) and calculating the Fourier transform in x and y gives

$$U(k_x, k_y, \omega) = \pi F_0 \sigma_x \sigma_y \tilde{W} \frac{e^{-k_x^2 \sigma_x^2/4} e^{-k_y^2 \sigma_y^2/4}}{\mu(k_x^2 + k_y^2) - \rho\omega^2}. \quad (38)$$

The 2D-FT describing wave propagation along the x axis, $U^{(x)}(k_x, \omega)$, can be determined from Eq. (38) by calculating the inverse Fourier transform in y using Eq. (1.4.15) from Erdélyi¹⁷ and evaluating the result with $y = 0$,

$$\begin{aligned} U^{(x)}(k_x, \omega) &= \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} U(k_x, k_y, \omega) e^{ik_y y} dk_y \right\} \Big|_{y=0} \\ &= \frac{\pi F_0 \sigma_x \sigma_y \tilde{W}}{2\mu} \frac{e^{-\sigma_x^2 k_x^2/4} e^{\sigma_y^2 (k_x^2 - \rho\omega^2/\mu)/4}}{\sqrt{k_x^2 - \rho\omega^2/\mu}} \\ &\quad \times \text{Erfc}\left(\frac{\sigma_y}{2} \sqrt{k_x^2 - \rho\omega^2/\mu}\right). \end{aligned} \quad (39)$$

For the case with $\sigma_x = \sigma_y$, the excitation force (37) is cylindrically symmetric, and Eq. (39) agrees with Eq. (33). As with the cylindrically symmetric case, the 2D-FT expressions for shear wave propagation along the $+x$ axis and for particle velocity can be determined using Eqs. (34) and (36), respectively.

IV. METHODS

For the analysis presented in this paper, 2D-FT data were calculated using a material model from which the true values of $c(\omega)$ and $\alpha(\omega)$ could be determined using the model parameters. The specific material model used here is a three-parameter, standard linear model⁶ that can be represented by a spring with stiffness μ_1 in parallel with the series combination of a spring with stiffness μ_2 and dashpot with viscosity η . The stress-strain relation for this model is given by⁶

$$\sigma + \frac{\eta}{\mu_2} \dot{\sigma} = \mu_1 \varepsilon + \eta \left(1 + \frac{\mu_1}{\mu_2}\right) \dot{\varepsilon}, \quad (40)$$

and the shear modulus is given by Eq. (3),

$$\mu(\omega) = \frac{\mu_1\mu_2 + i\omega\eta(\mu_1 + \mu_2)}{\mu_2 + i\omega\eta}. \quad (41)$$

The true speed and attenuation were calculated from $\mu(\omega)$ using Eqs. (5) and (6). The choice of the three-parameter model used here is not critical; it was selected simply because model parameters that characterize healthy and sick liver have previously been identified by Nightingale *et al.*¹¹ Alternative material models, such as a Voigt model, have also been used to analyze shear wave propagation in tissue^{7,9,26} and could have been used for the analysis in this study.

Results are presented in Sec. V for two materials with the model parameters μ_1 , μ_2 , and η listed in Table I. These sets of parameters were chosen from among the 58 sets of simulation parameters identified by Nightingale *et al.*¹¹ that had spatial and temporal extents of 2D-FT data similar to data observed in experimental measurements in healthy and diseased liver. In particular, the linear dispersion analysis using the model parameters for material M1 gives values for the phase velocity at 200 Hz, $c(200 \text{ Hz})$, and dispersion slope dc/df that are similar to values observed in healthy liver, while the model parameters for material M2 give values similar to those observed in diseased liver. Figure 2 shows plots of the real and imaginary parts of $\mu(\omega)$ for the two material models. The phase velocity $c(f)$ and shear attenuation $\alpha(f)$ calculated using the model parameters are shown as the true results in red in Fig. 6. We also note that the values of $c(f)$ and $\alpha(f)$ for the two materials are similar to values (not shown) calculated using a Voigt model with parameters representative of the parameters measured in liver by Chen *et al.*²⁶

Calculations of 2D-FT data were performed using a Gaussian excitation width $\sigma_x = 1 \text{ mm}$ and three values of the Gaussian excitation width σ_y : $\sigma_y = 1 \text{ mm}$, $\sigma_y = 2 \text{ mm}$, and $\sigma_y = 4 \text{ mm}$. These widths were chosen to correspond to a cylindrically symmetric excitation, an excitation with aspect ratio of 2:1 which corresponds approximately to the excitation geometry at the focal depth for measurements in liver,¹¹ and an excitation with aspect ratio 4:1 corresponding to a point in the acoustic intensity field far from the focal point.

The time dependence $W(t)$ in Eq. (37) of the excitation force was assumed to have a unit magnitude and was applied during the interval $0 \leq t \leq T$ so that $\tilde{W}(\omega)$ in Eq. (39) is given by

$$\tilde{W}(\omega) = \frac{i}{\omega} (e^{-i\omega T} - 1). \quad (42)$$

The excitation duration T was set to a value of $180 \mu\text{s}$ to agree with the duration used in the *in vivo* measurements in

TABLE I. Model parameters for materials M1 and M2.

Parameter	Material M1	Material M2
μ_1	2 kPa	4 kPa
μ_2	8 kPa	16 kPa
η	1.2 Pa s	4 Pa s

FIG. 2. Real (left) and imaginary (right) parts of the shear modulus $\mu(f)$ calculated from Eq. (41) for materials M1 and M2 in Table I.

human liver described by Wang *et al.*¹⁴ and Palmeri *et al.*¹⁵ and the finite element simulations of these measurements in Nightingale *et al.*¹¹ The window $W(t)$ and its Fourier transform $\tilde{W}(f)$ are plotted in Fig. 3.

All calculations were performed using MATLAB (The MathWorks, Natick, MA). In particular, we note that the sign conventions used in the MATLAB functions `fft` and `ifft` agree with Eqs. (8) and (9), respectively. The Faddeeva function $w(z) = e^{-z^2} \text{Erfc}(-iz)$ was evaluated using the Faddeeva package from Johnson.²⁷

Results are presented in Sec. V for the 2D-FT particle velocity signal calculated using Eqs. (39) and (36) on a grid of points with temporal frequency spacing of 0.001 kHz and spatial frequency spacing of 0.001 cycle/mm. For 2D-FT signals observed along the $+x$ axis, Eqs. (34) and (35) were evaluated by calculating the discrete Fourier transform (DFT) using the MATLAB functions `fft` and `ifft`. No filtering or additional processing was applied in the calculation of the DFT. The number of samples used for the DFT calculations was found by performing the calculations using a variable number of samples and observing that the phase velocity $c(f)$ and shear attenuation $\alpha(f)$ signals converged to within 0.05% of their limiting values when the calculations were performed using 20 000 samples. We note that with the spatial frequency spacing of 0.001 cycle/mm, 20 000 samples corresponds to a sample spacing in the spatial domain of 0.05 mm.

V. RESULTS

Figure 4 shows sample 2D-FT particle velocity signals calculated for materials M1 (left) and M2 (right) from Table I with Gaussian excitation widths of $\sigma_x = 1 \text{ mm}$ and $\sigma_y = 2 \text{ mm}$. For each material, the top left plot shows $|V^{(x)}(k, \omega)|^2$, the magnitude squared of the 2D-FT particle velocity signal calculated using Eqs. (39) and (36). As expected, signal is

FIG. 3. Excitation window $W(t)$ (left) and the real and imaginary parts of its Fourier transform $\tilde{W}(f)$ (right) from Eq. (42) for the case of an excitation duration $T = 180 \mu\text{s}$.

FIG. 4. Examples of 2D-FT signals calculated for materials M1 (left side) and M2 (right side) from Table I with Gaussian excitation widths of $\sigma_x = 1$ mm and $\sigma_y = 2$ mm. For each material, the top left 2D-FT image shows $|V^{(x)}(k, \omega)|^2$ calculated using Eqs. (39) and (36) for shear wave propagation observed along both the $+x$ and $-x$ axes, and the top right 2D-FT image shows $|V^{(+x)}(k, \omega)|^2$ obtained by applying Eq. (34) for shear waves observed only along the $+x$ axis. These signals are normalized by the maximum $|V^{(x)}(k, \omega)|^2$ signal at a frequency of 100 Hz. The bottom row shows profiles from both of the 2D-FT images for frequencies of $f = 100$ Hz, $f = 250$ Hz, and $f = 400$ Hz.

observed in all four quadrants of the Fourier domain because this expression describes shear wave propagation observed along both $+x$ and $-x$ axes. The top row in Fig. 4 also shows $|V^{(+x)}(k, \omega)|^2$ obtained by applying Eq. (34) to select only the signal observed along the $+x$ axis. The 2D-FT signals for each material are normalized by the maximum signal at $f = 100$ Hz from $|V^{(x)}(k, \omega)|^2$. In particular, we observe that even after applying Eq. (34), significant signal remains in the $(k > 0, \omega > 0)$ and $(k < 0, \omega < 0)$ quadrants due to wave motion excited by the extended source. This signal would not be observed if a directional filter²⁸ had been applied by setting the 2D-FT signal to zero in these quadrants. The bottom row of Fig. 4 shows the profiles observed at frequencies of $f = 100$ Hz, 250 Hz, and 400 Hz. We observe that application of Eq. (34) causes the position of the peak signal to shift slightly and thus affects the value of phase velocity calculated using Eq. (13).

Figure 5 shows the spatial-temporal shear wave signals $v(x, t)$ that have been calculated from the 2D-FT signals shown in Fig. 4 using the inverse Fourier transform,

$$v(x, t) = \mathcal{F}_x^{-1} \mathcal{F}_t^{-1} [V^{(+x)}(k, \omega)]. \quad (43)$$

As with the evaluation of Eq. (34), these operations were performed numerically using a discrete Fourier transform. We observe that these calculated signals appear similar to signals observed in experimental measurements or finite element simulations such as those shown in Nightingale *et al.*¹¹

Figure 6 shows the results for the 2D-FT analysis of shear wave propagation in materials M1 and M2 from Table I for Gaussian excitation widths $\sigma_x = 1$ mm and $\sigma_y = 1$ mm, $\sigma_y = 2$ mm, and $\sigma_y = 4$ mm. The top row of images shows the 2D-FT particle velocity signals $|V^{(+x)}(k, \omega)|^2$ calculated using

Eqs. (39), (36), and (34). Each of these signals are normalized in the same manner as described for the 2D-FT signals shown in Fig. 4.

The second row of Fig. 6 shows plots of the phase velocity $c(f)$ for the true results (red) calculated using Eq. (5), the 2D-FT results (black) determined from the 2D-FT signal using Eq. (13), and the 2D-FT results (blue) obtained by including the weighting factor of $\sqrt{x}$ from Nenadic *et al.*¹³ using Eq. (35). The third row of plots shows the

FIG. 5. Particle velocity signals $v(x, t)$ calculated using Eq. (43) for shear wave propagation observed along the $+x$ axis in materials M1 (left) and M2 (right) following excitations with Gaussian widths of $\sigma_x = 1$ mm and $\sigma_y = 2$ mm. These signals have been normalized by the maximum value of $v(x, t)$ at a lateral position of $x = 0$ mm. The bottom plots show the normalized particle velocity signals as a function of time for $x = 3$ mm, $x = 6$ mm, and $x = 9$ mm.

FIG. 6. Results obtained using the 2D-FT analysis of shear wave propagation in materials M1 and M2 from Table I for Gaussian excitation widths $\sigma_x = 1$ mm and $\sigma_y = 1$ mm, $\sigma_y = 2$ mm, and $\sigma_y = 4$ mm. The top row shows 2D-FT data calculated using Eqs. (39), (36), (34), and normalized as shown in Fig. 4. The second row shows the true phase velocity $c(f)$ (red) calculated using Eq. (5) and measured using Eq. (13) from unweighted 2D-FT data (black) and $\sqrt{x}$ weighted 2D-FT data (blue) calculated using Eq. (35). The third row shows percentage biases of the measured $c(f)$ relative to the true values. The fourth row shows true values of shear attenuation $\alpha(f)$ (red) calculated using Eq. (6) and measured values determined using Eq. (14) from unweighted 2D-FT data (black) and $\sqrt{x}$ weighted 2D-FT data (blue). The fifth row shows the percentage biases of the measured $\alpha(f)$ relative to the true values.

bias in $c(f)$ results obtained from the 2D-FT analysis expressed as a percentage of the true result. We observe that, in general, the speeds obtained using the 2D-FT analysis do not agree with the true results with biases exceeding 9% for material M1, and 3% for material M2, at frequencies greater than 400 Hz. The results obtained using the $\sqrt{x}$ weighted 2D-FT signals are in somewhat better agreement with the true results with biases less than 8% for both materials over the entire frequency range shown. In addition, we observe that, in general, the biases increase as a function of frequency and depend on the aspect ratio of the Gaussian excitation widths.

Similarly, rows 4 and 5 of Fig. 6 show the values of shear attenuation calculated using Eq. (14) and the percentage bias relative to the true results obtained using Eq. (6). For the results determined using the unweighted 2D-FT data, the biases are fairly large and exceed 20% for both materials over a large fraction of the frequency range plotted. Above a

frequency of roughly 40 Hz, the biases obtained using the $\sqrt{x}$ weighted 2D-FT data are considerably smaller, with values less than 4% for the case with $\sigma_y = 1$ mm and less than 18% for the cases with $\sigma_y = 2$ mm and $\sigma_y = 4$ mm for both materials. As with the phase velocity results above, we conclude that the biases introduced by the 2D-FT analysis depend on the frequency and the aspect ratio of the Gaussian sources, and that biases observed with the $\sqrt{x}$ weighted 2D-FT data are considerably smaller than obtained with the unweighted 2D-FT data.

Table II shows the results obtained by analyzing the phase velocity data shown in Fig. 6 using the linear dispersion model (2). Specifically, this model was fit to the $c(f)$ data over a frequency range $100 \text{ Hz} \leq f \leq 400 \text{ Hz}$, and values of $c(200 \text{ Hz})$ and dc/df determined from the fit are shown in Table II for the cases of materials M1 and M2 and the three Gaussian excitation widths $\sigma_y = 1$ mm, $\sigma_y = 2$ mm, and $\sigma_y = 4$ mm. The results in parentheses in Table II show the

TABLE II. Results of the linear dispersion analysis. The values in parentheses give the percentage biases relative to the true values of $c(200 \text{ Hz}) = 1.71 \text{ m/s}$ and $dc/df = 2.16 \text{ m/s/kHz}$ for material M1, and $c(200 \text{ Hz}) = 2.81 \text{ m/s}$ and $dc/df = 4.18 \text{ m/s/kHz}$ for material M2.

	Material M1			Material M2		
	$\sigma_y = 1 \text{ mm}$	$\sigma_y = 2 \text{ mm}$	$\sigma_y = 4 \text{ mm}$	$\sigma_y = 1 \text{ mm}$	$\sigma_y = 2 \text{ mm}$	$\sigma_y = 4 \text{ mm}$
$c(200 \text{ Hz}) \text{ (m/s)}$						
2D-FT	1.76 (2.6%)	1.79 (4.3%)	1.80 (4.9%)	2.82 (0.4%)	2.87 (2.2%)	2.92 (4.0%)
$\sqrt{x}$ weighted 2D-FT	1.73 (0.7%)	1.75 (2.1%)	1.77 (3.0%)	2.80 (-0.3%)	2.83 (0.9%)	2.88 (2.5%)
$dc/df \text{ (m/s/kHz)}$						
2D-FT	2.93 (35.4%)	2.96 (36.6%)	2.67 (23.6%)	4.63 (10.8%)	4.81 (15.2%)	4.77 (14.1%)
$\sqrt{x}$ weighted 2D-FT	2.50 (15.4%)	2.59 (19.6%)	2.45 (13.0%)	4.33 (3.7%)	4.51 (7.9%)	4.56 (9.0%)

percentage bias of the unweighted and $\sqrt{x}$ weighted 2D-FT results relative to the true results of $c(200 \text{ Hz}) = 1.71 \text{ m/s}$ and $dc/df = 2.16 \text{ m/s/kHz}$ for material M1, and $c(200 \text{ Hz}) = 2.81 \text{ m/s}$ and $dc/df = 4.18 \text{ m/s/kHz}$ for material M2.

As expected, we observe that the biases in the values of $c(200 \text{ Hz})$ obtained using the linear dispersion model are comparable to the biases observed in the values of $c(f)$ in Fig. 6 with somewhat smaller biases for the $\sqrt{x}$ weighted 2D-FT results. The biases in the dispersion slope are fairly large with values from the unweighted 2D-FT analysis of 24%–37% for material M1 and 10%–15% for material M2. The results obtained using the $\sqrt{x}$ weighted 2D-FT analysis are smaller with values of 13%–20% for material M1 and 4%–9% for material M2. These large biases in the dispersion slope are a result of the frequency dependent bias in the phase velocity observed in Fig. 6.

Figure 7 shows results obtained when the $\sqrt{x}$ weighting factor in Eq. (35) is replaced by a weighting factor with the form x^p where the power p can be varied. The effect of varying this power was evaluated by constructing 2D-FT data sets using the weighting factor x^p and calculating the percentage biases in $c(f)$ and $\alpha(f)$ measured from the 2D-FT data using Eqs. (13) and (14). The total bias for the weighting factor x^p was found by calculating the root-mean-square value of the percentage biases in the $c(f)$ and $\alpha(f)$ measurements over a frequency range $100 \text{ Hz} \leq f \leq 400 \text{ Hz}$. The top row of plots in Fig. 7 shows the total bias as a function of weighting power p for materials M1 and M2 from Table I and the three Gaussian excitation functions shown in Fig. 6 with $\sigma_x = 1 \text{ mm}$ and $\sigma_y = 1 \text{ mm}$ (left), $\sigma_y = 2 \text{ mm}$ (center), and $\sigma_y = 4 \text{ mm}$ (right). As the power of the weighting term is varied, the total bias reaches a minimum value and thereby defines an optimal power with minimum total bias. The bottom left plot in Fig. 7 shows the value of the optimal power of the weighting function as the value of σ_y , and thus the aspect ratio of the source, is varied. The bottom right plot shows the residual total bias remaining after constructing the 2D-FT using the weighting function x^p with optimal power.

VI. DISCUSSION

In this paper, we have derived an analytic expression for the Fourier description of shear wave propagation in a viscoelastic material by writing the equation of motion in the Fourier domain and solving it for the case of a spatially

asymmetric Gaussian source. This result has been used to evaluate the measurement of the frequency dependent phase velocity $c(\omega)$ and shear attenuation $\alpha(\omega)$ from the 2D-FT description of spatial-temporal shear wave data. By using a specific viscoelastic model, biases in the values of $c(\omega)$ and $\alpha(\omega)$ determined from the 2D-FT analysis can be compared to the true values calculated from the model parameters. In particular, by using the analytic expression for 2D-FT data presented in this study, it is possible to avoid confounding factors such as the finite spatial or temporal ranges in experimental or simulation data.

Results presented in Fig. 6 show that the unweighted 2D-FT analysis method gives biased results for the measurement of $c(\omega)$ and $\alpha(\omega)$. A wide range of biases are observed with values that can exceed 10% for $c(\omega)$ and 40% for $\alpha(\omega)$

FIG. 7. Results obtained using a weighting factor of the form x^p with a variable power p in Eq. (35). The top row shows the total bias calculated as described in Sec. V for materials M1 and M2 from Table I and Gaussian sources with $\sigma_x = 1 \text{ mm}$ and $\sigma_y = 1 \text{ mm}$ (left), $\sigma_y = 2 \text{ mm}$ (center), and $\sigma_y = 4 \text{ mm}$ (right). The bottom left plot shows the optimal power corresponding to the minimum total bias plotted as a function of σ_y . The bottom right plot shows the total bias achieved using the weighting function with optimal power (solid lines) and also the total bias obtained using $\sqrt{x}$ (i.e., $p = 0.5$) weighted 2D-FT data (dashed lines) and unweighted (i.e., $p = 0$) 2D-FT data (dotted-dashed lines).

depending on the specific frequency, aspect ratio of the source, and material considered. For the phase velocity $c(\omega)$, these biases generally increase with frequency and, as seen in Table II, this behavior can give large biases in the dispersion slope when using a linear model to analyze the dispersion. These biases are sufficiently large that they could be an important contribution to the biases observed in 2D-FT analysis of simulation data by Nightingale *et al.*¹¹

As shown in Fig. 6 and Table II, the biases in $c(\omega)$ and $\alpha(\omega)$ can be reduced considerably by applying a weighting factor of $\sqrt{x}$ in the spatial domain as described by Nenadic *et al.*^{12,13} However, like the results obtained with unweighted 2D-FT data, the biases in $c(f)$ increase systematically with frequency and can still give large percentage biases in the dispersion slope dc/df . The $\sqrt{x}$ weighting factor was suggested based on the difference between a plane wave described by Eq. (7) and expression (17) describing the asymptotic form of a wave propagating away from a cylindrical source. However, for shear wave propagation observed near the source and for spatially asymmetric sources, the wavefunction would be expected to deviate from the asymptotic form, and suggests that a different weighting factor could reduce the bias further. The optimal weighting factor would be expected to depend on the relative contributions of shear wave signal observed close to, and far from, the source and, thus, on the shear wave attenuation of the material.

Figure 7 shows the results obtained by considering a variable weighting factor of the form x^p in Eq. (35). This form was chosen because it allows a continuous variation of the weighting factor including a value of unity (i.e., $p = 0$) as expected for the plane wave (7) and a value of $\sqrt{x}$ (i.e., $p = 0.5$) as expected for the asymptotic form of the cylindrically expanding wave (17). In particular, the optimal power corresponding to the minimum total bias of the measured phase velocity and attenuation is found to vary as the aspect ratio of the source varies. For an aspect ratio $\sigma_y/\sigma_x = 1$, the source is cylindrical, and the optimal power is approximately equal to 0.5 as expected from the asymptotic form (17). As the aspect ratio increases, the increased values of σ_y cause the source to appear more planar, and the optimal power decreases. This decrease is somewhat counterintuitive because sources with aspect ratios $\sigma_y/\sigma_x > 1$ still appear cylindrical at sufficiently large distances where the asymptotic form (17) would be appropriate. However, because of the attenuation in the viscoelastic material, the relative contribution of the shear wave signal from these asymptotic ranges will be small, and the source appears more planar due to the contributions of larger signals from regions near the source. Thus, in Fig. 7, as the aspect ratio of the source increases, the total bias obtained using unweighted 2D-FT data decreases and the total bias obtained using $\sqrt{x}$ weighted 2D-FT data increases. The relative contributions of shear wave signal from regions near to, and far from, the source will also depend on the value of shear wave attenuation. Thus, as seen for the case of materials M1 and M2 in Fig. 7, the optimal weighting factor depends on the properties of the material under investigation. From Fig. 7, we note that for typical geometries employed in abdominal imaging with a source aspect ratio of $\sigma_y/\sigma_x = 2$, the use of the optimal power

p gives total biases of 4.9% and 1.8% for materials M1 and M2, respectively, compared to biases of 7.8% and 5.1%, respectively, obtained using the $\sqrt{x}$ weighting with $p = 0.5$.

In addition, we have also investigated the application of the x^p weighting factor to the 58 simulation data sets that modeled the experimental measurements in human liver described by Nightingale *et al.*¹¹ After applying the weighting factor in the spatial domain, these data sets were analyzed using the same linear dispersion analysis procedure described by Nightingale *et al.*¹¹ to determine the phase velocity $c(200 \text{ Hz})$ and dispersion slope dc/df . Varying the weighting power p gave a minimum total bias with a weighting power $p = 0.41$ and average biases for $c(200 \text{ Hz})$ and dc/df of 3% and 21%, respectively. These values can be compared with the biases of 9% and 51%, respectively, originally obtained by Nightingale *et al.*¹¹ using unweighted data, and also biases obtained using $\sqrt{x}$ weighted data of 3% and 24%, respectively. We also note from the discussion in the previous paragraph that the optimal weighting factor depends on the material properties due to different relative contributions of signals from locations near to, and far from, the excitation. Thus, it is expected that these biases could be reduced further by introducing a material-specific weighting factor. However, a systematic investigation of these higher order effects is the subject of future investigation.

Finally, we note that the analytic solution (39) can be checked for the case of an elastic material with no dispersion or attenuation so that the shear modulus μ is real and independent of frequency. Then the factor of $1/\sqrt{k^2 - \rho\omega^2/\mu}$ in Eq. (39) will diverge for $k^2 = \rho\omega^2/\mu$, and the 2D-FT signal will be proportional to the delta function $\delta(k \pm \omega/\sqrt{\mu/\rho})$. Applying the 2D-FT analysis to this signal gives the phase velocity $c = \sqrt{\mu/\rho}$ and attenuation $\alpha = 0$ as expected.

In addition, we also note that Parker and Baddour²⁹ have presented analytic expressions describing shear wave propagation in a dispersive medium following a localized, impulsive excitation. Those results rely on Theorems 3 and 4 from Baddour³⁰ whose proofs use contour integration and require that (using Baddour's notation) the integrand

$$\frac{\phi(z)}{z^2 - k_s} H_n^{(1,2)}(zr)z$$

vanishes on the contour for $\text{Im}(z) \rightarrow \pm \infty$, and that the function $\phi(z)$ is an odd function of z for the case $n = 0$ as needed here for the inverse Hankel transform in Eq. (33). Neither of these conditions hold for the Gaussian source functions considered here.

VII. CONCLUSION

In this paper, we derive the equation of motion describing shear wave propagation in a viscoelastic material in the Fourier domain and solve the equation for the case of an asymmetric Gaussian source. The result is an exact, analytic expression for the 2D-FT description of the propagating shear wave. This solution is used to evaluate the possibility of measuring the complex, frequency dependent shear modulus $\mu(\omega)$ in a viscoelastic material by determining the phase

velocity $c(\omega)$ and shear attenuation $\alpha(\omega)$ using an analysis procedure derived under the assumption the shear wave is described as a plane wave, or as the asymptotic form of a wave expanding radially from a cylindrically symmetric source.^{12,13} Biases introduced by the analysis procedure are determined by calculating the exact 2D-FT signal using a specific viscoelastic model and comparing values of $c(\omega)$ and $\alpha(\omega)$ to the true values determined from the model parameters. Results indicate that the measurement of $c(\omega)$ and $\alpha(\omega)$ using the 2D-FT analysis method gives biased results, with the size of the bias depending on the analysis frequency, the aspect ratio of the excitation function, and the specific material model. For measurements of the phase velocity $c(\omega)$, the systematic increase in bias with frequency leads to large biases in the dispersion slope that are similar to the biases observed in the analysis of simulation data sets modeling experimental data in human liver by Nightingale *et al.*¹¹ These biases can be reduced by weighting the shear wave data in the spatial domain by a factor of x^p with the power p determined from the aspect ratio of the excitation function and material properties.

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